

# KMS STATES ON THE $C^*$ -ALGEBRA OF A HIGHER-RANK GRAPH AND PERIODICITY IN THE PATH SPACE

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ABSTRACT. We study the KMS states of the  $C^*$ -algebra of a strongly connected finite  $k$ -graph. We find that there is only one 1-parameter subgroup of the gauge action that can admit a KMS state. The extreme KMS states for this preferred dynamics are parameterised by the characters of an abelian group that captures the periodicity in the infinite-path space of the graph. We deduce that there is a unique KMS state if and only if the  $k$ -graph  $C^*$ -algebra is simple, giving a complete answer to a question of Yang. When the  $k$ -graph  $C^*$ -algebra is not simple, our results reveal a phase change of an unexpected nature in its Toeplitz extension.

## 1. INTRODUCTION

Higher-rank graphs ( $k$ -graphs) are higher-dimensional analogues of directed graphs (the 1-graphs). Each  $k$ -graph  $\Lambda$  has a  $C^*$ -algebra  $C^*(\Lambda)$  generated by a family of partial isometries satisfying relations analogous to the Cuntz-Krieger relations for a directed graph [16]. These graphs and their algebras have attracted a great deal of attention, and the algebras provide illustrative examples for various active areas of research [22, 23, 24, 25, 29]. Much of the structure theory of graph algebras carries over to  $k$ -graphs, though often with significant changes and considerable difficulty. It took quite a while, for example, to find a necessary and sufficient condition for simplicity [27], and even for 2-graphs with a single vertex, this condition is hard to verify [8].

Here we study the KMS states for a natural dynamics on the  $C^*$ -algebra of a  $k$ -graph. When a  $C^*$ -algebra  $A$  represents the observables in a physical model, time evolution is modelled by a continuous action of  $\mathbb{R}$  (a dynamics) on  $A$ . The equilibrium states are the states on  $A$  that satisfy a commutation relation (the KMS condition) involving a parameter called the inverse temperature. The KMS condition makes sense for any action of  $\mathbb{R}$  on any  $C^*$ -algebra  $A$ , no matter where  $A$  comes from, and the behaviour of the KMS states always seems to reflect important structural properties of  $A$ . In recent years there has been a flurry of activity in which various authors have studied the KMS states on families of Toeplitz algebras arising in number theory [1, 18, 7], in the representation theory of self-similar groups [19], and in graph algebras [10, 15, 12, 14, 5]. These dynamics all manifest a phase transition in which a simplex of KMS states collapses to a simplex of lower dimension at a critical inverse temperature.

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The  $C^*$ -algebra of a finite  $k$ -graph admits a preferred dynamics  $\alpha$ , which we discuss at the start of §7. Yang has studied this dynamics for  $k$ -graphs with a single vertex [31, 32]. She has made a conjecture about the KMS states and has verified this conjecture for  $k = 2$  [33]. Here we determine the full simplex of KMS states on  $(C^*(\Lambda), \alpha)$  for a large class of finite  $k$ -graphs, including all  $k$ -graphs with one vertex. This allows us to verify Yang's conjecture for all  $k$ , and in far greater generality than it was posed. It also completes the description of the KMS states for the preferred dynamics on the Toeplitz algebra of  $\Lambda$ . Many examples exhibit an unexpected phase transition in which the simplex expands dramatically at the critical inverse temperature instead of collapsing.

Our results deal with finite  $k$ -graphs that are strongly connected in the sense that there is a directed path from  $v$  to  $w$  for each pair  $v, w$  of vertices. Each  $r \in [0, \infty)^k$  determines a homomorphism of  $\mathbb{R}$  into  $\mathbb{T}^k$ , and composing this with the gauge action on  $C^*(\Lambda)$  yields a dynamics  $\alpha$ . We study KMS states of  $(C^*(\Lambda), \alpha)$ . Previous analyses [9, 15, 13] for finite 1-graphs depend on Perron-Frobenius theory. So we start in Section 3 by developing a Perron-Frobenius theory for families of pairwise commuting non-negative matrices. In Section 4 we apply our Perron-Frobenius theory to the coordinate matrices of  $k$ -graphs. We characterise the vectors  $r$  for which the associated dynamics admits KMS states on the Toeplitz algebra  $\mathcal{T}C^*(\Lambda)$ . We show that only one dynamics admits KMS states on  $C^*(\Lambda)$ . We call this the preferred dynamics.

Our main result describes the KMS states of  $C^*(\Lambda)$  in terms of states on the  $C^*$ -algebra of an abelian group  $\text{Per } \Lambda$  that captures periodicity in the infinite-path space  $\Lambda^\infty$ . We describe  $\text{Per } \Lambda$  and its properties in Section 5, and construct in Section 6 an injection  $\pi_U : C^*(\text{Per } \Lambda) \rightarrow C^*(\Lambda)$ . Our main theorem, Theorem 7.1, says that the map  $\phi \mapsto \phi \circ \pi_U$  is an isomorphism from the KMS simplex of  $C^*(\Lambda)$  to the state space of  $C^*(\text{Per } \Lambda)$ . (The inverse is described later in Remark 10.4.) For  $k = 1$ , our theorem recovers the characterisations of KMS states for Cuntz algebras [21] and Cuntz-Krieger algebras [9], and we obtain a new description of the unique KMS state as an integral of vector states.

The proof of our main theorem occupies Sections 8–10. In Section 8 we show that the KMS states of  $C^*(\Lambda)$  all induce the same probability measure  $M$  on the spectrum  $\Lambda^\infty$  of the diagonal. We deduce in Theorem 9.1 a formula for a KMS state  $\phi$  in terms of  $\phi \circ \pi_U$ . In Section 10 we construct a particular KMS state  $\phi_1$  of  $C^*(\Lambda)$  as an integral against  $M$  of vector states. Unlike for  $k = 1$  [9, 5], this KMS state is not always supported on the fixed-point algebra for the gauge action. Composing  $\phi_1$  with gauge automorphisms then yields more KMS states  $\phi_z$  (Corollary 10.3). To prove our main theorem, we use Theorem 9.1 to see that  $\phi \mapsto \phi \circ \pi_U$  is an affine injection, and then establish surjectivity by showing that every pure state of  $C^*(\text{Per } \Lambda)$  has the form  $\phi_z \circ \pi_U$ .

In Section 11, we discuss three applications of our main result. First, we prove in Theorem 11.1 that  $C^*(\Lambda)$  has a unique gauge-invariant KMS state, and that this KMS state is a factor state if and only if  $\Lambda$  is aperiodic. Restricting this result to  $k$ -graphs with one vertex confirms Yang's conjecture in [33]. Second, we describe the phase transition in  $\mathcal{T}C^*(\Lambda)$  at the critical inverse temperature 1. For many  $k$ -graphs, the KMS simplex expands at the critical inverse temperature from a  $(|\Lambda^0| - 1)$ -dimensional simplex to an infinite-dimensional simplex. Third, we show that the KMS simplex of  $C^*(\Lambda)$  is highly symmetric: it carries a free and transitive action of  $(\text{Per } \Lambda)^\wedge$ . We conclude in Section 12 by relating our results to Neshveyev's analysis of KMS states on groupoid  $C^*$ -algebras [20].

## 2. BACKGROUND

**2.1. Higher-rank graphs.** A higher-rank graph of rank  $k$ , or  $k$ -graph, is a countable category  $\Lambda$  equipped with a functor  $d : \Lambda \rightarrow \mathbb{N}^k$  satisfying the factorisation property: whenever  $d(\lambda) = m + n$  there exist unique  $\mu \in d^{-1}(m)$  and  $\nu \in d^{-1}(n)$  such that  $\lambda = \mu\nu$ . We denote  $d^{-1}(n)$  by  $\Lambda^n$ . The elements of  $\Lambda^0$  are precisely the identity morphisms, and we call them vertices. We refer to other elements of  $\Lambda$  as paths. We write  $r, s : \Lambda \rightarrow \Lambda^0$  for the maps determined by the domain and codomain maps in  $\Lambda$ . We assume that  $\Lambda^{e_i} \neq \emptyset$  for each generator  $e_i$  of  $\mathbb{N}^k$  (otherwise we can just regard  $\Lambda$  as a  $(k-1)$ -graph).

If  $\lambda = \mu\nu\tau \in \Lambda$  where  $d(\mu) = m$ ,  $d(\nu) = n - m$  and  $d(\tau) = d(\lambda) - n$ , then we denote  $\nu = \lambda(m, m+n)$ . We use the convention that for  $\lambda \in \Lambda$  and  $X \subseteq \Lambda$ ,

$$\lambda X = \{\lambda\mu : \mu \in X \text{ and } r(\mu) = s(\lambda)\}, \quad X\lambda = \{\mu\lambda : \mu \in X \text{ and } s(\mu) = r(\lambda)\},$$

and so forth. We write  $\Lambda^{\min}(\mu, \nu)$  for the set  $\{(\alpha, \beta) : \mu\alpha = \nu\beta \in \Lambda^{d(\mu) \vee d(\nu)}\}$ .

We say that  $\Lambda$  is *finite* if  $\Lambda^n$  is finite for all  $n \in \mathbb{N}^k$  and say it has *no sources* if  $v\Lambda^{e_i} \neq \emptyset$  for all  $v \in \Lambda^0$  and all  $e_i$ ; it follows that  $v\Lambda^n \neq \emptyset$  for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ . We say that  $\Lambda$  is *strongly connected* if, for all  $v, w \in \Lambda^0$ , the set  $v\Lambda w$  is nonempty.

**Lemma 2.1.** *Let  $\Lambda$  be a strongly connected  $k$ -graph. Then*

- (a)  $\Lambda$  has no sources and
- (b) for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ , the set  $\Lambda^n v \neq \emptyset$ .

*Proof.* For (a), let  $v \in \Lambda^0$  and  $i \in \{1, \dots, k\}$ . By our convention,  $\Lambda^{e_i} \neq \emptyset$ . Let  $\lambda \in \Lambda^{e_i}$ . Since  $\Lambda$  is strongly connected, there exists  $\mu \in v\Lambda r(\lambda)$ . By the factorisation property,  $\mu\lambda = \lambda'\mu'$  for some  $\lambda' \in v\Lambda^{e_i}$ . Thus  $v\Lambda^{e_i} \neq \emptyset$ . This gives (a). The proof of (b) is similar.  $\square$

**2.2. Higher-rank graph  $C^*$ -algebras.** Let  $\Lambda$  be a finite  $k$ -graph with no sources. A Cuntz-Krieger  $\Lambda$ -family is a collection  $\{t_\lambda : \lambda \in \Lambda\}$  of partial isometries in a  $C^*$ -algebra  $A$  such that

- (CK1) the elements  $\{t_v : v \in \Lambda^0\}$  are mutually orthogonal projections,
- (CK2)  $t_\mu t_\nu = t_{\mu\nu}$  when  $s(\mu) = r(\nu)$ ,
- (CK3)  $t_\mu^* t_\mu = t_{s(\mu)}$  for all  $\mu$ , and
- (CK4) for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ , we have  $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ .

The  $C^*$ -algebra  $C^*(\Lambda)$  of  $\Lambda$  is generated by a universal Cuntz-Krieger  $\Lambda$ -family  $\{s_\lambda\}$ . We write  $p_v := s_v$  for  $v \in \Lambda^0$ . The Cuntz-Krieger relations imply that for all  $\mu, \nu \in \Lambda$

$$s_\mu^* s_\nu = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)} s_\alpha s_\beta^*$$

(we interpret empty sums as zero). In particular, if  $d(\mu) = d(\nu)$ , then

$$s_\mu^* s_\nu = \delta_{\mu, \nu} p_{s(\mu)}.$$

Relations (CK1) and (CK2) then imply that  $C^*(\Lambda) = \overline{\text{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda, s(\mu) = s(\nu)\}$ . There is a strongly continuous action  $\gamma : \mathbb{T}^k \rightarrow \text{Aut } C^*(\Lambda)$  such that  $\gamma_z(p_v) = p_v$  and  $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$  for  $z \in \mathbb{T}^k$ . This action is called the *gauge action*.

**2.3. The Perron-Frobenius theorem.** There are several Perron-Frobenius theorems; the one we use here applies to irreducible matrices. Let  $S$  be a finite set. We say a matrix  $A \in M_S(\mathbb{C})$  is *non-negative* if  $A(s, t) \geq 0$  for all  $s, t \in S$  and is *positive* if  $A(s, t) > 0$  for all  $s, t \in S$ . A non-negative matrix  $A \in M_S$  is *irreducible* if for each  $s, t \in S$  there exists  $N \in \mathbb{N}$  such that  $A^N(s, t) > 0$ . Equivalently,  $A$  is irreducible if there is a finite subset  $F \subseteq \mathbb{N}$  such that  $\sum_{n \in F} A^n$  is positive.

Let  $A$  be an irreducible matrix. The Perron-Frobenius theorem (see, for example, [28, Theorem 1.5]) says that the spectral radius  $\rho(A)$  is an eigenvalue of  $A$  with a 1-dimensional eigenspace and a positive eigenvector; we call the unique positive eigenvector with eigenvalue  $\rho(A)$  and unit 1-norm the *unimodular Perron-Frobenius eigenvector* of  $A$ .

### 3. PERRON-FROBENIUS THEORY FOR COMMUTING MATRICES

In [14], we employed a version of the Perron-Frobenius theorem for pairwise commuting irreducible matrices [14, Lemma 2.1] to describe KMS states on the  $C^*$ -algebras of coordinatewise-irreducible  $k$ -graphs. David Pask subsequently pointed out to us that he and Kumjian had adapted a technique from Putnam [26] to prove a Perron-Frobenius theorem for strongly connected finite  $k$ -graphs in [17, Lemma 4.1]. In this section, we adapt Kumjian and Pask's ideas to formulate a Perron-Frobenius theorem for families of commuting non-negative matrices that are jointly irreducible in an appropriate sense. Our primary use for this theorem is in the context of finite  $k$ -graphs, and we deduce what we need to know about these in the next section. But our results are applicable to more general classes of matrices than those arising from  $k$ -graphs and may be of independent interest.

Let  $S$  be a finite set,  $\{A_1, \dots, A_k\} \subseteq M_S([0, \infty))$  a family of commuting matrices,  $n = (n_1, \dots, n_k) \in \mathbb{N}^k$  and  $F$  a finite subset of  $\mathbb{N}^k$ . We use the multi-index notation

$$A^n := \prod_{i=1}^k A_i^{n_i} \quad \text{and} \quad A_F := \sum_{n \in F} A^n.$$

We say that the family  $\{A_1, \dots, A_k\}$  is *irreducible* if each  $A_i \neq 0$  and there exists a finite subset  $F \subseteq \mathbb{N}^k$  such that  $A_F(s, t) > 0$  for all  $s, t \in S$ ; that is,  $A_F$  is positive. Observe that in an irreducible family of matrices, the individual  $A_i$  may not be irreducible. So an irreducible family of matrices is not the same thing as a family of irreducible matrices. For examples of this distinction arising from  $k$ -graphs, see Example 4.3.

**Proposition 3.1.** *Suppose that  $\{A_1, \dots, A_k\}$  is an irreducible family of nonzero commuting matrices in  $M_S([0, \infty))$ . Let  $F$  be a finite subset of  $\mathbb{N}^k$  such that  $A_F(s, t) > 0$  for all  $s, t \in S$  and let  $x$  be the unimodular Perron-Frobenius eigenvector of  $A_F$ .*

- (a) (i) *The vector  $x$  is the unique non-negative vector of unit 1-norm that is a common eigenvector of all the  $A_i$ .*
- (ii) *We have  $A_i x = \rho(A_i) x$  for each  $i$ , and each  $\rho(A_i) > 0$ .*
- (iii) *If  $z \in \mathbb{C}^S$  and  $A_i z = \rho(A_i) z$  for all  $1 \leq i \leq k$ , then  $z \in \mathbb{C} x$ .*
- (b) *Suppose that  $y \in [0, \infty)^S$  is non-zero and  $\lambda \in [0, \infty)^k$  satisfies  $A_i y \leq \lambda_i y$  for all  $1 \leq i \leq k$ .*
  - (i) *Then  $y > 0$  in the sense that each  $y_s > 0$ , and  $\lambda_i \geq \rho(A_i)$  for all  $1 \leq i \leq k$ .*
  - (ii) *If  $\lambda_i = \rho(A_i)$  for all  $1 \leq i \leq k$  and  $y$  has unit 1-norm, then  $y = x$ .*
- (c) *Let  $n \in \mathbb{N}^k$ . Then  $\rho(A^n) = \prod_{i=1}^k \rho(A_i)^{n_i} > 0$ .*

The following lemma helps in the proof of Proposition 3.1.

**Lemma 3.2.** *Let  $B \in M_S([0, \infty))$ , and suppose that  $x \in (0, \infty)^S$  and  $\lambda \geq 0$  satisfy  $Bx \leq \lambda x$ . Then  $\lambda \geq \rho(B)$ .*

*Proof.* Choose a sequence  $\{B_j\}$  in  $M_S((0, \infty))$  converging to  $B$ . Then  $B_j x \rightarrow Bx \leq \lambda x$ . Fix  $\epsilon > 0$ . The entries of  $x$  are strictly positive, and so  $B_j x < (\lambda + \epsilon)x$  for large  $j$ . Part (b) of the Subinvariance Theorem [28, Theorem 1.6] for the positive matrix  $B_j$  gives  $\lambda + \epsilon \geq \rho(B_j)$  for large  $j$ . Since the eigenvalues of a complex matrix vary continuously with its entries (see, for example, [11, Theorem B]), we have  $\rho(B_j) \rightarrow \rho(B)$  as  $j \rightarrow \infty$ . Hence  $\lambda + \epsilon \geq \rho(B)$ . Thus  $\lambda \geq \rho(B)$ .  $\square$

*Proof of Proposition 3.1.* Such a finite set  $F$  exists because  $\{A_1, \dots, A_k\}$  is irreducible.

