arXiv:1807.08067v1 [math.OA] 21 Jul 2018

THE SUSPENSION OF A GRAPH, AND ASSOCIATED C^*-ALGEBRAS

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Abstract. Given a directed graph E, we construct for each real number l a quiver whose vertex space is the topological realisation of E, and whose edges are directed paths of length l in the vertex space. These quivers are not topological graphs in the sense of Katsura, nor topological quivers in the sense of Muhly and Tomforde. We prove that when $l = 1$ and E is finite, the infinite-path space of the associated quiver is homeomorphic to the suspension of the one-sided shift of E. We call this quiver the suspension of E. We associate both a Toeplitz algebra and a Cuntz–Krieger algebra to each of the quivers we have constructed, and show that when $l = 1$ the Cuntz–Krieger algebra admits a natural faithful representation on the ℓ^2-space of the suspension of the one-sided shift of E. For graphs E in which sufficiently many vertices both emit and receive at least two edges, and for rational values of l, we show that the Toeplitz algebra and the Cuntz–Krieger algebra of the associated quiver are homotopy equivalent to the Toeplitz algebra and Cuntz–Krieger algebra respectively of a graph that can be regarded as encoding the l^{th} higher shift associated to the one-sided shift space of E.

1. Introduction

Each finite directed graph E determines a corresponding 1-sided shift space (X_E, σ). Cuntz–Krieger algebras [5], and more generally graph C^*-algebras [8, 18, 19], encode the dynamics (X_E, σ) C^*-algebraically, and there has been intense interest in these C^*-algebras ever since their introduction—see, for example [3, 4, 6, 7, 10, 30] and the bibliography of [25].

In addition to the shift space (X_E, σ) itself, a directed graph E determines a family of dynamical systems indexed by the rational numbers. Given a rational number m/n with $n > 0$, we first form the directed graph $D_n(E)$ obtained by inserting $n - 1$ new vertices along every edge of E—so that each edge of E corresponds to a path of length n in $D_n(E)$—and then consider the m^{th} higher-power graph $D_n(E)(0, m)$ of $D_n(E)$, whose vertices are those of $D_n(E)$ and whose edges are paths of length m in $D_n(E)$. The shift space associated to this graph can be regarded as encoding the m/n^{th} higher shift of E in the sense that the system $(X_{D_n(E)(0,m)}, \sigma^n_{D_n(E)(0,m)})$ decomposes into n disjoint copies of (X_E, σ_E^m).

All of these fractional higher shifts associated to a finite directed graph E are naturally encoded in the suspension of the base shift X_E. The suspension of X_E is the quotient $M(\sigma) := (X_E \times [0, \infty))/{\sim}$ by the equivalence relation in which $(x, t + m) \sim (\sigma^m(x), t)$ for any positive integer m. For $l \in [0, \infty)$, the map $(x, t) \mapsto (x, t + l)$ induces a continuous map lt_l on $M(\sigma)$. For positive rationals m/n, the restriction of $\mathrm{lt}_{m/n}$ to $\{[x, t] : t \in \frac{1}{n}\mathbb{Z}\}$

Date: July 24, 2018.
1991 *Mathematics Subject Classification.* Primary 46L05.
Key words and phrases. Graph algebra, Cuntz–Krieger algebra, symbolic dynamics, suspension flow.
This research was supported by the Australian Research Council.

is a copy of $(X_{D_n(E)(0,m)}, \sigma_{D_n(E)(0,m)})$. By analogy, we can regard the dynamics lt_l for arbitrary l as corresponding to the l^{th} higher shift of E.

In this paper we construct from each locally finite directed graph E with no sources a family of C^*-algebras parameterised by $l \in \mathbb{R}$, and we prove that for $l = m/n$ rational, and for graphs E in which enough vertices both emit and receive at least two edges, the corresponding C^*-algebra is isomorphic to $C([0,1], C^*(D_n(E)(0,m)))$.

Our approach is to associate to each directed graph E a family of quivers $\mathcal{S}^l E$, one for each real parameter l. By a quiver here we mean a quadruple (Q^0, Q^1, r, s) where Q^0 and Q^1 are topological spaces, and $r, s : Q^1 \to Q^0$ are continuous maps. The vertex space $\mathcal{S}^l E^0$ of the quiver $\mathcal{S}^l E$ is the topological realisation of E: it is the quotient of $E^1 \times [0,1]$ by the equivalence relation under which $(e,1)$ is glued to $(f,0)$ if $s(e) = r(f)$. The edge space $\mathcal{S}^l E^1$ is, roughly speaking, the collection of directed paths of length l in the vertex space. We link this construction to suspension flows by showing that if E is finite, then the infinite-path space of $\mathcal{S}^1 E$ is homeomorphic to the suspension flow $M(\sigma)$ of the one-sided shift associated to E. Based on this, we call $\mathcal{S}^1 E$ the *suspension* of E, and denote it $\mathcal{S}E$.

The quivers $\mathcal{S}^l E$, for $l \neq 0$, are typically not topological quivers in the sense of Muhly and Tomforde [23] because neither r nor s is typically an open map: if $l > 0$, then s is open map if and only if every vertex of E receives exactly one edge and r is open if and only if each vertex emits exactly one edge; if $l < 0$ then the roles of emitters and receivers are reversed. In particular, our quivers are not topological graphs in the sense of Katsura, and $C_c(\mathcal{S}^l E^1)$ does not admit a natural $C_0(\mathcal{S}^l E^0)$-valued inner-product. Thus Katsura's modification of Pimsner theory does not help us to construct a C^*-algebra from $\mathcal{S}^l E$. The source map for the natural groupoid associated to the shift map on the infinite-path space of $\mathcal{S}^l E$ is also typically not open, so this groupoid does not admit a Haar system, and we cannot employ Renault's theory of groupoid C^*-algebras to construct a C^*-algebra from $\mathcal{S}^l E$.