(a) Since  $x$  is a Perron-Frobenius eigenvector,  $x > 0$  by [28, Theorem 1.5 (b) and (f)]. Let  $i \in \{1, \dots, k\}$ .

We have

$$A_F(A_i x) = A_i(A_F x) = \rho(A_F)A_i x.$$

So  $A_i x$  is a non-negative eigenvector for  $A_F$  with eigenvalue  $\rho(A_F)$ . Since the eigenspace corresponding to  $\rho(A_F)$  is one-dimensional ([28, Theorem 1.5 (f)]) we have  $A_i x = \lambda_i x$  for some  $\lambda_i \in [0, \infty)$ . To prove uniqueness, we first claim that

$$(3.1) \quad y > 0 \text{ and } A_i y = \eta_i y \text{ for all } i \quad \implies \quad \sum_{n \in F} \prod_i \eta_i^{n_i} = \rho(A_F).$$

To see this, suppose that  $y > 0$  and  $A_i y = \eta_i y$  for all  $i$ . We have

$$(3.2) \quad A_F y = \left( \sum_{n \in F} \prod_i A_i^{n_i} \right) y = \left( \sum_{n \in F} \prod_i \eta_i^{n_i} \right) y.$$

Thus  $y$  is an eigenvector of  $A_F$  with eigenvalue  $\eta := \sum_{n \in F} \prod_{i=1}^k \eta_i^{n_i}$ . Since  $\|y\|_1 = 1$ , some  $y_s > 0$ . Since  $A_F$  is positive, we have  $A_F(s, s) > 0$  and so  $\eta y_s = (A_F y)_s \geq A_F(s, s)y_s > 0$ . So  $\eta > 0$ . Since  $y \geq 0$  and  $y \neq 0$ , the ‘‘if’’ direction of the last sentence of the Subinvariance Theorem [28, Theorem 1.6] gives  $\eta = \rho(A_F)$ .

Now suppose that  $y$  is a nonnegative unimodular common eigenvector of the  $A_i$ . Then (3.1) and (3.2) show that  $x$  and  $y$  are non-negative and of the same norm in the same one-dimensional eigenspace of  $A_F$ , hence are equal. This completes the proof of (ai).

Since the  $A_i$  and  $x$  are real and non-negative, and by definition of the spectral radius, each  $0 \leq \lambda_i \leq \rho(A_i)$ . Lemma 3.2 (applied to  $A_i$ ,  $x$  and  $\lambda_i$ ) implies that  $\lambda_i \geq \rho(A_i)$  as well, giving  $\lambda_i = \rho(A_i)$ . Thus  $A_i x = \rho(A_i)x$  for each  $i$ . Since  $x$  is positive and  $A_i \neq 0$  this forces each  $\rho(A_i) > 0$ . This proves (aii).

The claim (3.1) applied with  $y = x$  and  $\eta_i = \rho(A_i)$  now gives

$$(3.3) \quad \rho(A_F) = \sum_{n \in F} \prod_i \rho(A_i)^{n_i}.$$

For (aiii) suppose that  $A_i z = \rho(A_i)z$  for  $1 \leq i \leq k$ . Then

$$A_F z = \left( \sum_{n \in F} \prod_i \rho(A_i)^{n_i} \right) z = \rho(A_F)z$$

using (3.3). Thus  $z$  is an eigenvector of  $A_F$  with eigenvalue  $\rho(A_F)$ . The eigenspace of the Perron-Frobenius eigenvalue  $\rho(A_F)$  is one-dimensional, and hence  $z \in \mathbb{C}x$ .

(b) Fix  $s \in S$ . Since  $y \neq 0$ , there exists  $t \in S$  such that  $y_t > 0$ . Since  $\{A_1, \dots, A_k\}$  is an irreducible family, there exists  $n \in \mathbb{N}^k$  such that  $A^n(s, t) > 0$ . Then

$$\lambda^n y_s = \left( \prod_i \lambda_i^{n_i} \right) y_s \geq (A^n y)_s \geq A^n(s, t) y_t > 0.$$

Thus  $y_s > 0$  for all  $s \in S$ , so  $y > 0$ . Next, fix  $i$ . By assumption,  $\lambda_i \geq 0$  and  $A_i y \leq \lambda_i y$ . Thus Lemma 3.2 applied to  $A_i$ ,  $\lambda_i$  and  $y$  gives  $\lambda_i \geq \rho(A_i)$ . This establishes (bi).

Next, suppose that  $A_i y \leq \rho(A_i) y$  for  $1 \leq i \leq k$ . Then

$$A_F y \leq \left( \sum_{n \in F} \prod_i \rho(A_i)^{n_i} \right) y = \rho(A_F) y$$

using (3.3). So the ‘‘only-if’’ direction of the last sentence of the Subinvariance Theorem [28, Theorem 1.6] says that  $A_F y = \rho(A_F) y$ . Now  $x$  and  $y$  are non-negative of the same norm in the same one-dimensional eigenspace, hence are equal. This gives (bii).

(c) By (aai),  $x$  is a common eigenvector of the  $A_i$  with eigenvalue  $\rho(A_i)$ . Thus

$$A^n x = \left( \prod_i A_i^{n_i} \right) x = \left( \prod_i \rho(A_i)^{n_i} \right) x.$$

and hence  $\prod_i \rho(A_i)^{n_i} \leq \rho(A^n)$ . Since  $x > 0$ , Lemma 3.2 implies that  $\prod_i \rho(A_i)^{n_i} \geq \rho(A^n)$ . Now  $\rho(A^n) = \prod_i \rho(A_i)^{n_i} > 0$  because each  $\rho(A_i) > 0$  by (aai).  $\square$

#### 4. KMS STATES ON TOEPLITZ ALGEBRAS OF STRONGLY CONNECTED $k$ -GRAPHS

In this section, we apply the results of Section 3 to the coordinate matrices of finite  $k$ -graphs. We use them to improve the results of [14] about which dynamics on the Toeplitz algebra of a coordinatewise-irreducible  $k$ -graph admit KMS states. We will also use the results of this section extensively later to characterise the KMS states on the Cuntz-Krieger algebras of finite  $k$ -graphs.

Let  $\Lambda$  be a finite  $k$ -graph. For  $1 \leq i \leq k$ , let  $A_i$  be the matrix in  $M_{\Lambda^0}$  with entries  $A_i(v, w) = |v\Lambda^{e_i}w|$  for  $v, w \in \Lambda^0$ . We call the  $A_i$  the *coordinate matrices* of  $\Lambda$ .

**Lemma 4.1.** *Let  $\Lambda$  be a finite  $k$ -graph with coordinate matrices  $A_1, \dots, A_k$ . Then the  $A_i$  are nonzero pairwise-commuting matrices, and  $\Lambda$  is strongly connected if and only if  $\{A_1, \dots, A_k\}$  is an irreducible family of matrices.*

*Proof.* The  $A_i$  are nonzero by our convention that each  $\Lambda^{e_i}$  is nonempty. The factorisation property of  $\Lambda$  ensures that

$$(A_i A_j)(v, w) = |v\Lambda^{e_i+e_j}w| = (A_j A_i)(v, w),$$

so the  $A_i$  commute.

Suppose that  $\Lambda$  is strongly connected. Let  $v, w \in \Lambda^0$ . There exists  $n_{v,w} \in \mathbb{N}^k$  such that  $v\Lambda^{n_{v,w}}w \neq \emptyset$  because  $\Lambda$  is strongly connected. Now  $F := \{n_{v,w} : v, w \in \Lambda^0\}$  satisfies  $A_F(v, w) \geq A_{n_{v,w}}(v, w) > 0$  for all  $v, w$ . Thus  $\{A_1, \dots, A_k\}$  is an irreducible family.

Now suppose that  $\{A_1, \dots, A_k\}$  is an irreducible family. Choose  $F$  such that  $A_F$  is positive. For  $v, w \in \Lambda^0$ , we have  $A_F(v, w) \neq 0$  and so there exists  $n \in F$  such that  $|v\Lambda^n w| = A^n(v, w) \neq 0$ . So  $\Lambda$  is strongly connected.  $\square$

The next corollary sums up how we will use the results of Section 3.

**Corollary 4.2.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph. For  $1 \leq i \leq k$ , let  $A_i \in M_{\Lambda^0}([0, \infty))$  be the matrix with entries  $A_i(v, w) = |v\Lambda^{e_i}w|$ .*

- (a) Each  $\rho(A_i) > 0$ , and for  $n \in \mathbb{N}^k$ , we have  $\rho(A^n) = \prod_i \rho(A_i)^{n_i} > 0$ .
- (b) There exists a unique non-negative vector  $x^\Lambda \in [0, \infty)^{\Lambda^0}$  with unit 1-norm such that  $A_i x^\Lambda = \rho(A_i) x^\Lambda$  for all  $1 \leq i \leq k$ . Moreover,  $x^\Lambda > 0$  in the sense that  $x_v^\Lambda > 0$  for all  $v \in \Lambda^0$ .
- (c) If  $z \in \mathbb{C}^{\Lambda^0}$  and  $A_i z = \rho(A_i) z$  for all  $1 \leq i \leq k$ , then  $z \in \mathbb{C} x^\Lambda$ .
- (d) If  $y \in [0, \infty)^{\Lambda^0}$  has unit 1-norm and  $A_i y \leq \rho(A_i) y$  for all  $1 \leq i \leq k$ , then  $y = x^\Lambda$ .

*Proof.* Lemma 4.1 shows that the  $A_i$  are an irreducible family. So (a) is immediate from parts (aii) and (c) of Proposition 3.1. By Proposition 3.1(ai), the unimodular Perron-Frobenius eigenvector  $x^\Lambda$  of  $A_F$  is the unique non-negative, unimodular common eigenvector of the  $A_i$ . Since  $x^\Lambda$  is a Perron-Frobenius eigenvector for an irreducible matrix,  $x^\Lambda > 0$ . This gives (b). Parts (c) and (d) follow from parts (aiii) and (bii) of Proposition 3.1 respectively.  $\square$

*Example 4.3.* Corollary 4.2 is an improvement on [14, Proposition 7.1], which applies to finite  $k$ -graphs that are coordinatewise irreducible in the sense that each  $A_i$  is an irreducible matrix. To see that there are many strongly connected  $k$ -graphs that are not coordinatewise irreducible, consider strongly connected 1-graphs  $E, F$  and suppose that  $F$  has at least two vertices. Let  $\Lambda$  be the cartesian-product 2-graph  $\Lambda = E \times F$ . The connected components of the coordinate graph  $(\Lambda^0, \Lambda^{(1,0)}, r, s)$  are the sets

$$E^0 \times \{v\}, \quad v \in F^0.$$

So  $A_1$  is block-diagonal with blocks indexed by  $F^0$ , and in particular  $\Lambda$  is not coordinatewise irreducible. But it is strongly connected: take  $(u_1, v_1), (u_2, v_2) \in E^0 \times F^0$  and use that  $E$  and  $F$  are strongly connected to find  $\mu \in u_1 E^* u_2$  and  $\nu \in v_1 F^* v_2$ ; then  $(\mu, \nu) \in (u_1, v_1) \Lambda (u_2, v_2)$ .

**Definition 4.4.** Let  $\Lambda$  be a strongly connected finite  $k$ -graph. We call the vector  $x^\Lambda$  of Corollary 4.2 the *unimodular Perron-Frobenius eigenvector* of  $\Lambda$ .

We write  $\rho(\Lambda)$  for the vector  $(\rho(A_i)) \in [0, \infty)^k$ , and  $\ln \rho(\Lambda)$  for the vector  $(\ln \rho(A_i)) \in [-\infty, \infty)^k$ . For  $n \in \mathbb{N}^k$  we have  $A^n x^\Lambda = \rho(\Lambda)^n x^\Lambda$  where  $\rho(\Lambda)^n := \prod_{i=1}^k \rho(A_i)^{n_i}$  is defined using multi-index notation.

*Remark 4.5.* At first glance, Corollary 4.2, which allows Definition 4.4, appears very similar to Proposition 7.1 of [14] except that it has a weaker hypothesis. In Proposition 3.1, however, the individual  $A_i$  need not be irreducible, and so it does not make sense to discuss “the unique unimodular Perron-Frobenius eigenvectors of the  $A_i$ .” In [14, Proposition 7.1], the  $A_i$  and  $A^n$  are irreducible, so they each have a unique unimodular Perron-Frobenius eigenvector; the proposition asserts that these eigenvectors are all equal. When we apply Proposition 3.1 to the family of coordinate matrices  $A_i$  of a strongly connected graph, each  $A_i$  may have multiple linearly independent non-negative eigenvectors. The result asserts that there is a unique non-negative unimodular eigenvector  $x^\Lambda$  common to all the  $A_i$ , and that the spectral radius of each  $A^n$  is achieved at  $x^\Lambda$ .

We finish the section by using Proposition 3.1 to strengthen Corollaries 4.3 and 4.4 of [14]. Recall that a Toeplitz-Cuntz-Krieger  $\Lambda$ -family consists of partial isometries  $\{T_\lambda : \lambda \in \Lambda\}$  satisfying (CK1)–(CK3) and the additional relations

- (T4)  $T_v \geq \sum_{\lambda \in v \Lambda^n} T_\lambda T_\lambda^*$  for all  $v \in \Lambda^0$  and  $n \in \mathbb{N}^k$ ; and
- (T5)  $T_\mu^* T_\nu = \sum_{(\alpha, \beta) \in \Lambda^{\min(\mu, \nu)}} T_\alpha T_\beta^*$  for all  $\mu, \nu$ , where empty sums are interpreted as zero.

The Toeplitz algebra  $\mathcal{TC}^*(\Lambda)$  of  $\Lambda$  is generated by a universal Toeplitz-Cuntz-Krieger  $\Lambda$ -family  $\{t_\lambda\}$ . We write  $q_v := t_v$  for  $v \in \Lambda^0$ .

There is a strongly continuous action  $\gamma : \mathbb{T}^k \rightarrow \text{Aut } \mathcal{TC}^*(\Lambda)$  such that  $\gamma_z(q_v) = q_v$  and  $\gamma_z(t_\lambda) = z^{d(\lambda)}t_\lambda$  for  $z \in \mathbb{T}^k$ . This action is called the *gauge action*. We use the same letter  $\gamma$  for the gauge actions on  $C^*(\Lambda)$  and  $\mathcal{TC}^*(\Lambda)$ ; this is safe because the quotient map of  $\mathcal{TC}^*(\Lambda)$  onto  $C^*(\Lambda)$  intertwines the two.

**Corollary 4.6.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Let  $\beta \in [0, \infty)$ . Fix  $r \in \mathbb{R}^k$  and define  $\alpha : \mathbb{R} \rightarrow \text{Aut } \mathcal{TC}^*(\Lambda)$  by  $\alpha_t = \gamma_{e^{itr}}$ .*

- (a) *There exists a  $\text{KMS}_\beta$  state for  $(\mathcal{TC}^*(\Lambda), \alpha)$  if and only if  $\beta r \geq \ln \rho(\Lambda)$ .*
- (b) *If there is a  $\text{KMS}_\beta$  state for  $(\mathcal{TC}^*(\Lambda), \alpha)$  that factors through  $C^*(\Lambda)$ , then  $\beta r = \ln \rho(\Lambda)$ .*
- (c) *If  $\beta r = \ln \rho(\Lambda)$ , then every  $\text{KMS}_\beta$  state for  $(\mathcal{TC}^*(\Lambda), \alpha)$  factors through  $C^*(\Lambda)$ .*

*Proof.* (a) First suppose that  $\phi$  is a  $\text{KMS}_\beta$  state. Let  $v \in \Lambda^0$  and set  $m_v^\phi := \phi(q_v)$ . For  $1 \leq i \leq k$ , relation (T4) and the KMS condition give

$$(4.1) \quad \begin{aligned} 0 &\leq \phi\left(q_v - \sum_{\lambda \in v\Lambda^{e_i}} t_\lambda t_\lambda^*\right) = \phi(q_v) - \sum_{w \in \Lambda^0} |v\Lambda^{e_i}w| e^{-\beta r_i} \phi(t_\lambda^* t_\lambda) \\ &= \phi(q_v) - e^{-\beta r_i} \sum_{w \in \Lambda^0} A(v, w) \phi(t_{s(\lambda)}) = (m^\phi - e^{-\beta r_i} A_i m^\phi)_v. \end{aligned}$$

Hence  $A_i m^\phi \leq e^{\beta r_i} m^\phi$  for each  $i$ . Now Proposition 3.1(b), applied to  $A_i$ ,  $e^{\beta r_i}$  and  $m^\phi$ , implies that each  $e^{\beta r_i} \geq \rho(A_i)$ . Thus  $\beta r \geq \ln \rho(\Lambda)$ .

Second, suppose that  $\beta r \geq \ln \rho(\Lambda)$ . Choose a sequence  $\{r_n\}$  in  $\mathbb{R}^k$  converging to  $r$  from above and a sequence  $\beta_n$  converging to  $\beta$  from above. So each  $\beta_n r_n > \ln \rho(\Lambda)$ . For each  $n$  let  $\alpha^{r_n}$  be the dynamics  $\alpha_t^{r_n} = \gamma_{e^{itr_n}}$ . By [14, Theorem 6.1] there exists, for each  $n$ , a  $\text{KMS}_{\beta_n}$  state  $\phi_n$  of  $(\mathcal{TC}^*(\Lambda), \alpha^{r_n})$ . We have  $\alpha_t^{r_n}(t_\mu t_\nu^*) \rightarrow \alpha_t(t_\mu t_\nu^*)$  for all  $\mu, \nu$ , and so an  $\varepsilon/3$  argument shows that  $\|\alpha_t^{r_n}(a) - \alpha_t(a)\| \rightarrow 0$  for all  $a$ . Now [3, Proposition 5.3.25] shows that  $(\mathcal{TC}^*(\Lambda), \alpha)$  has a  $\text{KMS}_\beta$  state.