Instead, we associate a Toeplitz algebra and a Cuntz–Krieger algebra to each $\mathcal{S}^l E$ by considering natural actions by bounded linear operators of $C_0(\mathcal{S}^l E^0)$ and of $C_c(\mathcal{S}^l E^1)$ on the (nonseparable) Hilbert space $\ell^2(\mathcal{S}^l E^*)$ with basis indexed by all finite paths in $\mathcal{S}^l E$. We define the Toeplitz algebra $\mathcal{T}C^*(\mathcal{S}^l E)$ of $\mathcal{S}^l E$ to be the C^*-algebra generated by all of these operators. For each vertex ω of $\mathcal{S}^l E$, the subspace $\ell^2(\mathcal{S}^l E^* \omega)$, with basis indexed by paths whose source is ω, is invariant for $\mathcal{T}C^*(\mathcal{S}^l E)$. Thus, for each ω, we obtain a homomorphism from $\mathcal{T}C^*(\mathcal{S}^l E)$ to the Calkin algebra $\mathcal{Q}(\ell^2(\mathcal{S}^l E^* \omega)) = \mathcal{B}(\ell^2(\mathcal{S}^l E^* \omega))/\mathcal{K}(\ell^2(\mathcal{S}^l E^* \omega))$. We define the Cuntz–Krieger algebra $C^*(\mathcal{S}^l E)$ to be the image of $\mathcal{T}C^*(\mathcal{S}^l E)$ in $\bigoplus_\omega \mathcal{Q}(\ell^2(\mathcal{S}^l E^* \omega))$ under the direct sum of these homomorphisms. If E has just one vertex v and one edge e (so that $r(e) = s(e) = v$), then $\mathcal{S}^l E$ is the topological graph associated to the rotation-by-l homeomorphism of the circle $\mathbb{R}/\mathbb{Z}$, and $C^*(\mathcal{S}^l E)$ is isomorphic to the rotation algebra A_l (see Example 6.8).

Our main results give a complete description of each of $\mathcal{T}C^*(\mathcal{S}^{m/n} E)$ and $C^*(\mathcal{S}^{m/n} E)$ for integers $m \in \mathbb{Z}$ and $n \geq 1$ provided that enough vertices of E emit and receive at least two edges (the exact technical hypothesis is complicated, but a simple sufficient condition is that every vertex both emits and receives at least two edges). We prove that $\mathcal{T}C^*(\mathcal{S}^{m/n} E)$ is homotopy equivalent to $\mathcal{T}C^*(D_n(E)(0,m))$, and that $C^*(\mathcal{S}^{m/n} E)$ is isomorphic to $C([0,1], C^*(D_n(E)(0,m)))$. This suggests that the algebras $C^*(\mathcal{S}^l E)$ for irrational l are potential candidates for the role of the C^*-algebras of the l^{th} higher-power shifts of X_E.

We obtain our description of $\mathcal{T}C^*(\mathcal{S}^{m/n}E)$ and $C^*(\mathcal{S}^{m/n}E)$ in stages. We first reduce the problem to that of describing $\mathcal{T}C^*(\mathcal{S}^{m}E)$ and $C^*(\mathcal{S}^{m}E)$ for nonnegative integers m. To do this, we first eliminate the case $m = 0$ by showing that $\mathcal{T}C^*(\mathcal{S}^0E) \cong C_0(\mathcal{S}E^0) \otimes \mathcal{T}$, where $\mathcal{T}$ is the classical Toeplitz algebra, and that this isomorphism descends to an isomorphism $C^*(\mathcal{S}^0E) \cong C_0(\mathcal{S}E^0) \otimes C(\mathbb{T})$. We then show that $\mathcal{T}C^*(\mathcal{S}^lE) \cong \mathcal{T}C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$ and $C^*(\mathcal{S}^lE) \cong C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$ for negative l. We then show that for $m > 0$, we have $\mathcal{T}C^*(\mathcal{S}^{m/n}E) \cong \mathcal{T}C^*(\mathcal{S}^mD_n(E))$ and $C^*(\mathcal{S}^{m/n}E) \cong C^*(\mathcal{S}^mD_n(E))$. Thus for any graph E' and any rational m/n, we have $\mathcal{T}C^*(\mathcal{S}^{m/n}E') \cong \mathcal{T}C^*(\mathcal{S}^{|m|}E)$ and $C^*(\mathcal{S}^{m/n}E') \cong C^*(\mathcal{S}^{|m|}E)$ for a suitable graph E.