(b) Suppose that  $\phi$  is a  $\text{KMS}_\beta$  state of  $(\mathcal{TC}^*(\Lambda), \alpha)$  that factors through  $C^*(\Lambda)$ . Then we have equality in (4.1), and so  $m^\phi$  is a unimodular non-negative eigenvector of each  $A_i$  with eigenvalue  $e^{\beta r_i}$ . Thus Proposition 3.1(ai) and (a) imply that  $e^{\beta r_i} = \rho(A_i)$  for each  $i$ . Thus  $\beta r = \ln \rho(\Lambda)$ .

(c) Suppose that  $\beta r = \ln \rho(\Lambda)$  and that  $\phi$  is a  $\text{KMS}_\beta$  state of  $(\mathcal{TC}^*(\Lambda), \alpha)$ . Then (4.1) shows that  $\rho(\Lambda)_i m^\phi \geq A_i m^\phi$  for each  $i$ . Now Corollary 4.2(d) implies that  $m^\phi = x^\Lambda$ , and hence  $e^{\beta r_i} m^\phi = \rho(\Lambda)_i m^\phi = A_i m^\phi$  for all  $i$ . Hence [14, Proposition 4.1(b)] implies that  $\phi$  factors through  $C^*(\Lambda)$ .  $\square$

*Remark 4.7.* From the point of view developed by Bratteli, Elliott and Kishimoto [2], the collection  $\text{Lie}(\mathbb{T}^k)$  of continuous homomorphisms from  $\mathbb{R}$  to  $\mathbb{T}^k$  is the collection of possible finite inverse temperatures for KMS states for the gauge action  $\gamma$ . A KMS state for  $\gamma$  at inverse temperature  $\beta \in \text{Lie}(\mathbb{T}^k)$  is then a  $\text{KMS}_1$  state for the action  $\gamma \circ \beta$  of  $\mathbb{R}$ .

Embed  $\mathbb{R}^k$  in  $\text{Lie}(\mathbb{T}^k)$  via  $\beta \mapsto (t \mapsto e^{i\beta t})$ . Corollary 4.6(a) says that the gauge action on  $\mathcal{TC}^*(\Lambda)$  admits a KMS state at inverse temperature  $\beta \in \mathbb{R}^k$  if and only if  $\beta \in [\ln \rho(A_1), \infty) \times \cdots \times [\ln \rho(A_k), \infty)$ . Corollary 4.6(b) says that the KMS states that factor through  $C^*(\Lambda)$  are those at inverse temperature  $\ln \rho(\Lambda)$ . So from the point of view of [2], Corollary 4.6 identifies  $\beta = \ln \rho(\Lambda)$  as the critical inverse temperature for  $\gamma$ .



5. PERIODICITY OF  $k$ -GRAPHS

In this section we describe the periodicity group of a strongly connected finite  $k$ -graph  $\Lambda$ . This group is a key ingredient in the statement of our main theorem. The fundamental idea behind our analysis involves source- and range-preserving bijections between certain sets of paths, and comes from Davidson and Yang's analysis of periodicity in 2-graphs with one vertex [8].

Our results in this section and the next also follow from the more general results of [4] (see also [34]). Specifically, Lemma 5.1 and Proposition 5.2 follows from [4, Theorem 4.2(1)–(3)]; and Lemma 6.2 and Proposition 6.1 follow (with some effort) from [4, Theorem 4.2(5) and Proposition 3.3]. However, a simpler direct argument works for strongly connected finite  $k$ -graphs, and we present that instead.

To state our results we must briefly discuss infinite paths in  $k$ -graphs. The set

$$\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$$

becomes a  $k$ -graph with operations  $r(m, n) = (m, m)$ ,  $s(m, n) = (n, n)$ ,  $(m, n)(n, p) = (m, p)$  and  $d(m, n) = n - m$ . We identify  $\Omega_k^0$  with  $\mathbb{N}^k$  via  $(m, m) \mapsto m$ . An *infinite path* in a  $k$ -graph  $\Lambda$  is a functor  $x : \Omega_k \rightarrow \Lambda$  that intertwines the degree maps. We write  $\Lambda^\infty$  for the collection of all infinite paths and call this the *infinite-path space* of  $\Lambda$ . For  $x \in \Lambda^\infty$  we write  $r(x)$  for  $x(0)$ . For  $n \in \mathbb{N}^k$ , we write  $\sigma^n(x)$  for the infinite path such that  $\sigma^n(x)(p, q) = x(n + p, n + q)$ . If  $r(x) = s(\lambda)$ , then there is a unique infinite path  $\lambda x$  such that  $(\lambda x)(0, d(\lambda)) = \lambda$  and  $\sigma^{d(\lambda)}(\lambda x) = x$ . For  $\lambda \in \Lambda$  we define  $Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}$ . If  $\Lambda$  has no sources, then each  $Z(\lambda)$  is nonempty.

We say  $\Lambda$  is *aperiodic* if for each  $v \in \Lambda^0$ , there exists  $x \in Z(v)$  such that for all  $m \neq n \in \mathbb{N}^k$  we have  $\sigma^m(x) \neq \sigma^n(x)$ . By [27, Lemma 3.2],  $\Lambda$  is aperiodic if and only if there do not exist  $v \in \Lambda^0$  and  $m \neq n \in \mathbb{N}^k$  such that  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in Z(v)$ .

**Lemma 5.1.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph. Suppose that  $v \in \Lambda^0$  and  $m, n \in \mathbb{N}^k$  satisfy  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in Z(v)$ .*

- (a) *For all  $x \in \Lambda^\infty$  we have  $\sigma^m(x) = \sigma^n(x)$ .*
- (b) *For each  $\mu \in \Lambda^m$  there exists a unique  $\theta_{m,n}(\mu) \in \Lambda^n$  such that  $\mu x = \theta_{m,n}(\mu)x$  for all  $x \in Z(s(\mu))$ . The map  $\theta_{m,n} : \Lambda^m \rightarrow \Lambda^n$  is range- and source-preserving.*
- (c) *If  $w \in \Lambda^0$  and  $p \in \mathbb{N}^k$  also satisfy  $\sigma^n(x) = \sigma^p(x)$  for all  $x \in Z(w)$ , then  $\sigma^m(x) = \sigma^p(x)$  for all  $x \in \Lambda^\infty$ , and  $\theta_{n,p} \circ \theta_{m,n} = \theta_{m,p}$ .*
- (d) *Each  $\theta_{m,m} : \Lambda^m \rightarrow \Lambda^m$  is the identity map, and each  $\theta_{m,n} : \Lambda^m \rightarrow \Lambda^n$  is a bijection with  $\theta_{m,n}^{-1} = \theta_{n,m}$ .*

*Proof.* (a) Fix  $x \in \Lambda^\infty$ . Since  $\Lambda$  is strongly connected,  $v\Lambda r(x)$  has at least one element, say  $\lambda$ . So  $\lambda x \in Z(v)$ , and hence

$$\sigma^m(x) = \sigma^{m+d(\lambda)}(\lambda x) = \sigma^{d(\lambda)}(\sigma^m(\lambda x)) = \sigma^{d(\lambda)}(\sigma^n(\lambda x)) = \sigma^{n+d(\lambda)}(\lambda x) = \sigma^n(x).$$

(b) Fix  $\mu \in \Lambda^m$ . Since  $\Lambda$  is strongly connected, Lemma 2.1(b) shows that there exists  $\alpha \in \Lambda^n r(\mu)$ . Let  $\beta := (\alpha\mu)(0, m)$  and let  $\theta_{m,n}(\mu) := (\alpha\mu)(m, m+n)$ . Fix  $x \in Z(s(\mu))$ . By (a) applied to  $\alpha\mu x$ ,

$$\mu x = \sigma^n(\alpha\mu x) = \sigma^m(\alpha\mu x) = \sigma^m(\beta\theta_{m,n}(\mu)x) = \theta_{m,n}(\mu)x.$$

This implies in particular that  $Z(\mu) = Z(\theta_{m,n}(\mu))$ . Since the sets  $Z(\nu)$  for  $\nu \in \Lambda^n$  are mutually disjoint,  $\theta_{m,n}(\mu)$  is the unique element of  $\Lambda^n$  such that  $\mu x = \theta_{m,n}(\mu)x$ .

This gives a function  $\theta_{m,n} : \Lambda^m \rightarrow \Lambda^n$ . We have  $s(\theta_{m,n}(\mu)) = s(\mu)$  and  $r(\theta_{m,n}(\mu)) = r(\mu)$  by construction.

(c) Two applications of part (a) show that  $\sigma^m(x) = \sigma^n(x) = \sigma^p(x)$  for all  $x \in \Lambda^\infty$ . Let  $\mu \in \Lambda^m$  and  $x \in Z(s(\mu))$ . Then

$$\theta_{n,p}(\theta_{m,n}(\mu))x = \theta_{m,n}(\mu)x = \mu x = \theta_{m,p}(\mu)x.$$

Thus  $\theta_{n,p} \circ \theta_{m,n} = \theta_{m,p}$  by the uniqueness assertion in (b).

(d) Uniqueness in part (b) shows that  $\theta_{m,m}(\mu) = \mu$  for all  $\mu \in \Lambda^m$ . Now  $\theta_{n,m} \circ \theta_{m,n} = \theta_{m,m} = \text{id}_{\Lambda^m}$  by (c), and likewise  $\theta_{m,n} \circ \theta_{n,m} = \text{id}_{\Lambda^n}$ .  $\square$

**Proposition 5.2.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph.*

- (a) *Let  $m, n, p, q \in \mathbb{N}^k$ . Suppose that  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in \Lambda^\infty$ . If  $p - q = m - n$ , then  $\sigma^p(x) = \sigma^q(x)$  for all  $x \in \Lambda^\infty$ .*  
 (b) *The set*

$$\text{Per } \Lambda := \{m - n : m, n \in \mathbb{N}^k, \sigma^m(x) = \sigma^n(x) \text{ for all } x \in \Lambda^\infty\}$$

*is a subgroup of  $\mathbb{Z}^k$ .*

- (c) *Suppose that  $m - n \in \text{Per } \Lambda$  and that  $\mu \in \Lambda^m$ . Then  $\theta_{d(\alpha)+m, d(\alpha)+n}(\alpha\mu) = \alpha\theta_{m,n}(\mu)$  and  $\theta_{m+d(\beta), n+d(\beta)}(\mu\beta) = \theta_{m,n}(\mu)\beta$  for all  $\alpha \in \Lambda r(\mu)$  and  $\beta \in s(\mu)\Lambda$ .*

*Proof.* (a). Fix  $x \in \Lambda^\infty$ . Lemma 2.1(b) shows that there exists  $\alpha \in \Lambda^m r(x)$ . We calculate:

$$\sigma^p(x) = \sigma^{p+m}(\alpha x) = \sigma^m(\sigma^p(\alpha x)) = \sigma^n(\sigma^p(\alpha x)) = \sigma^{n+p}(\alpha x) = \sigma^{m+q}(\alpha x) = \sigma^q(x).$$

(b) We have  $0 \in \text{Per } \Lambda$ , and  $-p \in \text{Per } \Lambda$  whenever  $p \in \text{Per } \Lambda$ . If  $m - n, p - q \in \text{Per } \Lambda$ , then for  $x \in \Lambda^\infty$ ,  $\sigma^{p+m}(x) = \sigma^p(\sigma^m(x)) = \sigma^p(\sigma^n(x)) = \sigma^q(\sigma^n(x)) = \sigma^{q+n}(x)$ . Thus  $\text{Per } \Lambda$  is closed under addition, and hence is a subgroup of  $\mathbb{Z}^k$ .

(c) Let  $\alpha \in \Lambda r(\mu)$ , and fix  $x \in Z(s(\mu))$ . The defining property of  $\theta_{m,n}(\mu)$  implies that  $\theta_{m,n}(\mu)x = \mu x$ , and hence  $\alpha\theta_{m,n}(\mu)x = \alpha\mu x$ . Uniqueness in Lemma 5.1(b) gives  $\theta_{d(\alpha)+m, d(\alpha)+n}(\alpha\mu) = \alpha\theta_{m,n}(\mu)$ . A similar argument shows that  $\theta_{m+d(\beta), n+d(\beta)}(\mu\beta) = \theta_{m,n}(\mu)\beta$  for all  $\beta \in s(\mu)\Lambda$ .  $\square$

**Corollary 5.3.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Suppose that  $m - n \in \text{Per } \Lambda$  and  $\mu \in \Lambda^m$ . Let  $p := (m \vee n) - m$  and  $q := (m \vee n) - n$ . Then*

$$\Lambda^{\min}(\theta_{m,n}(\mu), \mu) = \{(\alpha, \theta_{q,p}(\alpha)) : \alpha \in s(\mu)\Lambda^q\}.$$

*Proof.* For the containment  $\subseteq$ , suppose that  $(\alpha, \beta) \in \Lambda^{\min}(\theta_{m,n}(\mu), \mu)$ . Then  $d(\alpha) = q$  and  $d(\beta) = p$  by definition, and Lemma 5.1(b) gives  $r(\alpha) = s(\theta_{m,n}(\mu)) = s(\mu)$ . For  $x \in Z(s(\alpha))$  we have

$$\alpha x = \sigma^n(\theta_{m,n}(\mu)\alpha x) = \sigma^n(\mu\beta x) = \sigma^m(\mu\beta x) = \beta x$$

because  $m - n \in \text{Per } \Lambda$ . Thus  $\beta = \theta_{q,p}(\alpha)$ .

For the containment  $\supseteq$ , fix  $\alpha \in s(\mu)\Lambda^q$ . Let  $x \in Z(s(\alpha))$ . We have

$$\theta_{m,n}(\mu)\alpha x = \mu\alpha x = \mu\theta_{q,p}(\alpha)x.$$

The factorisation property implies that  $\theta_{m,n}(\mu)\alpha = \mu\theta_{q,p}(\alpha)$ . Since  $n + q = m + p = m \vee n$  we have  $(\alpha, \theta_{q,p}(\alpha)) \in \Lambda^{\min}(\theta_{m,n}(\mu), \mu)$ .  $\square$

**Proposition 5.4.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Then  $\Lambda$  is aperiodic if and only if  $\text{Per } \Lambda = \{0\}$ .*

*Proof.* First suppose that  $\Lambda$  is aperiodic, and take  $m - n \in \text{Per } \Lambda$ . Proposition 5.2(a) implies that  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in \Lambda^\infty$ . Since  $\Lambda$  is aperiodic, this forces  $m = n$ . Hence  $\text{Per } \Lambda = \{0\}$ .

Now suppose that  $\Lambda$  is not aperiodic. The equivalence of (i) and (iii) in [27, Lemma 3.2] implies that there exist  $m \neq n \in \mathbb{N}^k$  and  $v \in \Lambda^0$  such that  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in Z(v)$ . Lemma 5.1(a) then implies that  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in \Lambda^\infty$ , and hence  $m - n \in \text{Per } \Lambda \setminus \{0\}$ .  $\square$

*Example 5.5.* Suppose that  $\Lambda$  is a finite 2-graph with one vertex. This puts us in the situation studied by Davidson and Yang in [8]. The group  $\text{Per } \Lambda$  is then the intersection, over all infinite paths  $x$  in  $\Lambda$ , of the associated symmetry groups  $H_x$  discussed in [8, Section 2]. Proposition 5.2(a) boils down to equivalence of (i) and (ii) in [8, Theorem 3.1]. The bijections  $\theta_{m,n}$  of Proposition 5.1(d) are the bijections  $\gamma$  of [8, Theorem 3.1(iii)].

## 6. A CENTRAL REPRESENTATION OF THE PERIODICITY GROUP

We now describe how the group  $\text{Per } \Lambda$  shows up in  $C^*(\Lambda)$ .

**Proposition 6.1.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph, and for  $m, n \in \mathbb{N}^k$  such that  $m - n \in \text{Per } \Lambda$ , let  $\theta_{m,n}$  be the bijection of Lemma 5.1. There is a unitary representation  $U$  of  $\text{Per } \Lambda$  in the centre of  $C^*(\Lambda)$  such that  $U_{m-n} = \sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^*$  whenever  $m - n \in \text{Per } \Lambda$ .*

**Lemma 6.2.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph. Suppose that  $m - n \in \text{Per } \Lambda$ . Then  $s_\mu s_\mu^* = s_{\theta_{m,n}(\mu)} s_{\theta_{m,n}(\mu)}^*$  for all  $\mu \in \Lambda^m$ . The element  $U := \sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^*$  is a unitary in  $C^*(\Lambda)$ .*

*Proof.* Let  $p = (m \vee n) - m$ ,  $q = (m \vee n) - n$  and  $\mu \in \Lambda^m$ . By Corollary 5.3,

$$\Lambda^{\min}(\mu, \theta_{m,n}(\mu)) = \{(\theta_{q,p}(\alpha), \alpha) : \alpha \in s(\mu)\Lambda^q\},$$

and in particular,  $\mu\theta_{q,p}(\alpha) = \theta_{m,n}(\mu)\alpha$  for all  $\alpha \in s(\mu)\Lambda^q$ . Using this at the fourth equality, we compute:

$$\begin{aligned} s_\mu s_\mu^* &= s_\mu \left( \sum_{\beta \in s(\mu)\Lambda^p} s_\beta s_\beta^* \right) s_\mu^* = s_\mu \left( \sum_{\alpha \in s(\mu)\Lambda^q} s_{\theta_{q,p}(\alpha)} s_{\theta_{q,p}(\alpha)}^* \right) s_\mu^* \\ &= \sum_{\alpha \in s(\mu)\Lambda^q} s_{\mu\theta_{q,p}(\alpha)} s_{\mu\theta_{q,p}(\alpha)}^* = \sum_{\alpha \in s(\mu)\Lambda^q} s_{\theta_{m,n}(\mu)\alpha} s_{\theta_{m,n}(\mu)\alpha}^* \\ &= s_{\theta_{m,n}(\mu)} \left( \sum_{\alpha \in s(\mu)\Lambda^q} s_\alpha s_\alpha^* \right) s_{\theta_{m,n}(\mu)}^* = s_{\theta_{m,n}(\mu)} s_{\theta_{m,n}(\mu)}^*. \end{aligned}$$

Since  $\theta_{m,n}$  is a source-preserving bijection we have

$$UU^* = \sum_{\mu, \eta \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^* s_{\theta_{m,n}(\eta)} s_\eta^* = \sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^* s_{\theta_{m,n}(\mu)} s_\mu^* = \sum_{v \in \Lambda^0} \sum_{\mu \in v\Lambda^m} s_\mu s_\mu^* = 1_{C^*(\Lambda)}.$$

The symmetric calculation gives  $UU^* = 1$ . Thus  $U$  is unitary.  $\square$

*Proof of Proposition 6.1.* We start by showing that  $\sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^*$  depends only on  $m - n$ . Suppose that  $m, n, p, q \in \mathbb{N}^k$  and  $m - n = p - q \in \text{Per } \Lambda$ . Then (CK4) followed by

Proposition 5.2(c) imply that

$$\sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^* = \sum_{\mu \in \Lambda^m} \sum_{\alpha \in s(\mu)\Lambda^p} s_\mu s_\alpha s_\alpha^* s_{\theta_{m,n}(\mu)}^* = \sum_{\eta \in \Lambda^{m+p}} s_\eta s_{\theta_{m+p,n+p}(\eta)}^*.$$

The same calculation with  $(m, n, p)$  replaced by  $(p, q, m)$  gives

$$\sum_{\nu \in \Lambda^p} s_\nu s_{\theta_{p,q}(\nu)}^* = \sum_{\zeta \in \Lambda^{p+m}} s_\zeta s_{\theta_{p+m,q+m}(\zeta)}^*.$$

Since  $n + p = q + m$ , the formula for  $U_{m-n}$  is well defined.

Lemma 6.2 implies that  $U_{m-n}$  is unitary. By Lemma 5.1(b),  $\theta_{n,m} = \theta_{m,n}^{-1}$ , and hence  $U_{m-n} = U_{n-m}^*$ . To see that  $g \mapsto U_g$  is a homomorphism, fix  $g, h \in \text{Per } \Lambda$ . To line things up, choose  $g_+, g_-, h_+, h_- \in \mathbb{N}^k$  such that  $g = g_+ - g_-$  and  $h = h_+ - h_-$ . Let  $m := g_+ + h_+$ ,  $n := g_- + h_+$  and  $p := g_- + h_-$ . Then  $g = m - n$  and  $h = n - p$ . For  $\mu \in \Lambda^m$  and  $\nu \in \Lambda^n$ , we have  $s_{\theta_{m,n}(\mu)}^* s_\nu = \delta_{\theta_{m,n}(\mu), \nu} p_{s(\mu)}$  by (CK3) and (CK4). This and Lemma 5.1(c) give

$$U_g U_h = \sum_{\mu \in \Lambda^m} \sum_{\nu \in \Lambda^n} s_\mu s_{\theta_{m,n}(\mu)}^* s_\nu s_{\theta_{n,p}(\nu)}^* = \sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{n,p}(\theta_{m,n}(\mu))}^* = \sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,p}(\mu)}^*.$$

Since  $m - p = m - n + n - p = g + h$ , we deduce that  $U_g U_h = U_{g+h}$ .

To see that the  $U_g$  are central, it suffices to show that  $U_g s_\lambda = s_\lambda U_g$  for all  $g \in \text{Per } \Lambda$  and  $\lambda \in \Lambda$ : since  $\text{Per } \Lambda$  is a group and  $U_{-g} = U_g^*$  we then have  $U_g s_\lambda^* = (s_\lambda U_{-g})^* = (U_{-g} s_\lambda)^* = s_\lambda^* U_g$ . Fix  $\lambda \in \Lambda$  and  $g \in \text{Per } \Lambda$ . Choose  $m, n \in \mathbb{N}^k$  such that  $g = m - n$ , and let  $p := m + d(\lambda)$  and  $q := n + d(\lambda)$ . By factoring  $\xi \in \Lambda^p$  into paths of degree  $d(\lambda)$  and  $m$ ,

$$U_g s_\lambda = \sum_{\xi \in \Lambda^p} s_\xi s_{\theta_{p,q}(\xi)}^* s_\lambda = \sum_{\eta \in \Lambda^{d(\lambda)}} \sum_{\mu \in s(\eta)\Lambda^m} s_{\eta\mu} s_{\theta_{p,q}(\eta\mu)}^* s_\lambda.$$

By Proposition 5.2(c), each  $\theta_{p,q}(\eta\mu) = \eta\theta_{m,n}(\mu)$ . Since  $s_{\eta\theta_{m,n}(\mu)}^* s_\lambda = \delta_{\eta,\lambda} s_{\theta_{m,n}(\mu)}^*$  we deduce that

$$U_g s_\lambda = \sum_{\mu \in s(\lambda)\Lambda^m} s_{\lambda\mu} s_{\theta_{m,n}(\mu)}^* = \sum_{\mu \in \Lambda^m} s_\lambda s_\mu s_{\theta_{m,n}(\mu)}^* = s_\lambda U_g. \quad \square$$

## 7. THE STATEMENT OF THE MAIN RESULT

Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Our main theorem, Theorem 7.1 below, describes the  $\text{KMS}_1$  states of  $C^*(\Lambda)$  for the *preferred dynamics* defined by

$$\alpha_t = \gamma_{\rho(\Lambda)it} \quad \text{for all } t \in \mathbb{R}$$

corresponding to  $r = \ln \rho(\Lambda)$ .

To see why we chose this dynamics and inverse temperature, take  $r \in \mathbb{R}^k$  and  $\beta \in [0, \infty)$  and let  $\alpha^r$  be the dynamics  $\alpha_t^r = \gamma_{e^{itr}}$ . Suppose that  $\phi$  is a  $\text{KMS}_\beta$  state for  $(C^*(\Lambda), \alpha^r)$ . Then Corollary 4.6(b) implies that  $\beta r = \ln \rho(\Lambda)$ . So  $\alpha_t = \alpha_{\beta t}^r$  for all  $t$ , and hence the  $\text{KMS}_\beta$  condition for  $\alpha^r$  is the  $\text{KMS}_1$  condition for  $\alpha$ . So  $\phi$  is a  $\text{KMS}_1$  state for  $(C^*(\Lambda), \alpha)$ .

There is a slight subtlety here when  $\rho(\Lambda) = (1, \dots, 1)$ . The preferred dynamics is then the trivial action, and so the  $\text{KMS}_1$  states described in Theorem 7.1 are traces, and are  $\text{KMS}_\beta$  states for all other values of  $\beta$ . If at least one  $\rho(\Lambda)_i$  is different from 1, then Corollary 4.6(b) shows that  $(C^*(\Lambda), \alpha)$  admits  $\text{KMS}_\beta$  states only for  $\beta = 1$ .

**Theorem 7.1.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Let  $\alpha$  be the preferred dynamics on  $C^*(\Lambda)$ . Let  $\pi_U : C^*(\text{Per } \Lambda) \rightarrow C^*(\Lambda)$  be the homomorphism of Proposition 6.1. Then  $\pi_U^* : \phi \mapsto \phi \circ \pi_U$  is an affine isomorphism of the  $\text{KMS}_1$  simplex of  $(C^*(\Lambda), \alpha)$  onto the state space of  $C^*(\text{Per } \Lambda)$ .*

The proof of Theorem 7.1 occupies the next three sections. The proof strategy is as follows. In Section 8, we show that the KMS states of  $(C^*(\Lambda), \alpha)$  all induce the same measure  $M$  on the spectrum of the abelian subalgebra  $\overline{\text{span}}\{s_\lambda s_\lambda^* \mid \lambda \in \Lambda\} \subseteq C^*(\Lambda)$ , and we characterise  $M$  in terms of the unimodular Perron-Frobenius eigenvector  $x^\Lambda$ . We use  $M$  in Section 9 to establish a formula for a KMS state  $\phi$  in terms of  $\phi \circ \pi_U$  (see Theorem 9.1). In Section 10 we use  $M$  again to construct a particular KMS state in Proposition 10.2. This state is not always supported on  $\overline{\text{span}}\{s_\lambda s_\lambda^*\}$ , so composing with the gauge automorphisms yields more  $\text{KMS}_1$  states (Corollary 10.3). We can then prove Theorem 7.1: we deduce from the formula for KMS states established in Theorem 9.1 that  $\pi_U^*$  is a continuous affine injection; we then use Corollary 10.3 to see that each pure state of  $C^*(\text{Per } \Lambda)$  is in the image of  $\pi_U^*$ , and deduce that  $\pi_U^*$  is surjective.

Before moving on to the first part of the proof of Theorem 7.1, a reality check is in order. If  $\Lambda$  is coordinatewise irreducible, in the sense that each  $A_i$  is an irreducible matrix, then it is also strongly connected. So both Theorem 7.1 and [14, Theorem 7.2] apply. The next remark reconciles the hypotheses of the two results.

*Remark 7.2.* Suppose that  $\Lambda$  is coordinatewise-irreducible. Theorem 7.1 says that if  $\text{Per } \Lambda$  is nontrivial, then  $(C^*(\Lambda), \alpha)$  has many KMS states. Theorem 7.2 of [14], on the other hand, says that if the coordinates of the vector  $\ln \rho(\Lambda)$  are rationally independent, then  $(C^*(\Lambda), \alpha)$  admits a unique KMS state. To reconcile the two results, we will show that if  $\text{Per } \Lambda$  is nontrivial, then the coordinates of  $\ln \rho(\Lambda)$  are rationally dependent.

Let  $m - n \in \text{Per } \Lambda \setminus \{0\}$ . By Lemma 5.1(d), there is a source- and range-preserving bijection of  $\Lambda^m$  onto  $\Lambda^n$ . For  $v, w \in \Lambda^0$ , we have  $A^m(v, w) = |v \Lambda^m w| = |v \Lambda^n w| = A^n(v, w)$ , and so  $A^m = A^n$ . Since  $\Lambda$  is strongly connected, Corollary 4.2(a) shows that

$$(7.1) \quad \rho(A)^m = \prod_i \rho(A_i)^{m_i} = \rho(A^m) = \rho(A^n) = \prod_i \rho(A_i)^{n_i} = \rho(A)^n.$$

Taking logarithms,

$$m \cdot \ln \rho(\Lambda) = \sum_{i=1}^k m_i \ln \rho(A_i) = \ln \left( \prod_{i=1}^k \rho(A_i)^{m_i} \right) = \ln \left( \prod_{i=1}^k \rho(A_i)^{n_i} \right) = n \cdot \ln \rho(\Lambda).$$

Thus the coordinates of  $\ln \rho(\Lambda)$  are rationally dependent.

## 8. MEASURES ON THE INFINITE-PATH SPACE

Let  $\Lambda$  be a strongly connected finite  $k$ -graph. Each KMS state of  $C^*(\Lambda)$  restricts to a state of the commutative subalgebra  $\overline{\text{span}}\{s_\lambda s_\lambda^* \mid \lambda \in \Lambda\}$ , and hence to a probability measure on its spectrum satisfying an invariance condition (see (8.2)). In this section we use our Perron-Frobenius theorem and results of Choksi [6] about measures on inverse-limit spaces to show that there is a unique measure satisfying (8.2). We will use this result in Section 9 to give a formula for a KMS state  $\phi$  in terms of its restriction to the image of  $C^*(\text{Per } \Lambda)$ , and again in Section 10 to construct KMS states.

Recall from [16] that the sets  $Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}$  indexed by  $\lambda \in \Lambda$  constitute a basis of compact open sets for a compact Hausdorff topology on  $\Lambda^\infty$ . Equip

the finite sets  $\Lambda^m$  with the discrete topology. Let  $m \leq n \in \mathbb{N}^k$  and define  $\pi_{m,n} : \Lambda^n \rightarrow \Lambda^m$  by  $\pi_{m,n}(\lambda) = \lambda(0, m)$ . Then  $(\Lambda^m, \pi_{m,n})$  is an inverse system of compact topological spaces and continuous, surjective maps. Using the universal property of the inverse limit it is routine to show that  $x \mapsto (x(0, m))_{m \in \mathbb{N}^k}$  is a homeomorphism of  $\Lambda^\infty$  onto the inverse limit  $\varprojlim (\Lambda^m, \pi_{m,n})$ .

There is an isomorphism of the commutative subalgebra  $\overline{\text{span}}\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}$  of  $C^*(\Lambda)$  onto  $C(\Lambda^\infty)$  that carries  $s_\lambda s_\lambda^*$  to  $1_{Z(\lambda)}$  (see, for example, Theorem 7.1 of [30]). Thus the Riesz Representation Theorem associates to each state  $\phi$  of  $C^*(\Lambda)$  a Borel probability measure  $M$  on  $\Lambda^\infty$  such that  $M(Z(\lambda)) = \phi(s_\lambda s_\lambda^*)$ .

Let  $\alpha$  denote the preferred dynamics on  $C^*(\Lambda)$ , and suppose that  $\phi$  is a  $\text{KMS}_1$  state of  $(C^*(\Lambda), \alpha)$ . The KMS condition ensures that, for  $\lambda \in \Lambda$ ,

$$(8.1) \quad \phi(s_\lambda s_\lambda^*) = \rho(\Lambda)^{-d(\lambda)} \phi(s_\lambda^* s_\lambda) = \rho(\Lambda)^{-d(\lambda)} \phi(p_{s(\lambda)}),$$

and hence the corresponding probability measure  $M$  on  $\Lambda^\infty$  satisfies

$$(8.2) \quad M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} M(Z(s(\lambda))) \quad \text{for all } \lambda \in \Lambda.$$

We now show that there is exactly one measure satisfying (8.2).

**Proposition 8.1.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Then there exists a unique Borel probability measure  $M$  on  $\Lambda^\infty$  that satisfies (8.2). Let  $x^\Lambda$  be the unimodular Perron-Frobenius eigenvector of  $\Lambda$ . We have*

$$(8.3) \quad M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x_{s(\lambda)}^\Lambda \quad \text{for all } \lambda.$$

*Proof.* We build a measure  $M$  satisfying (8.2) and (8.3) by viewing  $\Lambda^\infty$  as the inverse limit of the sets  $\Lambda^m$  under the maps  $\pi_{m,n} : \Lambda^n \rightarrow \Lambda^m$  for  $n \geq m \in \mathbb{N}^k$ . For  $S \subseteq \Lambda^m$ , define  $M_m(S) = \rho(\Lambda)^{-m} \sum_{\mu \in S} x_{s(\mu)}^\Lambda$ . Then  $M_m$  is a measure on  $\Lambda^m$ .

For  $m \leq n$  and  $\mu \in \Lambda^m$ , we have

$$\begin{aligned} M_n(\pi_{m,n}^{-1}(\{\mu\})) &= \sum_{\mu' \in s(\mu)\Lambda^{n-m}} \rho(\Lambda)^{-n} x_{s(\mu')}^\Lambda = \rho(\Lambda)^{-n} \sum_{w \in \Lambda^0} A^{n-m}(s(\mu), w) x_w^\Lambda \\ &= \rho(\Lambda)^{-n} (A^{n-m} x^\Lambda)_{s(\mu)} = \rho(\Lambda)^{-m} x_{s(\mu)}^\Lambda = M_m(\{\mu\}). \end{aligned}$$

Thus  $M_n(\pi_{m,n}^{-1}(S)) = M_m(S)$  for all  $S \subseteq \Lambda^m$  and the measure spaces  $((\Lambda^m, M_m), \pi_{m,n})$  form an inverse system. Theorem 2.2 of [6] implies that there is a Borel measure  $M$  on  $\Lambda^\infty = \varprojlim (\Lambda^m, \pi_{m,n})$  such that, for  $\mu \in \Lambda^m$ ,

$$M(Z(\mu)) = M_m(\{\mu\}) = \rho(\Lambda)^{-m} x_{s(\mu)}^\Lambda = \rho(\Lambda)^{-m} M(Z(s(\mu))).$$

Since  $M(\Lambda^\infty) = \sum_{v \in \Lambda^0} M(Z(v)) = \sum_{v \in \Lambda^0} x_v^\Lambda = 1$ , this  $M$  is a probability measure satisfying (8.2) and (8.3).

Now suppose that  $M'$  is a Borel probability measure satisfying (8.2). Define a vector  $y \in [0, \infty)^{\Lambda^0}$  by  $y_v = M'(Z(v))$  for each  $v$ . For  $1 \leq i \leq k$ , we have  $Z(v) = \bigsqcup_{\alpha \in v\Lambda^{e_i}} Z(\alpha)$ , and using (8.2), we have

$$\begin{aligned} \rho(A_i) y_v &= \rho(A_i) M'(Z(v)) = \rho(A_i) \sum_{\alpha \in v\Lambda^{e_i}} M'(Z(\alpha)) \\ &= \sum_{\alpha \in v\Lambda^{e_i}} M'(Z(s(\alpha))) = \sum_{w \in \Lambda^0} A_i(v, w) y_w = (A_i y)_v. \end{aligned}$$

So  $y$  is a non-negative eigenvector of each  $A_i$  with eigenvalue  $\rho(A_i)$  and unit 1-norm. Thus  $y = x^\Lambda$  by Corollary 4.2(b). Now (8.3) for  $M'$  follows from (8.2). Thus  $M = M'$ .  $\square$

Each  $\sigma^m : \Lambda^\infty \rightarrow \Lambda^\infty$  is continuous since it restricts to a homeomorphism of  $Z(\mu)$  for each  $\mu \in \Lambda^m$ . So each  $\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}$  is closed and hence Borel. We next show that when  $m - n \in \text{Per } \Lambda$ , the measure  $M$  is supported on  $\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}$ .