The bulk of the technical work in the paper goes into analysing $\mathcal{T}C^*(\mathcal{S}^mE)$ and then $C^*(\mathcal{S}^mE)$ for a locally finite directed graph E with no sinks or sources. We do this by showing that they are both $C(\mathbb{S})$-algebras, where $\mathbb{S}$ is the circle $\mathbb{R}/\mathbb{Z}$, and analysing their fibres. We make use of the higher dual graphs $E(1, m+1)$ and $E(0, m)$ studied by Bates [1]: $E(1, m+1)$ is the graph with vertices E^1 and edges E^{m+1}, and with range and source maps given by $\mu_1 \cdots \mu_{m+1} \mapsto \mu_1$ and $\mu_1 \cdots \mu_{m+1} \mapsto \mu_{m+1}$ respectively; and $E(0, m)$ is the graph with vertices E^0 and edges E^m and the usual range and source maps. It is relatively straightforward to show that the fibre of $\mathcal{T}C^*(\mathcal{S}^mE)$ over each $t \in \mathbb{S} \setminus \{0\}$ is canonically isomorphic to $\mathcal{T}C^*(E(1, m+1))$ and indeed that the ideal of $\mathcal{T}C^*(\mathcal{S}^mE)$ corresponding to $0 \in \mathbb{S}$ is isomorphic to $C_0((0, 1), \mathcal{T}C^*(E(1, m+1))$. The fibre over $0 \in \mathbb{S}$ is more complicated. Writing $\mathcal{S}^mE^*_t$ for the space of paths in $\mathcal{S}^mE$ whose source lies in the image of $E^1 \times \{t\}$ in $\mathcal{S}E^0$, we describe natural unitary transformations $U_t : \ell^2(E(1, m+1)^*) \to \ell^2(\mathcal{S}^mE^*_t)$ for $t \neq 0$. For each $a \in \mathcal{T}C^*(\mathcal{S}^mE)$ and $t \neq 0$, this allows us to view the image of a_t in the fibre of $\mathcal{T}C^*(\mathcal{S}^mE)$ over t as an operator on $\ell^2(E(1, m+1)^*)$. We prove that $a_0^- := \lim_{t\nearrow 0} a_t$ and $a_0^+ := \lim_{t\searrow 0} a_t$ exist in $\mathcal{B}(\ell^2(E(1, m+1)))$ and belong to the image of $\mathcal{T}C^*(E(1, m+1))$, and that the map $a \mapsto a_0^+ \oplus a_0^-$ descends to an injective homomorphism η from the fibre of $\mathcal{T}C^*(\mathcal{S}^mE)$ over 0 to $\mathcal{T}C^*(E(1, m+1)) \oplus \mathcal{T}C^*(E(1, m+1))$. Following the arguments of [1], we show that there is a canonical injection $\jmath_{1,m+1} : \mathcal{T}C^*(E(0, m)) \to \mathcal{T}C^*(E(1, m+1))$ that descends to the isomorphism $C^*(E(1, m+1)) \cong C^*(E(0, m))$ of [1, Theorem 3.1]. We then show that if the set of vertices of E that emit at least two edges has hereditary closure E^0 in $E(0, m)$, then the image of $\mathcal{T}C^*(\mathcal{S}^mE)_0$ under η is precisely $\mathcal{T}C^*(E(1, m+1)) \oplus \jmath_{1,m+1}(\mathcal{T}C^*(E(0, m)))$. It is then straightforward to obtain our description of $\mathcal{T}C^*(\mathcal{S}^mE)$, and we prove that it is homotopy equivalent to $\mathcal{T}C^*(\mathcal{S}E(0, m))$, allowing us to compute its K-theory. We prove that the inclusion $\mathcal{T}C^*(\mathcal{S}^mE) \hookrightarrow C([0, 1], \mathcal{T}C^*(E(1, m+1)))$ takes $\mathcal{T}C^*(\mathcal{S}^mE) \cap \bigoplus_{\omega \in \mathcal{S}^mE^0} \mathcal{K}(\ell^2(\mathcal{S}^mE^*\omega))$ into $C([0, 1], \mathcal{K}(\ell^2(E(1, m+1)^*)))$. We then deduce that $C^*(\mathcal{S}^mE) \cong C([0, 1], C^*(E(1, m+1))$.

By showing that $C^*(D_n(E)(1, m+1))$ is Morita equivalent to $C^*(E(0, m))$ when m, n are coprime, we describe $K_*(C^*(\mathcal{S}^lE))$ for rational l provided that E is locally finite and enough vertices of E admit and receive at least two edges. A sufficient condition is that E is a finite, strongly connected graph that is not a simple cycle and has period 1 (in the sense of Perron–Frobenius theory).

The paper is organised as follows. We present the background that we will need on directed graphs and their C^*-algebras, on topological graphs, and on $C(X)$-algebras in Section 2. We then present the definition of $\mathcal{S}E$ in Section 3. In Section 4 we describe the path space and the infinite-path space of $\mathcal{S}E$, and prove that the latter coincides with $M(\sigma)$ if E is finite. We define the C^*-algebras of $\mathcal{S}E$ in Section 5. In Section 6, we describe our general construction of $\mathcal{S}^lE$ and its C^*-algebras, and prove that the case $l = 1$

coincides with our previous definitions of $\mathcal{S}E$ and its C^*-algebras, and also that $l = -1$ corresponds to the suspension of the opposite graph of E. This is not the most efficient order of presentation: we could have simply defined $\mathcal{S}^l E$ and its C^*-algebras immediately after Section 2, and then defined $\mathcal{S}E := \mathcal{S}^1 E$. But we feel that $\mathcal{S}E$, which is the key point of contact with suspension flows, is important enough to warrant separate discussion, and also that the later definitions of $\mathcal{S}^l E$ and its C^*-algebras are better motivated by first discussing $\mathcal{S}E$ and its C^*-algebras. Section 6 also contains our reduction of the analysis of the C^*-algebras of $\mathcal{S}^l E$ for rational l to the analysis of the C^*-algebras of $\mathcal{S}^m E$ for positive integers m. Our analyses of $\mathcal{T}C^*(\mathcal{S}^m E)$ and $C^*(\mathcal{S}^m E)$ occupy Sections 7 and 8 respectively.