**Proposition 8.2.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph, and let  $M$  be the measure on  $\Lambda^\infty$  obtained from Proposition 8.1. For  $m, n \in \mathbb{N}^k$ , we have*

$$M(\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}) = \begin{cases} 1 & \text{if } m - n \in \text{Per } \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

The proof of the proposition requires the following two technical lemmas.

**Lemma 8.3.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph. Suppose  $g \in \mathbb{Z}^k \setminus \text{Per } \Lambda$ . Then there exist  $a \in \mathbb{N}^k \setminus \{0\}$  and, for each  $v \in \Lambda^0$ , a path  $\lambda_v \in v\Lambda^a$  such that for  $\mu, \nu \in \Lambda$  with  $s(\mu) = s(\nu)$  and  $d(\mu) - d(\nu) = g$  we have  $\Lambda^{\min}(\mu\lambda_{s(\mu)}, \nu\lambda_{s(\mu)}) = \emptyset$ .*

*Proof.* Let  $m := g \vee 0$  and  $n := -g \vee 0$ . Then  $g = m - n$  and whenever  $m', n' \in \mathbb{N}^k$  satisfy  $m' - n' = g$ , we have  $m' \geq m$  and  $n' \geq n$ .

Since  $g \notin \text{Per } \Lambda$ , there exists  $x \in \Lambda^\infty$  such that  $\sigma^m(x) \neq \sigma^n(x)$ . So there exists  $l \in \mathbb{N}^k \setminus \{0\}$  such that  $\sigma^m(x)(0, l) \neq \sigma^n(x)(0, l)$ . For each  $v \in \Lambda^0$  there exists  $\tau_v \in v\Lambda r(x)$  because  $\Lambda$  is strongly connected. Let  $a := m + n + l + \bigvee_{v \in \Lambda^0} d(\tau_v)$ . For each  $v \in \Lambda^0$  define

$$\lambda_v := \tau_v x(0, a - d(\tau_v)).$$

Fix  $\mu, \nu \in \Lambda$  such that  $d(\mu) - d(\nu) = g$  and  $s(\mu) = s(\nu) = v$ . Then  $d(\mu) \geq m$ ,  $d(\nu) \geq n$ , and there exists  $p \in \mathbb{N}^k$  such that  $d(\mu) = m + p$  and  $d(\nu) = n + p$ . Factorise  $\mu = \alpha\mu'$  and  $\nu = \beta\nu'$  where  $d(\alpha) = d(\beta) = p$ , so that  $d(\mu') = m$  and  $d(\nu') = n$ . If  $\alpha \neq \beta$ , then  $\Lambda^{\min}(\mu, \nu) = \emptyset$  and hence  $\Lambda^{\min}(\mu\lambda_v, \nu\lambda_v) = \emptyset$ . So we suppose that  $\alpha = \beta$ . Then  $\Lambda^{\min}(\mu\lambda_v, \nu\lambda_v) = \Lambda^{\min}(\mu'\lambda_v, \nu'\lambda_v)$ . We have

$$\begin{aligned} (\mu'\lambda_v)(m + n + d(\tau_v), m + n + d(\tau_v) + l) &= \lambda_v(n + d(\tau_v), n + d(\tau_v) + l) \\ &= x(n, n + l) = \sigma^n(x)(0, l). \end{aligned}$$

Similarly  $(\nu'\lambda_v)(m + n + d(\tau_v), m + n + d(\tau_v) + l) = \sigma^m(x)(0, l) \neq \sigma^n(x)(0, l)$  by choice of  $x$  and  $l$ , the factorisation property gives  $\Lambda^{\min}(\mu\lambda_v, \nu\lambda_v) = \emptyset$ .  $\square$

**Lemma 8.4.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph, and let  $M$  be the measure on  $\Lambda^\infty$  obtained in Proposition 8.1. Suppose that  $g \in \mathbb{Z}^k \setminus \text{Per } \Lambda$ . There exist  $a \in \mathbb{N}^k \setminus \{0\}$  and  $0 < K < 1$  such that whenever  $s(\mu) = s(\nu)$  and  $d(\mu) - d(\nu) = g$ , we have*

$$(8.4) \quad M\left(\bigcup_{\substack{\lambda \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset}} Z(\mu\lambda)\right) \leq K^j M(Z(\mu)) \quad \text{for all } j \in \mathbb{N}.$$

*Proof.* By Lemma 8.3 there exist  $a \in \mathbb{N}^k \setminus \{0\}$  and  $\lambda_v \in v\Lambda^a$  for each  $v \in \Lambda^0$  such that  $\Lambda^{\min}(\mu\lambda_v, \nu\lambda_v) = \emptyset$  whenever  $\mu, \nu \in \Lambda v$  satisfy  $d(\mu) - d(\nu) = g$ .

Let  $v \in \Lambda^0$ . Equation (8.3) implies that  $0 < M(Z(\lambda_v))$ . Thus  $M(Z(v) \setminus Z(\lambda_v)) < M(Z(v))$ . Since  $\Lambda^0$  is finite, there exists  $0 < K < 1$  such that

$$M(Z(v) \setminus Z(\lambda_v)) < KM(Z(v)) < M(Z(v)) \quad \text{for all } v \in \Lambda^0.$$

Fix  $\mu, \nu$  such that  $s(\mu) = s(\nu)$  and  $d(\mu) - d(\nu) = g$ . We prove (8.4) by induction on  $j$ . When  $j = 0$  both sides of (8.4) are just  $M(Z(\mu))$ , so the inequality is trivial.

Now suppose that (8.4) holds for some  $j \geq 0$ . If  $\eta, \zeta \in \Lambda$  satisfy  $\Lambda^{\min}(\eta, \zeta) = \emptyset$ , then  $\Lambda^{\min}(\eta\xi, \zeta\xi) = \emptyset$  for all  $\xi$ . Using this for the second equality, we calculate:

$$\begin{aligned} \bigcup_{\substack{\lambda \in s(\mu)\Lambda^{(j+1)a} \\ \Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset}} Z(\mu\lambda) &= \bigcup_{\eta \in s(\mu)\Lambda^{ja}} \bigcup_{\substack{\xi \in s(\eta)\Lambda^a \\ \Lambda^{\min}(\mu\eta\xi, \nu\eta\xi) \neq \emptyset}} Z(\mu\eta\xi) \\ &= \bigcup_{\substack{\eta \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\eta, \nu\eta) \neq \emptyset}} \bigcup_{\xi \in s(\eta)\Lambda^a} Z(\mu\eta\xi) \subseteq \bigcup_{\substack{\eta \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\eta, \nu\eta) \neq \emptyset}} \bigcup_{\xi \in s(\eta)\Lambda^a \setminus \{\lambda_{s(\eta)}\}} Z(\mu\eta\xi). \end{aligned}$$

Hence

$$\begin{aligned} M\left(\bigcup_{\substack{\lambda \in s(\mu)\Lambda^{(j+1)a} \\ \Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset}} Z(\mu\lambda)\right) &\leq \sum_{\substack{\eta \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\eta, \nu\eta) \neq \emptyset}} \rho(\Lambda)^{-d(\mu\eta)} \sum_{\xi \in s(\eta)\Lambda^a \setminus \{\lambda_{s(\eta)}\}} M(Z(\xi)) \\ &= \sum_{\substack{\eta \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\eta, \nu\eta) \neq \emptyset}} \rho(\Lambda)^{-d(\mu\eta)} M(Z(s(\eta)) \setminus Z(\lambda_{s(\eta)})) \\ &< K \sum_{\substack{\eta \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\eta, \nu\eta) \neq \emptyset}} \rho(\Lambda)^{-d(\mu\eta)} M(Z(s(\eta))) \\ &= KM\left(\bigcup_{\substack{\eta \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\eta, \nu\eta) \neq \emptyset}} Z(\mu\eta)\right) \\ &\leq K^{j+1}M(Z(\mu)) \end{aligned}$$

by the induction hypothesis.  $\square$

*Proof of Proposition 8.2.* Let  $m - n \in \text{Per } \Lambda$ . Then  $M(\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}) = M(\Lambda^\infty) = 1$  because  $M$  is a probability measure.

Now suppose that  $m - n \notin \text{Per } \Lambda$ . Let  $a$  and  $K$  be as in Lemma 8.4. Fix  $j \in \mathbb{N}$ . We claim that

$$\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\} \subseteq \bigcup_{\mu \in \Lambda^m, \nu \in \Lambda^n, s(\mu) = s(\nu)} \bigcup_{\substack{\lambda \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min}(\mu\lambda, \nu\lambda) \neq \emptyset}} Z(\mu\lambda).$$

To see this, let  $x \in \Lambda^\infty$  and suppose that  $\sigma^m(x) = \sigma^n(x)$ . Let  $\mu := x(0, m)$ ,  $\nu := x(0, n)$  and  $\lambda := \sigma^m(x)(0, ja) = \sigma^n(x)(0, ja)$ . Then  $x(0, (m \vee n) + ja) = \mu\lambda\alpha = \nu\lambda\beta$  for some  $\alpha, \beta$ , and then  $(\alpha, \beta) \in \Lambda^{\min}(\mu\lambda, \nu\lambda)$ . Since  $x \in Z(\mu\lambda)$ , this establishes the claim. Now Lemma 8.4 implies that for all  $j$ ,

$$M(\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}) \leq \sum_{\mu \in \Lambda^m, \nu \in \Lambda^n} K^j M(Z(\mu)) \leq |\Lambda^m| \cdot |\Lambda^n| \cdot K^j.$$

Since  $K < 1$ , the right-hand side goes to zero as  $j \rightarrow \infty$ .  $\square$



## 9. A FORMULA FOR KMS STATES ON THE CUNTZ-KRIEGER ALGEBRA

The next step in our proof of Theorem 7.1 is to establish a formula for a KMS state  $\phi$  of  $C^*(\Lambda)$  in terms of  $\phi \circ \pi_U$ . We will use this later to show that  $\pi_U^*$  is a continuous affine injection from  $\text{KMS}_1$  states of  $(C^*(\Lambda), \alpha)$  to states of  $C^*(\text{Per } \Lambda)$ .

**Theorem 9.1.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph, let  $x^\Lambda$  be the unimodular Perron-Frobenius eigenvector of  $\Lambda$ , and let  $\alpha$  be the preferred dynamics on  $C^*(\Lambda)$ . Let  $U : \text{Per } \Lambda \rightarrow C^*(\Lambda)$  be the unitary representation  $m-n \mapsto \sum_{\mu \in \Lambda^m} s_\mu s_{\theta_{m,n}(\mu)}^*$  of Proposition 6.1. If  $\phi$  is a  $\text{KMS}_1$  state for  $(C^*(\Lambda), \alpha)$ , then*

$$(9.1) \quad \phi(s_\mu s_\nu^*) = \begin{cases} \rho(\Lambda)^{-d(\mu)} x_{s(\mu)}^\Lambda \phi(U_{d(\mu)-d(\nu)}) & \text{if } d(\mu) - d(\nu) \in \text{Per } \Lambda \\ & \text{and } \theta_{d(\mu), d(\nu)}(\mu) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

*Remark 9.2.* On the face of it, the formula (9.1) doesn't appear to satisfy  $\phi(s_\mu s_\nu^*) = \phi(s_\nu s_\mu^*)$  (as a state must) because the coefficient  $\rho(\Lambda)^{-d(\mu)}$  doesn't appear to be symmetric in  $\mu$  and  $\nu$ . But all is well: (7.1) shows that  $\rho(\Lambda)^{-d(\mu)} = \rho(\Lambda)^{-d(\nu)}$  for  $d(\mu) - d(\nu) \in \text{Per } \Lambda$ .

Our proof of Theorem 9.1 requires a preliminary lemma.

**Lemma 9.3.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph. Let  $\alpha$  be the preferred dynamics on  $C^*(\Lambda)$ . Suppose that  $\phi$  is a  $\text{KMS}_1$  state for  $(C^*(\Lambda), \alpha)$ . Let  $\mu, \nu \in \Lambda$  with  $s(\mu) = s(\nu)$ . Then for every  $p \in \mathbb{N}^k$  we have*

$$|\phi(s_\mu s_\nu^*)| \leq \sum_{\substack{\lambda \in s(\mu)\Lambda^p \\ \Lambda^{\min(\mu\lambda, \nu\lambda)} \neq \emptyset}} \phi(s_{\mu\lambda} s_{\nu\lambda}^*).$$

*Proof.* First suppose that  $\rho(\Lambda)^{d(\mu)} \neq \rho(\Lambda)^{d(\nu)}$ . Applying the KMS condition twice, as in the end of the proof of [14, Proposition 3.1 (b)], gives  $\phi(s_\mu s_\nu^*) = \rho(\Lambda)^{d(\mu)-d(\nu)} \phi(s_\mu s_\nu^*)$ . Hence  $\phi(s_\mu s_\nu^*) = 0$ , and the result is trivial.

Second suppose that  $\rho(\Lambda)^{d(\mu)} = \rho(\Lambda)^{d(\nu)}$ . Applying (CK4) and the triangle inequality gives  $|\phi(s_\mu s_\nu^*)| \leq \sum_{\lambda \in s(\mu)\Lambda^p} |\phi(s_{\mu\lambda} s_{\nu\lambda}^*)|$ . As in the proof of [14, Lemma 5.3 (a)], the KMS condition combined with the relation  $s_\eta^* s_\zeta = \sum_{(\alpha, \beta) \in \Lambda^{\min(\eta, \zeta)}} s_\alpha s_\beta^*$  shows that  $\phi(s_{\mu\lambda} s_{\nu\lambda}^*) = 0$  whenever  $\Lambda^{\min(\mu\lambda, \nu\lambda)} = \emptyset$ . Hence

$$\sum_{\lambda \in s(\mu)\Lambda^p} |\phi(s_{\mu\lambda} s_{\nu\lambda}^*)| = \sum_{\substack{\lambda \in s(\mu)\Lambda^p \\ \Lambda^{\min(\mu\lambda, \nu\lambda)} \neq \emptyset}} |\phi(s_{\mu\lambda} s_{\nu\lambda}^*)|.$$

Since  $\rho(\Lambda)^{d(\mu)} = \rho(\Lambda)^{d(\nu)}$ , an argument using the Cauchy-Schwarz inequality (see [14, Lemma 5.2]) shows that each  $|\phi(s_{\mu\lambda} s_{\nu\lambda}^*)| \leq \phi(s_{\mu\lambda} s_{\mu\lambda}^*)$ , and the result follows.  $\square$

*Proof of Theorem 9.1.* Let  $M$  be the measure on  $\Lambda^\infty$  obtained in Proposition 8.1, so that  $\phi(s_\lambda s_\lambda^*) = M(Z(\lambda))$  for all  $\lambda \in \Lambda$ .

First suppose that  $d(\mu) - d(\nu) \notin \text{Per } \Lambda$ . Choose  $a \in \mathbb{N}^k$  and  $0 < K < 1$  as in Lemma 8.4. For  $j \in \mathbb{N}$ , Lemma 9.3 implies that

$$|\phi(s_\mu s_\nu^*)| \leq \sum_{\substack{\lambda \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min(\mu\lambda, \nu\lambda)} \neq \emptyset}} \phi(s_{\mu\lambda} s_{\nu\lambda}^*) = M\left(\bigcup_{\substack{\lambda \in s(\mu)\Lambda^{ja} \\ \Lambda^{\min(\mu\lambda, \nu\lambda)} \neq \emptyset}} Z(\mu\lambda)\right).$$

By choice of  $K$  and  $a$ , the right-hand side is dominated by  $K^j M(Z(\mu))$ . This goes to zero as  $j \rightarrow \infty$ , and so  $\phi(s_\mu s_\nu^*) = 0$ .

Now suppose that  $\mu \in \Lambda^m$  and  $\nu \in \Lambda^n$  with  $m - n \in \text{Per } \Lambda$ . We start by showing that

$$(9.2) \quad \phi(s_\mu s_\nu^*) = \delta_{\theta_{m,n}(\mu), \nu} \rho(\Lambda)^{-m} \phi(p_{s(\mu)} U_{m-n}).$$

Lemma 5.1 implies that  $s_\mu s_\mu^* = s_{\theta_{m,n}(\mu)} s_{\theta_{m,n}(\mu)}^*$ , and so the KMS condition implies that

$$\phi(s_\mu s_\nu^*) = \phi(s_{\theta_{m,n}(\mu)} s_{\theta_{m,n}(\mu)}^* s_\mu s_\nu^*) = \phi(s_\mu s_\nu^* s_{\theta_{m,n}(\mu)} s_{\theta_{m,n}(\mu)}^*) = \delta_{\nu, \theta_{m,n}(\mu)} \phi(s_\mu s_{\theta_{m,n}(\mu)}^*)$$

since  $d(\nu) = d(\theta_{m,n}(\mu))$ . The KMS condition gives

$$\phi(s_\mu s_{\theta_{m,n}(\mu)}^*) = \rho(\Lambda)^{-m} \phi(s_{\theta_{m,n}(\mu)}^* s_\mu) = \rho(\Lambda)^{-m} \phi\left(\sum_{(\alpha, \beta) \in \Lambda^{\min(\theta_{m,n}(\mu), \mu)}} s_\alpha s_\beta^*\right).$$

Let  $p := (m \vee n) - m$  and  $q := (m \vee n) - n$ . Corollary 5.3 implies that  $\Lambda^{\min(\theta_{m,n}(\mu), \mu)} = \{(\alpha, \theta_{q,p}(\alpha)) : \alpha \in s(\mu) \Lambda^q\}$ . Hence

$$\begin{aligned} \phi(s_\mu s_{\theta_{m,n}(\mu)}^*) &= \rho(\Lambda)^{-m} \phi\left(\sum_{\alpha \in s(\mu) \Lambda^q} s_\alpha s_{\theta_{q,p}(\alpha)}^*\right) \\ &= \rho(\Lambda)^{-m} \phi(p_{s(\mu)} U_{q-p}) = \rho(\Lambda)^{-m} \phi(p_{s(\mu)} U_{m-n}) \end{aligned}$$

since  $q - p = m - n$ . This gives (9.2).

To establish (9.1), it now suffices to show that  $\phi(p_v U_{n-m}) = x_v^\Lambda \phi(U_{n-m})$  for all  $v \in \Lambda^0$ . To see this, consider the vector  $(y_v^{n-m}) \in \mathbb{C}^{\Lambda^0}$  defined by  $y_v^{n-m} = \phi(p_v U_{n-m})$ . Fix  $1 \leq i \leq k$  and  $v \in \Lambda^0$ . Proposition 6.1 implies that  $U_{n-m}$  is central in  $C^*(\Lambda)$ . Using this and the Cuntz-Krieger relation and then the KMS condition, we calculate:

$$\begin{aligned} y_v^{n-m} &= \phi(p_v U_{n-m}) = \sum_{\lambda \in v \Lambda^{e_i}} \phi(s_\lambda s_\lambda^* U_{n-m}) = \sum_{\lambda \in v \Lambda^{e_i}} \phi(s_\lambda U_{n-m} s_\lambda^*) \\ &= \sum_{\lambda \in v \Lambda^{e_i}} \rho(\Lambda)_i^{-1} \phi(p_{s(\lambda)} U_{n-m}) = \rho(A_i)^{-1} \sum_{w \in \Lambda^0} A_i(v, w) y_w^{n-m} = \rho(A_i)^{-1} (A_i y^{n-m})_v. \end{aligned}$$

Hence  $y^{n-m}$  is an eigenvector of each  $A_i$  with eigenvalue  $\rho(A_i)$ . Corollary 4.2(c) now implies that  $y^{n-m} = z x^\Lambda$  for some  $z \in \mathbb{C}$ . Since  $x^\Lambda$  has unit 1-norm, we have

$$z = \sum_{v \in \Lambda^0} z x_v^\Lambda = \sum_{v \in \Lambda^0} y_v^{n-m} = \phi\left(\sum_{v \in \Lambda^0} p_v U_{n-m}\right) = \phi(U_{n-m}). \quad \square$$

## 10. CONSTRUCTING KMS STATES ON THE CUNTZ-KRIEGER ALGEBRA

In this section we construct a  $\text{KMS}_1$  state  $\phi_1$  of  $(C^*(\Lambda), \alpha)$  such that  $\pi_U^* \phi_1$  is the identity character of  $C^*(\text{Per } \Lambda)$ . We then show that every character of  $C^*(\text{Per } \Lambda)$  is obtained by composing  $\pi_U^* \phi_1$  with a gauge automorphism  $\gamma_z$ . At the end of the section we combine this with Theorem 9.1 to prove our main theorem.

Let  $\{h_x : x \in \Lambda^\infty\}$  be the orthonormal basis of point masses in  $\ell^2(\Lambda^\infty)$ . Recall from the proof of [16, Proposition 2.11] that there is a Cuntz-Krieger  $\Lambda$ -family  $\{S_\lambda : \lambda \in \Lambda\}$  in  $\mathcal{B}(\ell^2(\Lambda^\infty))$  such that  $S_\lambda h_x = \delta_{s(\lambda), r(x)} h_{\lambda x}$ . We then have  $S_\lambda^* h_x = \delta_{\lambda, x(0, d(\lambda))} h_{\sigma^{d(\lambda)}(x)}$ . The universal property of  $C^*(\Lambda)$  implies that there is a representation  $\pi_S : C^*(\Lambda) \rightarrow \mathcal{B}(\ell^2(\Lambda^\infty))$  such that  $\pi_S(s_\lambda) = S_\lambda$ . We call  $\pi_S$  the *infinite-path representation*.

**Lemma 10.1.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph, and let  $M$  be the measure on  $\Lambda^\infty$  obtained in Proposition 8.1.*

(a) Let  $\mu, \nu \in \Lambda$ . Then

$$\begin{aligned} M(\{x \in \Lambda^\infty : x = \mu y = \nu y \text{ for some } y \in \Lambda^\infty\}) \\ = \begin{cases} M(Z(\mu)) & \text{if } d(\mu) - d(\nu) \in \text{Per } \Lambda \text{ and } \theta_{d(\mu), d(\nu)}(\mu) = \nu \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b) Let  $\pi_S$  be the infinite-path representation. For  $a \in C^*(\Lambda)$ , the function  $f_a : x \mapsto (\pi_S(a)h_x | h_x)$  is  $M$ -integrable and

$$\left| \int_{\Lambda^\infty} (\pi_S(a)h_x | h_x) dM(x) \right| \leq \|a\|.$$

*Proof.* For convenience, write  $Z_{\mu, \nu} := \{x \in \Lambda^\infty : x = \mu y = \nu y \text{ for some } y \in \Lambda^\infty\}$ . Since  $Z_{\mu, \nu}$  is closed it is measurable.

First suppose that  $d(\mu) - d(\nu) \notin \text{Per } \Lambda$ . Then  $M(Z_{\mu, \nu}) \leq M(\{x \in \Lambda^\infty : \sigma^{d(\mu)}(x) = \sigma^{d(\nu)}(x)\}) = 0$  by Proposition 8.2. Thus  $M(Z_{\mu, \nu}) = 0$ .

Second, suppose that  $d(\mu) - d(\nu) \in \text{Per } \Lambda$  and  $\theta_{d(\mu), d(\nu)}(\mu) \neq \nu$ . Since  $Z_{\mu, \nu} \subseteq Z(\mu) \cap Z(\nu)$ , we deduce that  $Z_{\mu, \nu} = \emptyset$ , and  $M(Z_{\mu, \nu}) = 0$ .

Third, suppose that  $d(\mu) - d(\nu) \in \text{Per } \Lambda$  and  $\theta_{d(\mu), d(\nu)}(\mu) = \nu$ . If  $x \in Z(\mu)$ , then  $y = \sigma^{d(\mu)}(x)$  satisfies  $x = \mu y$ . So  $\mu y = \nu y$  by definition of  $\theta_{d(\mu), d(\nu)}$ , giving  $x \in Z_{\mu, \nu}$ . Thus  $Z_{\mu, \nu} = Z(\mu)$  and  $M(Z_{\mu, \nu}) = M(Z(\mu))$ . This gives (a).

For (b), observe that

$$(\pi_S(s_\mu s_\nu^*)h_x | h_x) = (S_\nu^* h_x | S_\mu^* h_x) = \begin{cases} 1 & \text{if } x = \mu y = \nu y \text{ for some } y \in \Lambda^\infty \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $f_{s_\mu s_\nu^*}$  is the characteristic function of the measurable set  $Z_{\mu, \nu}$ . Choose a sequence  $a_n$  of finite linear combinations of the  $s_\mu s_\nu^*$  such that  $a_n \rightarrow a$ . Then each  $f_{a_n}$  is a simple function. Continuity of  $\pi_S$  and of the inner product implies that  $f_{a_n} \rightarrow f_a$  pointwise on  $\Lambda^\infty$ . Thus  $f_a$  is measurable. Finally,

$$\left| \int_{\Lambda^\infty} f_a(x) dM(x) \right| \leq \int_{\Lambda^\infty} |(\pi_S(a)h_x | h_x)| dM(x) \leq \int_{\Lambda^\infty} \|a\| dM(x) = \|a\|. \quad \square$$

**Proposition 10.2.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph, and let  $M$  be the measure on  $\Lambda^\infty$  obtained in Proposition 8.1. Let  $\alpha$  be the preferred dynamics on  $C^*(\Lambda)$ . Let  $\pi_S$  be the infinite-path representation. Then there is a  $\text{KMS}_1$  state  $\phi$  of  $(C^*(\Lambda), \alpha)$  with formula*

$$(10.1) \quad \phi(a) := \int_{\Lambda^\infty} (\pi_S(a)h_x | h_x) dM(x) \text{ for } a \in C^*(\Lambda).$$

*Proof.* Lemma 10.1 implies that (10.1) defines a norm-decreasing map  $\phi : C^*(\Lambda) \rightarrow \mathbb{C}$ . This  $\phi$  is linear and positive. It is a state because

$$\phi(1) = \int_{\Lambda^\infty} (\pi_S(1)h_x | h_x) dM(x) = \int_{\Lambda^\infty} \|h_x\|^2 dM(x) = 1.$$

It remains to verify the KMS condition. Unfortunately [14, Proposition 3.1(b)] does not apply since the coordinates of  $\rho(\Lambda)$  may not be rationally independent; indeed KMS states may not be supported on the diagonal subalgebra. So we have to check the KMS condition from first principles.

Suppose that  $s(\mu) = s(\nu)$  and  $s(\eta) = s(\zeta)$ . We must show that

$$(10.2) \quad \phi(s_\mu s_\nu^* s_\eta s_\zeta^*) = \rho(\Lambda)^{-(d(\mu) - d(\nu))} \phi(s_\eta s_\zeta^* s_\mu s_\nu^*).$$

Suppose first that  $d(\mu) - d(\nu) + d(\eta) - d(\zeta) \notin \text{Per } \Lambda$ . Applying (CK4), we obtain

$$s_\mu s_\nu^* s_\eta s_\zeta^* = \sum_{\xi \in s(\nu)\Lambda^{d(\eta)}} \sum_{\omega \in s(\eta)\Lambda^{d(\nu)}} s_{\mu\xi} s_{\nu\xi}^* s_{\eta\omega} s_{\zeta\omega}^* = \sum_{\nu\xi = \eta\omega \in \Lambda^{d(\nu)+d(\eta)}} s_{\mu\xi} s_{\zeta\omega}^*.$$

Each  $d(\mu\xi) - d(\zeta\omega) = d(\mu) - d(\nu) + d(\eta) - d(\zeta) \notin \text{Per } \Lambda$ , and so Lemma 10.1 implies that  $\phi(s_\mu s_\nu^* s_\eta s_\zeta^*) = 0$ . Symmetry gives  $\phi(s_\eta s_\zeta^* s_\mu s_\nu^*) = 0$ , so both sides of (10.2) are zero.

Now suppose that  $d(\mu) - d(\nu) + d(\eta) - d(\zeta) \in \text{Per } \Lambda$ . Let  $q = d(\mu) \vee d(\nu)$ . Then

$$\phi(s_\mu s_\nu^* s_\eta s_\zeta^*) = \sum_{\kappa \in s(\eta)\Lambda^q} \phi(s_\mu s_\nu^* s_{\eta\kappa} s_{\zeta\kappa}^*),$$

and

$$\rho(\Lambda)^{-(d(\mu)-d(\nu))} \phi(s_\eta s_\zeta^* s_\mu s_\nu^*) = \sum_{\kappa \in s(\eta)\Lambda^q} \rho(\Lambda)^{-(d(\mu)-d(\nu))} \phi(s_{\eta\kappa} s_{\zeta\kappa}^* s_\mu s_\nu^*).$$

Thus it suffices to establish (10.2) under the additional hypothesis that  $d(\eta), d(\zeta) \geq d(\mu) \vee d(\nu)$ . Then  $d(\eta) \geq d(\nu)$ , and we have

$$\begin{aligned} \phi(s_\mu s_\nu^* s_\eta s_\zeta^*) &= \begin{cases} \phi(s_{\mu\tau} s_\zeta^*) & \text{if } \eta = \nu\tau \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \int_{\Lambda^\infty} (S_\zeta^* h_x \mid S_{\mu\tau}^* h_x) dM(x) & \text{if } \eta = \nu\tau \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} M(\{x \in \Lambda^\infty : x = \mu\tau y = \zeta y \text{ for some } y\}) & \text{if } \eta = \nu\tau \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If  $\eta = \nu\tau$ , then

$$d(\mu\tau) - d(\zeta) = d(\mu) + d(\tau) - d(\nu) + d(\nu) - d(\zeta) = d(\mu) - d(\nu) + d(\eta) - d(\zeta) \in \text{Per } \Lambda$$

by assumption. Thus Lemma 10.1 gives

$$\phi(s_\mu s_\nu^* s_\eta s_\zeta^*) = \begin{cases} M(Z(\mu\tau)) & \text{if } \eta = \nu\tau, d(\mu\tau) - d(\zeta) \in \text{Per } \Lambda, \theta_{d(\mu\tau), d(\zeta)}(\mu\tau) = \zeta \\ 0 & \text{otherwise.} \end{cases}$$

A similar argument gives

$$\phi(s_\eta s_\zeta^* s_\mu s_\nu^*) = \begin{cases} M(Z(\eta)) & \text{if } \zeta = \mu\beta, d(\eta) - d(\nu\beta) \in \text{Per } \Lambda, \theta_{d(\eta), d(\nu\beta)}(\eta) = \nu\beta \\ 0 & \text{otherwise.} \end{cases}$$

We check that the conditions appearing in the right-hand sides of these expressions for  $\phi(s_\mu s_\nu^* s_\eta s_\zeta^*)$  and  $\phi(s_\eta s_\zeta^* s_\mu s_\nu^*)$  match up. Suppose that the three conditions of the first expression hold:

$$(10.3) \quad \eta = \nu\tau, \quad d(\mu\tau) - d(\zeta) \in \text{Per } \Lambda \quad \text{and} \quad \theta_{d(\mu\tau), d(\zeta)}(\mu\tau) = \zeta.$$

Then  $d(\tau) - (d(\zeta) - d(\mu)) \in \text{Per } \Lambda$ . Let  $\beta := \theta_{d(\tau), d(\zeta)-d(\mu)}(\tau)$  (this makes sense since  $d(\zeta) \geq d(\mu)$ ). Proposition 5.2(c) shows that

$$\zeta = \theta_{d(\mu\tau), d(\zeta)}(\mu\tau) = \theta_{d(\mu)+d(\tau), d(\mu)+(d(\zeta)-d(\mu))}(\mu\tau) = \mu\beta.$$

We have

$$d(\nu\beta) - d(\eta) = d(\nu\tau) - d(\tau) + d(\beta) - d(\eta) = d(\beta) - d(\tau) = d(\zeta) - d(\mu\tau) \in \text{Per } \Lambda.$$

Proposition 5.2(c) then gives  $\theta_{d(\nu\beta),d(\eta)}(\nu\beta) = \nu\theta_{d(\beta),d(\tau)}(\beta)$ , and by Lemma 5.1(d) this is  $\nu\theta_{d(\tau),d(\beta)}^{-1}(\beta) = \nu\tau$ , which equals  $\eta$  by assumption. Another application of Lemma 5.1(d) yields  $\nu\beta = \theta_{d(\eta),d(\nu\beta)}(\eta)$ . So the three conditions of the second expression hold:

$$(10.4) \quad \zeta = \mu\beta, \quad d(\eta) - d(\nu\beta) \in \text{Per } \Lambda \quad \text{and} \quad \theta_{d(\eta),d(\nu\beta)}(\eta) = \nu\beta.$$

A symmetric argument shows that (10.4) implies (10.3).

To establish (10.2), first suppose that (10.3) fails. Then so does (10.4), and both sides of (10.2) are zero. Now suppose that (10.3) holds. Then so does (10.4), and so

$$\begin{aligned} \phi(s_\mu s_\nu^* s_\eta s_\zeta^*) &= M(Z(\mu\tau)) = \rho(\Lambda)^{-d(\mu\tau)} M(Z(s(\tau))) \\ &= \rho(\Lambda)^{d(\eta)-d(\mu\tau)} M(Z(\eta)) = \rho(\Lambda)^{-(d(\mu)-d(\nu))} \phi(s_\eta s_\zeta^* s_\mu s_\nu^*). \end{aligned} \quad \square$$

Since the  $\text{KMS}_1$  state  $\phi$  of Proposition 10.2 may not be supported on  $\overline{\text{span}}\{s_\lambda s_\lambda^*\}$  we can now perturb by gauge automorphisms  $\gamma_z$  to obtain new  $\text{KMS}_1$  states.

**Corollary 10.3.** *Suppose  $\Lambda$  is a strongly connected finite  $k$ -graph. Let  $x^\Lambda$  be the unimodular Perron-Frobenius eigenvector of  $\Lambda$ . Let  $\alpha$  be the preferred dynamics on  $C^*(\Lambda)$ . For each  $z \in \mathbb{T}^k$  there is a  $\text{KMS}_1$  state  $\phi_z$  of  $(C^*(\Lambda), \alpha)$  satisfying*

$$(10.5) \quad \phi_z(s_\mu s_\nu^*) = \begin{cases} \rho(\Lambda)^{-d(\mu)} z^{d(\mu)-d(\nu)} x_{s(\mu)}^\Lambda & \text{if } d(\mu) - d(\nu) \in \text{Per } \Lambda \\ & \text{and } \theta_{d(\mu),d(\nu)}(\mu) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $\phi$  be the  $\text{KMS}_1$  state of Proposition 10.2. Let  $z \in \mathbb{T}^k$  and  $\phi_z = \phi \circ \gamma_z$ . Then  $\phi_z$  is a state. Using the KMS condition for  $\phi$ , we calculate:

$$\begin{aligned} \phi_z(s_\mu s_\nu^* s_\eta s_\zeta^*) &= z^{d(\mu)-d(\nu)+d(\eta)-d(\zeta)} \phi(s_\mu s_\nu^* s_\eta s_\zeta^*) \\ &= z^{d(\mu)-d(\nu)+d(\eta)-d(\zeta)} \rho(\Lambda)^{-(d(\mu)-d(\nu))} \phi(s_\eta s_\zeta^* s_\mu s_\nu^*) \\ &= \rho(\Lambda)^{-(d(\mu)-d(\nu))} \phi_z(s_\eta s_\zeta^* s_\mu s_\nu^*). \end{aligned}$$

Hence  $\phi_z$  is a  $\text{KMS}_1$  state of  $(C^*(\Lambda), \alpha)$ .