2. Background

We recall some background about directed graphs and their C^*-algebras, on topological graphs, and on $C(X)$-algebras.

2.1. **Graphs.** We take our conventions for graph C^*-algebras from [25]. Throughout this paper, all of the graphs that we consider are directed graphs in the sense that the edges have an orientation. We omit the adjective throughout.

A *graph* is a quadruple $E = (E^0, E^1, r, s)$ consisting of finite or countably infinite sets E^0 and E^1, and functions $r, s : E^1 \to E^0$. We think of the elements of E^0 as vertices, and draw them as dots, and we regard the elements of E^1 as directed edges connecting vertices, and draw each as an arrow from the vertex $s(e)$ the vertex $r(e)$.

It is convenient to think of E^0 as the objects and E^1 as the indecomposable morphisms in the countable category E^* of finite paths in E as follows. For $n \geq 2$, we define $E^n := \{\mu_1\mu_2 \dots \mu_n : \mu_i \in E^1 \text{ and } s(\mu_i) = r(\mu_{i+1}) \text{ for all } i\}$, and refer to the elements of E^n as *paths of length* n in E. We think of vertices as paths of length 0 and edges as paths of length 1, and then write $E^* := \bigcup_{n=0}^\infty E^n$ for the path space of E. For $v \in E^0$, we write $r(v) = s(v) = v$, and for $n \geq 1$ and $\mu \in E^n$ we write $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$. We can concatenate $\mu, \nu \in E^* \setminus E^0$ to form the path $\mu\nu$ if $r(\nu) = s(\mu)$. For $v \in E^0$ and $\mu \in E^*$, the concatenation $v\mu$ is defined if $r(\mu) = v$, in which case, we have $v\mu = \mu$, and similarly the concatenation μv is defined if $v = s(\mu)$, in which case $\mu v = \mu$. If $\mu \in E^n$, we write $|\mu| = n$.

Given $U, V \subseteq E^*$, we define $UV := \{\mu\nu : \mu \in U, \nu \in V, \text{ and } s(\mu) = r(\nu)\}$. When U is a singleton $U = \{\mu\}$, we write μV rather than $\{\mu\}V$ for the set $\{\mu\nu : \nu \in V \text{ and } r(\nu) = s(\mu)\}$. In particular, for $v \in E^0$ we have

$$vE^1 = \{e \in E^1 : r(e) = v\} \qquad \text{and} \qquad E^1 v = \{e \in E^1 : s(e) = v\}.$$

We extend this notational convention in the obvious ways, so that for example if $v, w \in E^0$ then $vE^1 w = vE^1 \cap E^1 w$.

The *adjacency matrix* of the graph E is the integer matrix A_E given by $A_E(v, w) = |vE^1 w|$. We then have $A_E^n(v, w) = |vE^n w|$ for all v, w, n.

We say that E is *finite* if E^0 and E^1 are both finite sets. We say that E is *locally finite* if each $E^1 v \cup vE^1$ is a finite set; that is, if the row-sums and column sums of the matrix A_E are finite. A *sink* in E is a vertex v such that $E^1 v = \emptyset$, and a source is a vertex v such that $vE^1 = \emptyset$. In this paper, we are concerned exclusively with graphs that are locally-finite and have no sources.

A *cycle* in E is a path $\mu \in E^* \setminus E^0$ such that $r(\mu) = s(\mu)$. We say that μ has an *entrance* if there exists $i \le |\mu|$ such that $|r(\mu_i)E^1| \ge 2$.

2.2. **Infinite paths.** An *infinite path* in E is a string $x = x_1x_2x_3\cdots$ of edges of E such that $x_i \in E^1 r(x_{i+1})$ for all i. We write E^∞ for the set of all infinite paths in E and call it the *infinite-path space* of E. For $x \in E^\infty$, we define $r(x) = r(x_1) \in E^0$; and for $\mu \in E^*$ and $x \in E^\infty$ with $r(x) = s(\mu)$ we write μx for the infinite path $\mu_1 \cdots \mu_n x_1 x_2 \cdots$. We write $\mu E^\infty := \{\mu x : x \in E^\infty\}$.

We endow E^∞ with the topology that it inherits as a subspace of $\prod_{i=1}^\infty E^\infty$, a basis for which is the collection $\{\mu E^\infty : \mu \in E^*\}$. The set μE^∞ is called the *cylinder set* of μ and is often denoted $Z(\mu)$ elsewhere in the literature. When E is locally finite, the sets μE^∞ are compact open sets in E^∞, and the topology is a locally compact Hausdorff totally disconnected topology.

The *shift map* $\sigma : E^\infty \to E^\infty$ is defined by $\sigma(x)_i = x_{i+1}$, so $\sigma(x_1x_2x_3\cdots) = x_2x_3\cdots$, and $\sigma(ex) = x$ for all $x \in E^\infty$ and $e \in E^1$ with $s(e) = r(x)$. This σ is a local homeomorphism, as it restricts to a homeomorphism $eE^\infty \to s(e)E^\infty$ for each $e \in E^1$.

2.3. **Graph C^*-algebras.** Let E be a locally finite graph with no sources. A *Toeplitz–Cuntz–Krieger* family for E in a C^*-algebra A consists of a map $t : E^1 \to A$, written $e \mapsto t_e$ and a map $q : E^0 \to A$, written $v \mapsto q_v$ such that the elements q_v are mutually orthogonal projections in A, and such that

(TCK1) $t_e^* t_e = q_{s(e)}$ for all $e \in E^1$, and
(TCK2) $q_v \ge \sum_{e \in vE^1} t_e t_e^*$ for all $v \in E^0$.

A *Cuntz–Krieger* family for E is a Toeplitz–Cuntz–Krieger family (t, q) for E such that

(CK) $q_v = \sum_{e \in vE^1} t_e t_e^*$ for all $v \in E^0$.

Relations (TCK1) and (TCK2) imply that for each $\mu \in E^n$ the element $t_\mu = t_{\mu_1} t_{\mu_2} \ldots t_{\mu_n}$ is a partial isometry. As a notational convenience, we write $t_v := q_v$ for $v \in E^0$. With this notation, the C^*-algebra generated by the elements t_e and the elements q_v is equal to the closed linear span

$$C^*(t, q) = \overline{\operatorname{span}}\{t_\mu t_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\},$$

and we have $t_\mu t_\nu = \delta_{s(\mu), r(\nu)} t_{\mu\nu}$. An induction shows that $t_\mu^* t_\mu = t_{s(\mu)}$ for all $\mu \in E^*$, and we have

$$t_\mu t_\nu^* t_\eta t_\zeta^* = \begin{cases} t_\mu t_{\zeta\nu'}^* & \text{if } \nu = \eta\nu' \\ t_{\mu\eta'} t_\zeta^* & \text{if } \eta = \nu\eta' \\ 0 & \text{otherwise.} \end{cases}$$

There is a C^*-algebra $\mathcal{T}C^*(E)$ generated by a Toeplitz–Cuntz–Krieger family (T, Q) that is universal in the sense that given any other Toeplitz–Cuntz–Krieger family (t, q) in a C^*-algebra A, there is a homomorphism $\pi_{t,q} : \mathcal{T}C^*(E) \to A$ such that $\pi_{t,q}(T_e) = t_e$ and $\pi_{t,q}(Q_v) = q_v$. The universal property ensures that there is an action $\gamma : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}C^*(E))$ called the *gauge action* such that $\gamma_z(T_e) = zT_e$ and $\gamma_z(Q_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$.

There is a faithful representation $\pi : \mathcal{T}C^*(E) \to \mathcal{B}(\ell^2(E^*))$ called the path-space representation, and determined by $\pi(T_e)h_\mu = \delta_{s(e),r(\mu)} h_{e\mu}$ and $\pi(Q_v)h_\mu = \delta_{v,r(\mu)} h_\mu$. (The existence of π follows from the universal property, and injectivity follows from an application of [9, Theorem 4.1].)

There is also a C^*-algebra $C^*(E)$ generated by a Cuntz–Krieger E-family (s,p) that is universal in the sense that given any other Cuntz–Krieger family (s',p') there is a homomorphism of $C^*(E)$ taking each s_e to s'_e and each p_v to p'_v. This $C^*(E)$ is isomorphic to the quotient of $\mathcal{T}C^*(E)$ by the ideal I_E generated by the projections $\Delta_v := Q_v - \sum_{e\in vE^1} T_eT_e^*$ indexed by $v \in E^0$.

Let E be a locally finite graph with no sources and let (t,q) be a Cuntz–Krieger E-family. Since the projections Δ_v are fixed by the gauge action on $\mathcal{T}C^*(E)$, it descends to an action, also called the gauge action and denoted γ, on $C^*(E)$. The gauge-invariant uniqueness theorem [11, 25] states that if there is an action $\beta : \mathbb{T} \to \operatorname{Aut}(C^*(t,q))$ such that $\beta_z(t_e) = zt_e$ for all $e \in E^1$, then $\pi_{q,t} : C^*(E) \to C^*(t,q)$ is injective if and only if each q_v is nonzero. The Cuntz–Krieger uniqueness theorem [5, 25] says that if every cycle in E has an entrance, then $\pi_{q,t}$ is injective if and only if each q_v is nonzero.

With a little work, one can check that the path-space representation of $\mathcal{T}C^*(E)$ carries the ideal I_E to $\pi(\mathcal{T}C^*(E)) \cap \mathcal{K}(\ell^2(E^*))$. It follows that there is a homomorphism from $C^*(E)$ to the Calkin algebra $\mathcal{Q}(\ell^2(E^*)) := \mathcal{B}(\ell^2(E^*))/\mathcal{K}(\ell^2(E^*))$ given by $p_v \mapsto Q_v + \mathcal{K}(\ell^2(E^*))$ and $s_e \mapsto T_e + \mathcal{K}(\ell^2(E^*))$. We call this homomorphism the *Calkin representation* of $C^*(E)$. An argument using the gauge-invariant uniqueness theorem shows that the Calkin representation of $C^*(E)$ is injective. The subspaces $\ell^2(E^*v) \subseteq \ell^2(E^*)$ indexed by $v \in E^0$ are invariant for π, and we have $\pi(\mathcal{T}C^*(E)) \cap \mathcal{K}(\ell^2(E^*)) = \bigoplus_{v\in E^0} \mathcal{K}(\ell^2(E^*v))$. So the Calkin representation can be regarded as an injective homomorphism of $C^*(E)$ into $\bigoplus_{v\in E^0} \mathcal{Q}(\ell^2(E^*v))$ given by $s_e \mapsto \bigoplus_v (T_e|_{\ell^2(E^*v)} + \mathcal{K}(\ell^2(E^*v)))$.