Let  $\mu, \nu \in \Lambda$  and let  $M$  be the measure on  $\Lambda^\infty$  obtained in Proposition 8.1. The formula for  $\phi$  in Proposition 10.2 gives

$$\phi_z(s_\mu s_\nu^*) = \int_{\Lambda^\infty} (\pi_S(\gamma_z(s_\mu s_\nu^*)) h_x \mid h_x) dM(x) = z^{d(\mu)-d(\nu)} \int_{\Lambda^\infty} (\pi_S(s_\mu s_\nu^*) h_x \mid h_x) dM(x).$$

By Lemma 10.1, this is

$$= \begin{cases} z^{d(\mu)-d(\nu)} M(Z(\mu)) & \text{if } d(\mu) - d(\nu) \in \text{Per } \Lambda \text{ and } \theta_{d(\mu),d(\nu)}(\mu) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Now (10.5) follows from (8.3).  $\square$

*Proof of Theorem 7.1.* It is clear that  $\phi \mapsto \phi \circ \pi_U$  is continuous and affine. To see that it is injective, suppose that  $\phi$  and  $\phi'$  are  $\text{KMS}$  states of  $C^*(\Lambda)$  such that  $\phi \circ \pi_U = \phi' \circ \pi_U$ . Then the formula (9.1) implies that  $\phi(s_\mu s_\nu^*) = \phi'(s_\mu s_\nu^*)$  for all  $\mu, \nu$ , and so  $\phi = \phi'$ .

To prove that  $\pi_U^*$  is surjective, we first show that every pure state of  $C^*(\text{Per } \Lambda)$  belongs to the range of  $\pi_U^*$ . Fix a pure state  $\chi$  of  $C^*(\text{Per } \Lambda)$ . Since  $C^*(\text{Per } \Lambda)$  is commutative,  $\chi$  is a 1-dimensional representation and hence determines a character, also denoted  $\chi$ , of  $\text{Per } \Lambda$ . Choose  $z \in \mathbb{T}^k$  such that  $z^m = \chi(m)$  for all  $m \in \text{Per } \Lambda$ . Let  $\phi_z$  be the  $\text{KMS}_1$  state

of Corollary 10.3. Let  $i_{\text{Per } \Lambda} : \text{Per } \Lambda \rightarrow C^*(\text{Per } \Lambda)$  be the universal unitary representation. For  $m - n \in \text{Per } \Lambda$ , we have

$$\phi_z \circ \pi_U(i_{\text{Per } \Lambda}(m - n)) = \phi_z(U_{m-n}) = \sum_{\mu \in \Lambda^m} \phi_z(s_\mu s_{\theta_{m,n}(\mu)}^*).$$

Applying the formula for  $\phi_z$  from (10.5) to each term gives

$$\begin{aligned} \phi_z \circ \pi_U(i_{\text{Per } \Lambda}(m - n)) &= \sum_{\mu \in \Lambda^m} \rho(\Lambda)^{-m} z^{m-n} x_{s(\mu)}^\Lambda = \rho(\Lambda)^{-m} \chi(m - n) \sum_{\mu \in \Lambda^m} x_{s(\mu)}^\Lambda \\ &= \rho(\Lambda)^{-m} \chi(m - n) \sum_{v, w \in \Lambda^0} A^m(v, w) x_w^\Lambda \\ &= \rho(\Lambda)^{-m} \chi(m - n) \sum_{v \in \Lambda^0} (A^m x^\Lambda)_v \\ &= \rho(\Lambda)^{-m} \chi(m - n) \sum_{v \in \Lambda^0} \rho(\Lambda)^m x_v^\Lambda = \chi(m - n). \end{aligned}$$

Hence  $\chi = \phi_z \circ \pi_U = \pi_U^*(\phi_z)$ .

Since  $\pi_U^*$  is affine, every convex combination of pure states of  $C^*(\text{Per } \Lambda)$  is in the range of  $\pi_U^*$ . Now fix a state  $\psi$  of  $C^*(\text{Per } \Lambda)$ . The Krein-Milman theorem implies that there is a sequence  $(\psi_n)$  of convex combinations of pure states of  $C^*(\text{Per } \Lambda)$  such that  $\psi_n \rightarrow \psi$ . Each  $\psi_n$  is in the range of  $\pi_U^*$ , so it suffices to show that the range of  $\pi_U^*$  is closed. The  $\text{KMS}_1$  simplex of  $(C^*(\Lambda), \alpha)$  is compact [3, Theorem 5.3.30(1)], and so its image under the continuous map  $\pi_U^*$  is also compact. Since the state space of  $C^*(\text{Per } \Lambda)$  is Hausdorff, we deduce that the image of  $\pi_U^*$  is closed.  $\square$

*Remark 10.4.* Theorem 9.1 shows how to describe the inverse of  $\pi_U^*$ . Let  $\psi$  be a state of  $C^*(\text{Per } \Lambda)$ . Then  $\phi := (\pi_U^*)^{-1}(\psi)$  satisfies  $\phi \circ \pi_U = \psi$  and so  $\phi(U_{m-n}) = \phi \circ \pi_U(i_{\text{Per } \Lambda}(m - n)) = \psi(i_{\text{Per } \Lambda}(m - n))$ . So (9.1) shows that

$$\phi(s_\mu s_\nu^*) = \begin{cases} \rho(\Lambda)^{-d(\mu)} \psi(i_{\text{Per } \Lambda}(d(\mu) - d(\nu))) x_{s(\mu)}^\Lambda & \text{if } d(\mu) - d(\nu) \in \text{Per } \Lambda \\ & \text{and } \theta_{d(\mu), d(\nu)}(\mu) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

## 11. CONSEQUENCES OF OUR MAIN THEOREM

**A question of Yang.** In [31, 33], Yang studies a particular state  $\omega$  on the  $C^*$ -algebra of a finite  $k$ -graph with one vertex. She asks whether this  $\omega$  is a factor state if and only if  $\Lambda$  is aperiodic. We will use the following theorem to give an affirmative answer for a much broader class of  $k$ -graphs. We explain precisely how our theorem relates to Yang's conjecture in Remark 11.2.

Given a state  $\phi$  of a  $C^*$ -algebra  $A$ , we write  $\pi_\phi$  for the associated GNS representation of  $A$ . Recall that  $\phi$  is a factor state if the double-commutant  $\pi_\phi(A)''$  is a factor.

**Theorem 11.1.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph. Let  $\alpha$  be the preferred dynamics on  $C^*(\Lambda)$ , and let  $x^\Lambda$  be the unimodular Perron-Frobenius eigenvector of  $\Lambda$  (see Definition 4.4). Let  $\gamma$  denote the gauge action of  $\mathbb{T}^k$  on  $C^*(\Lambda)$ . There is a  $\text{KMS}_1$  state  $\omega$  of  $(C^*(\Lambda), \alpha)$  such that*

$$(11.1) \quad \omega(s_\mu s_\nu^*) = \delta_{\mu, \nu} \rho(\Lambda)^{-d(\mu)} x_{s(\mu)}^\Lambda \quad \text{for all } \mu, \nu.$$

This  $\omega$  is the unique  $\gamma$ -invariant KMS state of  $(C^*(\Lambda), \alpha)$ , and restricts to a trace on the fixed-point algebra  $C^*(\Lambda)^\gamma$ . The following are equivalent:

- (a)  $\Lambda$  is aperiodic;
- (b)  $C^*(\Lambda)$  is simple;
- (c)  $\omega$  is a factor state;
- (d)  $\omega$  is the only  $\text{KMS}_1$  state of  $(C^*(\Lambda), \alpha)$ .

*Proof.* Let  $\text{Tr}$  be the trace on  $C^*(\text{Per } \Lambda)$  corresponding to Haar measure on  $(\text{Per } \Lambda)^\wedge$ . Then  $\text{Tr}(i_{\text{Per } \Lambda}(g)) = \delta_{g,0}$  for  $g \in \text{Per } \Lambda$ . Remark 10.4 shows that there is a  $\text{KMS}_1$  state of  $(C^*(\Lambda), \alpha)$  satisfying

$$\omega(s_\mu s_\nu^*) = \begin{cases} \rho(\Lambda)^{-d(\mu)} \text{Tr}(i_{\text{Per } \Lambda}(d(\mu) - d(\nu))) x_{s(\mu)}^\Lambda & \text{if } d(\mu) - d(\nu) \in \text{Per } \Lambda \\ & \text{and } \theta_{d(\mu), d(\nu)}(\mu) = \nu \\ 0 & \text{otherwise,} \end{cases}$$

and that  $\omega \circ \pi_U = \text{Tr}$ . Lemma 5.1(d) shows that  $\theta_{m,m} = \text{id}_{\Lambda^m}$  for each  $m \in \mathbb{N}^k$ , so  $\omega$  satisfies (11.1).

The formula (11.1) shows that  $\omega(\gamma_z(s_\mu s_\nu^*)) = \omega(s_\mu s_\nu^*)$  for all  $\mu, \nu$ , and so  $\omega$  is gauge-invariant. For uniqueness, suppose that  $\omega'$  is a gauge-invariant  $\text{KMS}_1$  state of  $C^*(\Lambda, \alpha)$ . For  $m, n \in \mathbb{N}^k$  with  $m - n \in \text{Per } \Lambda$ , and for  $z \in \mathbb{T}^k$ , we have

$$\omega'(U_{m-n}) = \omega'(\gamma_z(U_{m-n})) = \omega' \left( \sum_{\mu \in \Lambda^m} \gamma_z(s_\mu s_{\theta_{m,n}(\mu)}^*) \right) = z^{m-n} \omega'(U_{m-n}).$$

So if  $\omega'(U_{m-n}) \neq 0$  then  $z^{m-n} = 1$  for all  $z \in \mathbb{T}^k$ , forcing  $m - n = 0$ . Hence  $\omega' \circ \pi_U = \text{Tr} = \omega \circ \pi_U$ , and Theorem 7.1 implies that  $\omega' = \omega$ .

Since the dynamics  $\alpha$  is a subgroup of the gauge action, every element of  $C^*(\Lambda)^\gamma$  is fixed by  $\alpha$ . In particular, every element of  $C^*(\Lambda)^\gamma$  is analytic, and the KMS condition implies that each  $\omega(ab) = \omega(b\alpha_i(a)) = \omega(ba)$ , so  $\omega$  is a trace on  $C^*(\Lambda)^\gamma$ .

It remains to establish that the conditions (a)–(d) are equivalent. We will prove (a)  $\iff$  (b), then (a)  $\iff$  (d), and then (c)  $\iff$  (d).

For (a)  $\iff$  (b), observe that since  $\Lambda$  is strongly connected it is cofinal (see [16, Definition 4.7]). So combining [27, Theorem 3.1] and (iii)  $\iff$  (i) of [27, Lemma 3.2] shows that  $C^*(\Lambda)$  is simple if and only if  $\Lambda$  is aperiodic.

For (a)  $\implies$  (d), observe that since  $\Lambda$  is aperiodic, we have  $\text{Per } \Lambda = \{0\}$  by Proposition 5.4. So  $\text{Tr}$  is the unique state of  $C^*(\text{Per } \Lambda)$ , and Theorem 7.1 implies that  $\omega$  is the unique  $\text{KMS}_1$  state of  $(C^*(\Lambda), \alpha)$ . For (d)  $\implies$  (a), observe that if  $\omega$  is the only  $\text{KMS}_1$  state of  $(C^*(\Lambda), \alpha)$ , then Theorem 7.1 shows that  $\text{Tr} := \omega \circ \pi_U$  is the only state of  $C^*(\text{Per } \Lambda)$  and hence  $\text{Per } \Lambda = \{0\}$ . So Proposition 5.4 implies that  $\Lambda$  is aperiodic.

For (c)  $\iff$  (d), first recall that the pure states of  $C^*(\text{Per } \Lambda)$  are the states obtained from integration against point-mass measures on  $(\text{Per } \Lambda)^\wedge$ . Since  $\text{Tr}$  is obtained from integration against Haar measure, we deduce that  $\text{Tr}$  is a pure state if and only if it is the only state of  $C^*(\text{Per } \Lambda)$ . So Theorem 7.1 shows that  $\omega$  is an extreme point of the  $\text{KMS}_1$  simplex of  $C^*(\Lambda, \alpha)$  if and only if it is the unique  $\text{KMS}_1$  state. Theorem 5.3.30(3) of [3] implies that a  $\text{KMS}_1$  state is a factor state if and only if it is an extreme  $\text{KMS}_1$  state, giving (c)  $\iff$  (d).  $\square$

We now discuss how this result relates to Yang's work.

*Remark 11.2.* Let  $\Lambda$  be a row-finite  $k$ -graph with one vertex. Then [16, Lemma 3.2] implies that  $C^*(\Lambda)^\gamma$  is a UHF algebra, and so has a unique trace  $\tau$ . Let  $\Phi : C^*(\Lambda) \rightarrow C^*(\Lambda)^\gamma$  be the conditional expectation obtained from averaging over  $\gamma$  as on page 6 of [16]. In [33] (see also [31, 32]), Yang studies the state  $\tau \circ \Phi$ .

We claim that the gauge-invariant KMS<sub>1</sub> state  $\omega$  described in Theorem 11.1 is equal to  $\tau \circ \Phi$ . To see this, observe that the formula (11.1) shows that  $\omega = \omega|_{C^*(\Lambda)^\gamma} \circ \Phi$ . Theorem 11.1 implies that  $\omega|_{C^*(\Lambda)^\gamma}$  is a trace. Since  $\tau$  is the unique trace on  $C^*(\Lambda)^\gamma$ , we deduce that  $\omega|_{C^*(\Lambda)^\gamma} = \tau$  and hence that  $\omega = \tau \circ \Phi$ .

Since  $\Lambda$  has one vertex, each  $A_i$  is the  $1 \times 1$  matrix  $(|\Lambda^{e_i}|)$ . So  $\rho(\Lambda)$  is the vector, denoted  $m$  in [33], with entries  $|\Lambda^{e_i}|$ . So Yang's formula [33, Equation (2)] for the modular automorphism group  $\sigma$  of the extension of  $\omega$  to  $\pi_\omega(C^*(\Lambda))''$  shows that  $\sigma$  agrees with the preferred dynamics  $\alpha$  on  $C^*(\Lambda)$ . Consequently, restricting Theorem 11.1 to  $k$ -graphs with one vertex improves [33, Theorem 5.3] by proving its first assertion without the hypothesis that  $\{n \in \mathbb{Z}^k : \rho(\Lambda)^n = 1\}$  has rank at most 1. This confirms, for strongly-connected  $k$ -graphs, the first part of the conjecture stated for single-vertex 2-graphs in [32, Remark 5.5].

**The phase change for the preferred dynamics on the Toeplitz algebra.** For KMS states for the gauge actions on the Toeplitz algebras of finite graphs [12, 13], the phase-changes that occur with decreasing inverse temperature are from larger to smaller KMS simplices. Here we show that for many  $k$ -graphs, there is a phase change of a very different nature at the critical temperature for the preferred dynamics. In general, all sorts of phase changes can happen as inverse temperatures approach a critical one (see, for example, [2]), but this is the first time we have seen this phenomenon for graph algebras. Recall that a  $k$ -graph is *periodic* if it is not aperiodic.

**Corollary 11.3.** *Suppose that  $\Lambda$  is a strongly connected finite  $k$ -graph and that  $\rho(\Lambda)_i > 1$  for all  $i$ . Denote by  $\alpha$  the preferred dynamics on  $\mathcal{TC}^*(\Lambda)$ . For  $\beta \in \mathbb{R}$ , let  $E_\beta$  be the set of extreme points of the KMS <sub>$\beta$</sub>  simplex of  $(\mathcal{TC}^*(\Lambda), \alpha)$ . Then*

$$|E_\beta| = \begin{cases} |\Lambda^0| & \text{if } \beta > 1 \\ \infty & \text{if } \beta = 1 \text{ and } \Lambda \text{ is periodic} \\ 1 & \text{if } \beta = 1 \text{ and } \Lambda \text{ is aperiodic.} \\ 0 & \text{if } \beta < 1. \end{cases}$$

*Proof.* Suppose that  $\beta > 1$ . Then  $\beta \ln \rho(A_i) > \ln \rho(A_i)$  for all  $i$ . Since  $\Lambda$  is strongly connected it has no sources by Lemma 2.1. Thus [14, Theorem 6.1(c)] applies and shows that  $|E_\beta| = |\Lambda^0|$ .

Now suppose that  $\beta = 1$ . Then Corollary 4.6(c) implies that the quotient map from  $\mathcal{TC}^*(\Lambda)$  to  $C^*(\Lambda)$  induces a bijection between  $E_1$  and the extreme KMS<sub>1</sub> states of  $(C^*(\Lambda), \alpha)$ . Hence Theorem 7.1 gives a bijection from  $E_1$  to the pure states of  $C^*(\text{Per } \Lambda)$ . If  $\Lambda$  is periodic, then  $\text{Per } \Lambda$  is a nontrivial subgroup of  $\mathbb{Z}^k$  by Lemma 5.1(a), and so has infinitely many pure states. If  $\Lambda$  is aperiodic, then  $\text{Per } \Lambda = \{0\}$ , and so  $C^*(\text{Per } \Lambda)$  has a unique state.

If  $\beta < 1$ , then Corollary 4.6(a) applied with  $r = \ln \rho(\Lambda)$  implies that  $(\mathcal{TC}^*(\Lambda), \alpha)$  admits no KMS states.  $\square$

*Example 11.4.* It is easy to construct examples exhibiting the phase change to an infinite-dimensional KMS<sub>1</sub> simplex described in Corollary 11.3. To see this, consider a finite



directed graph  $E$  whose vertex matrix  $A_E$  is irreducible and satisfies  $\rho(A_E) > 1$ . The path category  $E^*$  is a 1-graph. Define  $f : \mathbb{N}^2 \rightarrow \mathbb{N}$  by  $f(m, n) = m + n$ , and let  $\Lambda$  be the pullback 2-graph  $f^*E^*$  of [16, Definition 1.9]. Then  $\Lambda^0 = E^0 \times \{0\}$ , and each  $(v, 0)\Lambda^{e_i}(w, 0) = vE^1w \times \{e_i\}$ . So  $A_1 = A_2 = A_E$  is irreducible, and so  $\Lambda$  is strongly connected. Corollary 3.5(iii) of [16] shows that  $C^*(\Lambda) \cong C^*(E) \otimes C(\mathbb{T})$ , which is not simple. So the equivalence (b)  $\iff$  (a) of Theorem 11.1 shows that  $\Lambda$  is periodic.