A set $H \subseteq E^0$ is called *hereditary* if $s(HE^1) \subseteq H$, or equivalently $HE^* \subseteq E^*H$.

2.4. **Dual graphs.** Given a locally finite graph E with no sources, the *dual graph* $\widehat{E}$ is the graph $\widehat{E} = (E^1, E^2, \hat{r}, \hat{s})$ where $\hat{r}(ef) = e$ and $\hat{s}(ef) = f$ for all $ef \in E^2$. The properties of being row-finite or locally finite and of having no sinks or no sources pass from E to $\widehat{E}$ and vice versa.

There is a homeomorphism $E^\infty \cong \widehat{E}^\infty$ that carries the infinite path $x = x_1x_2x_3\cdots$ of E to the infinite path $\hat{x} = (x_1x_2)(x_2x_3)(x_3x_4)\cdots$ of $\widehat{E}$.

Corollary 2.5 of [2] shows that there is an isomorphism $C^*(\widehat{E}) \cong C^*(E)$ satisfying $s_{ef} \mapsto s_e s_f s_f^*$ and $p_e \mapsto s_e s_e^*$ for all $ef \in \widehat{E}^1$ and all $e \in \widehat{E}^0$.

More generally (see [1]), we can construct from any pair of integers $0 < p < q$ a new graph $E(p,q)$ from E. We define $E(p,q)$ to be the graph with

$$E(p,q)^0 := E^p \quad\text{and}\quad E(p,q)^1 := E^q,$$

with range and source maps given by

$$r_{p,q}(e_1\cdots e_q) := \begin{cases} e_1\cdots e_p & \text{if } p \geq 1\\ r(e_1) & \text{if } p = 0\end{cases} \quad\text{and}\quad s_{p,q}(e_1\cdots e_q) := \begin{cases} e_{q-p+1}\cdots e_q & \text{if } p \geq 1\\ s(e_q) & \text{if } p = 0.\end{cases}$$

So $E(0,1) \cong E$, $E(1,2) \cong \widehat{E}$, and $E(0,p)$ is the p^{th} higher-power graph (E^0, E^p, r, s). Bates shows in [1, Theorem 3.1] that $C^*(E(p+1, q+1)) \cong C^*(E(p,q))$ for all $0 < p < q$, and hence, by induction, that $C^*(E(p,q)) \cong C^*(E(0, q-p))$ for all $0 < p < q$, generalising the usual isomorphism $C^*(\widehat{E}) \cong C^*(E)$ of [2, Corollary 2.5]. We will need the following analogue of this result for Toeplitz algebras.

Recall that for any graph E, we denote by I_E the ideal of $\mathcal{T}C^*(E)$ generated by the projections $\Delta_v := Q_v - \sum_{e\in vE^1} T_eT_e^*$ indexed by $v \in E^0$, and that then $C^*(E) = \mathcal{T}C^*(E)/I_E$.

Lemma 2.1. *Let E be a row-finite graph with no sources and fix integers $0 < p < q$. For $v \in E^0$ let $q_v := \sum_{\mu \in vE^p} Q_\mu \in \mathcal{T}C^*(E(p,q))$, and for $\mu \in E^{q-p} = E(0,q-p)^1$, let $t_\mu := \sum_{\nu \in s(\mu)E^p} T_{\mu\nu} \in \mathcal{T}C^*(E(p,q))$. Then there is an injective homomorphism $\jmath_{p,q} : \mathcal{T}C^*(E(0,q-p)) \hookrightarrow \mathcal{T}C^*(E(p,q))$ such that $\jmath_{p,q}(Q_v) = q_v$ for $v \in E^0$, and $\jmath_{p,q}(T_e) = t_e$ for $e \in E^1$. We have*

$$\jmath_{p,q}(\Delta_v) = \sum_{\mu \in vE^p} \Delta_\mu \quad \text{for all } v \in E^0, \tag{2.1}$$

and $\jmath_{p,q}(I_{E(0,q-p)}) \subseteq I_{E(p,q)}$. There is an isomorphism $\tilde{\jmath}_{p,q} : C^*(E(0,q-p)) \to C^*(E(p,q))$ such that $\tilde{\jmath}_{p,q}(a + I_{E(0,q-p)}) = \jmath_{p,q}(a) + I_{E(p,q)}$ for all $a \in \mathcal{T}C^*(E(0,q-p))$.

Proof. The elements q_v are mutually orthogonal projections in $\mathcal{T}C^*(E(p,q))$ because $\{Q_\mu : \mu \in E(p,q)^0\}$ is a collection of mutually orthogonal projections. For $\mu, \nu \in E^{q-p}$, we have

$$t_\mu^* t_\nu = \sum_{\alpha \in s(\mu)E^p, \beta \in s(\nu)E^p} T_{\mu\alpha}^* T_{\nu\beta} = \sum_{\alpha \in s(\mu)E^p, \beta \in s(\nu)E^q} \delta_{\mu\alpha,\nu\beta} Q_\alpha = \delta_{\mu,\nu} \sum_{\alpha \in s(\mu)E^p} Q_\alpha = \delta_{\mu,\nu} q_{s(\mu)}.$$

Hence (t, q) satisfies (TCK1), and the partial isometries $\{t_\mu : \mu \in E^{q-p}\}$ have mutually orthogonal range projections. For $v \in E^0$ and $\mu \in vE^{q-p}$, we have

$$q_v t_\mu = \sum_{\nu \in vE^{q-p}} Q_\nu \sum_{\alpha \in s(\mu)E^p} T_{\mu\alpha} = \sum_{\nu \in vE^{q-p}, \alpha \in s(\mu)E^p} \delta_{\mu,\nu} T_{\mu\alpha} = \sum_{\alpha \in s(\mu)E^p} T_{\mu\alpha} = t_\mu.$$

In particular, each $t_\mu t_\mu^* \le q_{r(\mu)}$, and since the $t_\mu t_\mu^*$ are mutually orthogonal, it follows that (q, t) satisfies (TCK2). So the universal property of $\mathcal{T}C^*(E(0,q-p))$ gives a homomorphism $\jmath_{p,q} : \mathcal{T}C^*(E(0,q-p)) \to \mathcal{T}C^*(E(p,q))$ such that $\jmath_{p,q}(Q_v) = q_v$ for $v \in E^0$, and $\jmath_{p,q}(T_e) = t_e$ for $e \in E^1$.

To see that this homomorphism is injective, observe that for $v \in E^0$, we have

$$\begin{aligned} q_v - \sum_{\nu \in vE^{q-p}} t_\nu t_\nu^* &= \sum_{\mu \in vE^p} Q_\mu - \sum_{\nu \in vE^{q-p}} \sum_{\alpha \in s(\nu)E^p} T_{\nu\alpha} T_{\nu\alpha}^* \\ &= \sum_{\mu \in vE^p} Q_\mu - \sum_{\lambda \in vE^q} T_\lambda T_\lambda^* = \sum_{\mu \in vE^p} \Big(Q_\mu - \sum_{\beta \in s(\mu)E^{q-p}} T_{\mu\beta} T_{\mu\beta}^* \Big). \end{aligned} \tag{2.2}$$

Let $\pi : \mathcal{T}C^*(E(p,q)) \to \mathcal{B}(\ell^2(E(p,q)^*))$ be the path-space representation. For each $\mu \in vE^p$, we have $\pi(Q_\mu - \sum_{\alpha \in s(\mu)E^{q-p}} T_{\mu\alpha} T_{\mu\alpha}^*) = \theta_{h_\mu, h_\mu} \in \mathcal{K}(\ell^2(E(p,q)^*))$, and hence $\pi(Q_\mu - \sum_{\alpha \in s(\mu)E^{q-p}}) \neq 0$. Hence $q_v - \sum_{\mu \in vE^{q-p}} t_\mu t_\mu^* \neq 0$. Theorem 4.1 of [9] therefore shows that $\jmath_{p,q}$ is injective.

The calculation (2.2) establishes (2.1), and since $\jmath_{p,q}(I_E)$ is an ideal of $\jmath(\mathcal{T}C^*(E(0,q-p)))$, it follows that it is contained in the ideal of $\mathcal{T}C^*(E(p,q))$ generated by the Δ_μ, which is $I_{E(p,q)}$ by definition. So $\jmath_{p,q}$ descends to a homomorphism $\tilde{\jmath}_{p,q} : C^*(E(0,q-p)) \to C^*(E(p,q))$ satisfying

$$\tilde{\jmath}_{p,q}(p_v) = \tilde{\jmath}_{p,q}(Q_v + I_{E(0,q-p)}) = \jmath_{p,q}(Q_v) + I_{E(p,q)} = \sum_{\mu \in vE^p} p_\mu, \text{ and}$$

$$\tilde{\jmath}_{p,q}(s_\nu) = \tilde{\jmath}_{p,q}(T_\nu + I_{E(0,q-p)}) = \jmath_{p,q}(T_\nu) + I_{E(p,q)} = \sum_{\alpha \in s(\nu)E^p} s_{\nu\alpha}.$$

This is an isomorphism because it agrees on generators with that obtained from [1, Corollary 3.3]. □

2.5. **Topological graphs and their C^*-algebras.** We present here a very brief introduction to those parts of Katsura's theory of topological graphs and their C^*-algebras that we will need later. For a more comprehensive overview, see [25, Chapter 8]; for full details, see [14, 15, 16, 17].

As defined by Katsura [14], a *topological graph* is a quadruple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous map, and $s : E^1 \to E^0$ is a local homeomorphism.

The associated *graph bimodule* is defined as follows. The space $C_c(E^1)$ is a $C_0(E^0)$-bimodule with respect to the actions $\big(a \cdot \xi\big)(e) = a(r(e))\xi(e)$ and $\big(\xi \cdot a\big)(e) = \xi(e)a(s(e))$ for $a \in C_0(E^0)$ and $\xi \in C_c(E^1)$. The formula $\langle \xi \mid \eta\rangle_{C_0(E^0)}(v) := \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$ defines a $C_0(E^0)$-valued inner-product on $C_c(E^1)$, and the completion $X(E)$ of $C_c(E^1)$ in the norm $\|\xi\| = \|\langle \xi, \xi\rangle_A\|^{1/2}$ is a Hilbert $C_0(E^0)$-bimodule called the *graph correspondence* of E. Katsura proves that $X(E)$ is equal to the space $C_d(E^1)$ of functions

$$C_d(E^1) := \Big\{\xi \in C_0(E^1) : \Big(v \mapsto \sum_{s(e)=v} |\xi(e)|^2\Big) \in C_0(E)^0\Big\}. \tag{2.3}$$

The left action of $C_0(E^0)$ on $X(E)$ determines a homomorphism $\phi : C_0(E^0) \to \mathcal{L}(X(E))$ by $\phi(a)\xi = a\cdot\xi$. If $r : E^1 \to E^0$ is a proper map in the sense that the preimages of compact sets are compact, then ϕ takes values in $\mathcal{K}(X(E))$.