**Symmetries of the KMS simplex.** We show next that the gauge action on  $C^*(\Lambda)$  induces a free and transitive action of  $(\text{Per } \Lambda)^\wedge$  on the  $\text{KMS}_1$  simplex of  $C^*(\Lambda)$ . Recall that  $(\text{Per } \Lambda)^\perp$  denotes the collection of characters of  $\mathbb{Z}^k$  which are identically 1 on  $\text{Per } \Lambda$ . Identifying  $\widehat{\mathbb{Z}^k}$  with  $\mathbb{T}^k$ , we have

$$(\text{Per } \Lambda)^\perp = \{z \in \mathbb{T}^k : z^n = 1 \text{ for all } n \in \text{Per } \Lambda\}.$$

There is a homomorphism  $q : \mathbb{T}^k \rightarrow (\text{Per } \Lambda)^\wedge$  such that  $q(z)(g) = z^g$ , and  $\ker q = (\text{Per } \Lambda)^\perp$ .

**Proposition 11.5.** *Let  $\Lambda$  be a strongly connected finite  $k$ -graph.*

- (a) *For  $z, w \in \mathbb{T}^k$ , the states  $\phi_z$  and  $\phi_w$  of Corollary 10.3 are equal if and only if  $z\bar{w} \in (\text{Per } \Lambda)^\perp$ .*
- (b) *There is a homeomorphism  $h$  of  $(\text{Per } \Lambda)^\wedge$  onto the set  $E$  of extreme points of the  $\text{KMS}_1$  simplex of  $C^*(\Lambda)$  such that  $h(q(z)) = \phi_z$  for all  $z \in \mathbb{T}^k$ .*
- (c) *The gauge action  $\gamma$  induces a free and transitive action  $\tilde{\gamma}^*$  of  $(\text{Per } \Lambda)^\wedge$  on  $E$  such that  $\tilde{\gamma}_\chi^*(h(\rho)) = h(\chi\rho)$  for  $\chi, \rho \in (\text{Per } \Lambda)^\wedge$ .*

*Proof.* (a) Suppose that  $z\bar{w} \in (\text{Per } \Lambda)^\perp$ . Then  $z^{d(\mu)-d(\nu)} = w^{d(\mu)-d(\nu)}$  whenever  $d(\mu) - d(\nu) \in \text{Per } \Lambda$ . Hence (10.5) implies that  $\phi_z = \phi_w$ .

Now suppose that  $z\bar{w} \notin (\text{Per } \Lambda)^\perp$ . Take  $m - n \in \text{Per } \Lambda$  with  $(z\bar{w})^{m-n} \neq 1$ , and let  $\mu \in \Lambda^m$ . Let  $x^\Lambda$  be the unimodular Perron-Frobenius eigenvector of  $\Lambda$ . Corollary 4.2(b) implies that  $x_{s(\mu)}^\Lambda \neq 0$ , and so  $z^{m-n}\rho(\Lambda)^{-m}x_{s(\mu)}^\Lambda \neq w^{m-n}\rho(\Lambda)^{-m}x_{s(\mu)}^\Lambda$ . Hence (10.5) implies that  $\phi_z(s_\mu s_{\theta_{m,n}(\mu)}^*) \neq \phi_w(s_\mu s_{\theta_{m,n}(\mu)}^*)$ .

(b) Part (a) implies that the formula  $h(q(z)) = \phi_z$  determines a well-defined bijection from  $(\text{Per } \Lambda)^\wedge$  to  $E$ . Suppose that  $\chi_n \rightarrow \chi$  in  $(\text{Per } \Lambda)^\wedge$ , and choose  $z_n \in \mathbb{T}^k$  such that  $q(z_n) = \chi_n$ . By passing to a subsequence we may assume that the  $z_n$  converge to some  $z \in \mathbb{T}^k$ . We then have  $q(z) = \chi$ . The formula (10.5) shows that  $\phi_{z_n}(s_\mu s_\nu^*) \rightarrow \phi_z(s_\mu s_\nu^*)$  for all  $\mu, \nu$ , and an  $\varepsilon/3$ -argument then shows that  $\phi_{z_n} \rightarrow \phi_z$ , and so  $h(\chi_n) \rightarrow h(\chi)$ . Thus  $h$  is a continuous bijection, and so a homeomorphism since  $(\text{Per } \Lambda)^\wedge$  is compact.

(c) Formula (10.5) implies that  $\phi_w \circ \gamma_z = \phi_{wz}$ . So if  $z'\bar{z} \in (\text{Per } \Lambda)^\perp$ , then part (a) implies that  $\phi_w \circ \gamma_z = \phi_w \circ \gamma_{z'}$  for all  $w$ . Hence the action  $\gamma^*$  of  $\mathbb{T}^k$  on  $E$  induced by  $\gamma$  descends to an action  $\tilde{\gamma}^*$  of  $(\text{Per } \Lambda)^\wedge$  satisfying  $\tilde{\gamma}_{q(z)}^*(h(q(w))) = h(q(z)q(w))$  as required. This action is free and transitive because left translation in  $(\text{Per } \Lambda)^\wedge$  is free and transitive.  $\square$

## 12. THE GROUPOID MODEL

In [20], Neshveyev studies KMS states for dynamics on groupoid  $C^*$ -algebras arising from continuous  $\mathbb{R}$ -valued cocycles on groupoids. The Cuntz-Krieger algebra of a  $k$ -graph  $\Lambda$  admits a groupoid model with such a dynamics, and in this section we check that our Theorem 7.1 agrees with Neshveyev's [20, Theorem 1.3] for these examples.

**Neshveyev's theorem.** Let  $G$  be a locally compact second-countable étale groupoid and  $c : G \rightarrow \mathbb{R}$  a continuous cocycle. There is a dynamics  $\alpha^c$  on  $C^*(G)$  such that  $\alpha_t^c(f)(g) = e^{itc(g)}f(g)$  for  $f \in C_c(G)$  and  $g \in G$ .

Let  $U$  be an open bisection of  $G$  and write  $T^U : r(U) \rightarrow s(U)$  for the homeomorphism  $r(g) \mapsto s(g)$  for  $g \in U$ . Recall from [20, page 4] that a measure  $\mu$  on  $G^{(0)}$  is said to be quasi-invariant with Radon-Nikodym cocycle  $e^{-\beta c}$  if  $\frac{dT^U_* \mu}{d\mu}(s(g)) = e^{-\beta c(g)}$  for every open bisection  $U$  and every  $g \in U$ .

For  $x$  in the unit space  $G^{(0)}$ , write  $G_x^x$  for the stability subgroup  $\{g \in G : r(g) = x = s(g)\}$  and  $G_x$  for the subset  $\{g \in G : s(g) = x\}$  of  $G$ . Theorem 1.3 of [20] describes the  $\text{KMS}_\beta$  states of  $(C^*(G), \alpha^c)$  in terms of pairs  $(\mu, \psi)$  consisting of a quasi-invariant probability measure  $\mu$  on  $G^{(0)}$  with Radon-Nikodym cocycle  $e^{-\beta c}$  and a  $\mu$ -measurable field  $\psi = (\psi_x)_{x \in G^{(0)}}$  of states  $\psi_x : C^*(G_x^x) \rightarrow \mathbb{C}$  such that for  $\mu$ -almost all  $x \in G^0$  we have

$$(12.1) \quad \psi_x(u_g) = \psi_{r(h)}(u_{hgh^{-1}}) \text{ for all } g \in G_x^x \text{ and } h \in G_x.$$

(There is a second condition which we can ignore because for non-zero inverse temperatures  $\beta$  the properties of  $\mu$  ensure that it is always satisfied.) Neshveyev's theorem does not distinguish between measurable fields that agree  $\mu$ -almost everywhere.

**The path groupoid.** Let  $\Lambda$  be a row-finite  $k$ -graph with no sources. The set

$$G := \{(x, m - n, y) : x, y \in \Lambda^\infty, m, n \in \mathbb{N}^k \text{ and } \sigma^m(x) = \sigma^n(y)\}$$

is a groupoid with range and source maps  $r(x, g, y) = (x, 0, x)$ ,  $s(x, g, y) = (y, 0, y)$ , composition  $(x, g, y)(y, h, z) = (x, g + h, z)$  and inverses  $(x, g, y)^{-1} = (y, -g, x)$ . We identify  $G^{(0)}$  with  $\Lambda^\infty$  via  $(x, 0, x) \mapsto x$ .

For  $\lambda, \eta \in \Lambda$  with  $s(\lambda) = s(\eta)$ , define

$$Z(\lambda, \eta) = \{(x, d(\lambda) - d(\eta), y) \in G : x \in Z(\lambda), y \in Z(\eta) \text{ and } \sigma^{d(\lambda)}(x) = \sigma^{d(\eta)}(y)\}.$$

By Proposition 2.8 of [16], the sets  $Z(\lambda, \eta)$  form a basis for a locally compact Hausdorff topology on  $G$ . With this topology  $G$  is a second-countable étale groupoid, called the *path groupoid*. Each  $Z(\eta, \lambda)$  is a compact open bisection. By Corollary 3.5 of [16] there is an isomorphism of  $C^*(\Lambda)$  onto the  $C^*$ -algebra  $C^*(G)$  of  $G$  such that

$$(12.2) \quad t_\lambda \mapsto 1_{Z(\lambda, s(\lambda))}.$$

**Theorem 7.1 and Neshveyev's theorem.** Let  $\Lambda$  be a strongly connected finite  $k$ -graph and let  $G$  be its path groupoid.

There is a locally constant cocycle  $c : G \rightarrow \mathbb{R}$  given by  $c(x, n, y) = n \cdot \ln \rho(\Lambda)$ . This cocycle induces a dynamics  $\alpha^c : \mathbb{R} \rightarrow \text{Aut } C^*(G)$  such that  $\alpha_t^c(f)(x, n, y) = e^{itc(x, n, y)}f(x, n, y) = \rho(\Lambda)^{itn}f(x, n, y)$  for  $f \in C_c(G)$ . It is straightforward to check that the isomorphism of  $C^*(\Lambda)$  onto  $C^*(G)$  characterised by (12.2) intertwines  $\alpha^c$  and the preferred dynamics  $\alpha$  on  $C^*(\Lambda)$ .

It follows from (8.2) that the measure  $M$  on  $\Lambda^\infty$  of Proposition 8.1 is quasi-invariant with Radon-Nikodym cocycle  $e^{-c}$ ; the next lemma implies that  $M$  is the only such measure and investigates its support further. For the latter, we note that if  $g \in \text{Per } \Lambda$ , then there exist  $m, n \in \mathbb{N}^k$  such that  $g = m - n$  and  $\sigma^m(x) = \sigma^n(x)$  for all  $x \in \Lambda^\infty$ . Thus for each  $x \in \Lambda^\infty$ ,

$$\{x\} \times \text{Per } \Lambda \times \{x\} \subseteq G_x^x = \{(x, g, x) \in G : x \in \Lambda^\infty\}.$$

**Lemma 12.1.** *Suppose that  $\mu$  is a non-zero quasi-invariant probability measure on  $G^{(0)} = \Lambda^\infty$  with Radon-Nikodym cocycle  $e^{-c}$ . Then  $\mu$  is the measure  $M$  of Proposition 8.1 and*

$$(12.3) \quad M(\{x \in \Lambda^\infty : \{x\} \times \text{Per } \Lambda \times \{x\} = G_x^x\}) = 1.$$

*Proof.* Let  $v \in \Lambda^0$  and  $\lambda \in v\Lambda$ . Then  $Z(\lambda, s(\lambda))$  is a bisection with  $r(Z(\lambda, s(\lambda))) = Z(\lambda)$  and  $s(Z(\lambda, s(\lambda))) = Z(s(\lambda))$ . By the quasi-invariance of  $\mu$  we have

$$(12.4) \quad \mu(Z(\lambda)) = e^{-d(\lambda) \cdot \ln \rho(\Lambda)} \mu(Z(s(\lambda))) = \rho(\Lambda)^{-d(\lambda)} \mu(Z(s(\lambda))).$$

In particular, if  $\lambda \in v\Lambda^{e_i}$  then  $\mu(Z(\lambda)) = \rho(A_i)^{-1} \mu(Z(s(\lambda)))$ . Thus

$$(12.5) \quad \begin{aligned} \mu(Z(v)) &\geq \mu\left(\bigsqcup_{w \in \Lambda^0} \bigsqcup_{\lambda \in v\Lambda^{e_i} w} Z(\lambda)\right) = \sum_{w \in \Lambda^0} \sum_{\lambda \in v\Lambda^{e_i} w} \mu(Z(\lambda)) \\ &= \rho(A_i)^{-1} \sum_{w \in \Lambda^0} A_i(v, w) \mu(Z(w)). \end{aligned}$$

Set  $m := (\mu(Z(v))) \in [0, \infty)^{\Lambda^0}$ . Then (12.5) says that  $m$  satisfies  $\rho(A_i)m \geq A_i m$ . Also,  $\sum_{v \in \Lambda^0} m_v = \mu(\bigsqcup_{v \in \Lambda^0} Z(v)) = \mu(\Lambda^\infty) = 1$ . Thus Corollary 4.2(d) implies that  $m$  is the Perron-Frobenius eigenvector  $x^\Lambda$  of  $\Lambda$ . Now (12.4) shows that  $\mu(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x_{s(\lambda)}^\Lambda$ , and this is  $M(Z(\lambda))$  by (8.2). Since the  $Z(\lambda)$  form a basis for the topology on  $\Lambda^\infty$  we have  $\mu = M$ .

Finally,

$$\begin{aligned} &\{x \in \Lambda^\infty : \{x\} \times \text{Per } \Lambda \times \{x\} = G_x^x\} \\ &= \{x \in \Lambda^\infty : m, n \in \mathbb{N}^k \text{ and } \sigma^m(x) = \sigma^n(x) \implies m - n \in \text{Per } \Lambda\} \\ &= \bigcap_{m, n \in \mathbb{N}^k} \{x \in \Lambda^\infty : m - n \notin \text{Per } \Lambda \implies \sigma^m(x) \neq \sigma^n(x)\} \\ &= \bigcap_{m, n \in \mathbb{N}^k, m-n \notin \text{Per } \Lambda} \{x \in \Lambda^\infty : \sigma^m(x) \neq \sigma^n(x)\} \\ &= \Lambda^\infty \setminus \bigcup_{m, n \in \mathbb{N}^k, m-n \notin \text{Per } \Lambda} \{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}. \end{aligned}$$

By Proposition 8.2, if  $m - n \notin \text{Per } \Lambda$ , then  $M(\{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}) = 0$ . Since  $\{(m, n) : m - n \notin \text{Per } \Lambda\}$  is countable, this gives

$$M\left(\bigcup_{m, n \in \mathbb{N}^k, m-n \notin \text{Per } \Lambda} \{x \in \Lambda^\infty : \sigma^m(x) = \sigma^n(x)\}\right) = 0$$

and (12.3) follows.  $\square$

Now let  $(\mu, \psi)$  be one of Neshveyev's pairs for  $(C^*(G), \alpha^c)$ . By Lemma 12.1,  $\mu = M$  and  $M(\{x \in \Lambda^\infty : \{x\} \times \text{Per } \Lambda \times \{x\} = G_x^x\}) = 1$ . Thus we may assume that  $\psi_x = 0$  unless  $\{x\} \times \text{Per } \Lambda \times \{x\} = G_x^x$ . For each  $x \in \Lambda^\infty$ , let  $\iota_x : C^*(\text{Per } \Lambda) \rightarrow C^*(\{x\} \times \text{Per } \Lambda \times \{x\})$  be the isomorphism such that  $\iota_x(u_n) = u_{(x, n, x)}$ . For  $a \in C_c(\text{Per } \Lambda)$ , the  $M$ -measurability of  $\psi$  implies that  $x \mapsto \psi_x(\iota_x(a))$  is  $M$ -measurable. Thus there is a state  $\rho$  of  $C^*(\text{Per } \Lambda)$  such that

$$\rho(a) = \int_{\Lambda^\infty} \psi_x(\iota_x(a)) dM(x).$$

for  $a \in C^*(\text{Per } \Lambda)$ .

Conversely, let  $\rho$  be a state of  $C^*(\text{Per } \Lambda)$ . The measure  $M$  is quasi-invariant with Radon-Nikodym cocycle  $e^{-c}$ . Define

$$\rho_x = \begin{cases} \rho \circ \iota_x^{-1} & \text{if } \{x\} \times \text{Per } \Lambda \times \{x\} = G_x^x \\ 0 & \text{else.} \end{cases}$$

For  $f \in C_c(G)$ , the map  $x \mapsto \sum_{m \in \text{Per } \Lambda} f(x, m, x) \psi_x(u_{(x,m,x)}) = \sum_{m \in \text{Per } \Lambda} f(x, m, x) \rho(u_m)$  is continuous, hence measurable, and so  $(\rho_x)$  is a measurable field. Equation (12.1) follows because  $\rho_x(u_{(x,m,x)}) = \rho(u_m) = \rho_y(u_{(y,m,y)})$ . So  $(M, (\rho_x))$  is one of Neshveyev's pairs. Thus, reassuringly, our Theorem 7.1 and Neshveyev's [20, Theorem 1.3] say the same. To prove Theorem 7.1 using the groupoid approach we would have had to do much the same work (except for Proposition 10.2) and we would have lost the transparency of the direct proof.

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