A representation of $X(E)$ in a C^*-algebra A is a pair (π, ψ) such that $\pi : C_0(E^0) \to A$ is a homomorphism, $\psi : X(E) \to A$ is a linear map, and we have $\pi(a)\psi(\xi) = \psi(a \cdot \xi)$, $\psi(\xi)\pi(a) = \psi(\xi \cdot a)$ and $\psi(\xi)^*\psi(\eta) = \pi(\langle \xi, \eta\rangle_{C_0(E^0)})$ for all $\xi, \eta \in X(E)$. The pair (π, ψ) induces a homomorphism $\psi^{(1)} : \mathcal{K}(X(E)) \to A$ such that $\psi^{(1)}(\theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$ for all $\xi, \eta \in X(E)$ (see [24, Lemma 3.2]). The representation (ψ, π) is *Cuntz–Pimsner covariant* if we have $\psi^{(1)}(\phi(a)) = \pi(a)$ whenever $\phi(a) \in \mathcal{K}(X(E))$; in particular, if $r : E^1 \to E^0$ is proper, then (ψ, π) is Cuntz–Pimsner covariant if $\psi^{(1)} \circ \phi = \pi$.

The topological graph C^*-algebra, denoted here by $C^*(E)$ is the C^*-algebra generated by a universal Cuntz–Pimsner covariant representation of E. Its Toeplitz algebra $\mathcal{T}C^*(E)$ is the C^*-algebra generated by a universal representation of E.

If E is a graph and Y is a locally compact Hausdorff space then the topological graph $E \times Y$ is defined by $(E \times Y)^0 := E^0 \times Y$, $(E \times Y)^1 := E^1 \times Y$ and $r(e, y) := (r(e), y)$ and $s(e, y) := (s(e), y)$. If E is locally finite, then $r : (E \times Y)^1 \to (E \times Y)^0$ is a proper map. The universal properties of $C^*(E \times Y)$ and of the tensor product $C^*(E) \otimes C_0(Y)$ imply (see [15, Proposition 7.7]) that there is an isomorphism $C^*(E) \otimes C_0(Y) \cong C^*(E \times Y)$ that carries $s_e \otimes h$ to $\psi\big((f, y) \mapsto \delta_{e,f}h(y)\big)$ and carries $p_v \otimes h$ to $\pi\big((w, y) \mapsto \delta_{v,w}h(y)\big)$ for all $e \in E^1$, $v \in E^0$ and $h \in C_0(Y)$.

2.6. $C(X)$**-algebras.** We need only the bare bones of the theory of $C(X)$-algebras here. For details, see [32, Appendix C].

Let X be a locally compact Hausdorff space. A C^*-algebra A is called a $C(X)$-algebra if there exists a nondegenerate homomorphism $\iota : C(X) \to \mathcal{ZM}(A)$ of $C(X)$ into the centre of the multiplier algebra of A. For each $x \in X$, the maximal ideal $J_x = \{f \in C(X) : f(x) = 0\}$ of $C(X)$ generates an ideal $I_x = \iota(J_x)A$ of A. We define $A_x := A/I_x$ to be the corresponding quotient. For $a \in A$, the map $x \mapsto \|a+I_x\|$ is upper semicontinuous. There is a unique topology on $\mathcal{A} := \bigsqcup_{x\in X} A_x$ under which the functions $x \mapsto a + I_x$ are all continuous. In this topology,

$$\lim_{n\to\infty} \|a + I_{x_n}\| = 0 \qquad \Longrightarrow \qquad \lim_{n\to\infty} a + I_{x_n} = 0. \tag{2.4}$$

So $\mathcal{A}$ is an upper-semicontinuous bundle of C^*-algebras over X, and each $a \in A$ determines a section $\gamma_a : x \mapsto a + I_x$ of $\mathcal{A}$ that vanishes at infinity in the sense that for each $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that $\|\gamma_a(x)\| < \varepsilon$ for all $x \notin K$. The map $a \mapsto \gamma_a$ is an isomorphism of A onto the algebra $\Gamma_0(X, \mathcal{A})$ of continuous sections of $\mathcal{A}$ that vanish at infinity. So every $C(X)$-algebra is the algebra of sections of an upper-semicontinuous bundle of C^*-algebras over X. Conversely, if $\mathcal{A}$ is an upper-semicontinuous bundle over X then there is a nondegenerate homomorphism $\iota : C(X) \to \mathcal{ZM}(\Gamma_0(X, \mathcal{A}))$ characterised by $(\iota(f)\gamma)(x) = f(x)\gamma(x)$.

3. The suspension of a graph

In this section we define the suspension $\mathcal{S}E$ of a graph E and describe its basic properties. Our motivation is the relationship between the infinite-path space of $\mathcal{S}E$ and the suspension flow of the shift-space associated to E, which we establish in Section 4. The constructions in this section will be subsumed by the more-general construction of the quivers $\mathcal{S}^l E$ parameterised by $l \in \mathbb{R}$ in Section 6.

Let E be a locally finite graph with no sources. Let $\sim$ denote the smallest equivalence relation on $((E^* \setminus E^0) \times [0, 1]) \sqcup E^*$ such that

$$(\mu f, 0) \sim \mu \text{ and } (e\mu, 1) \sim \mu \quad \text{for all } \mu \in E^*,\ e \in E^1 r(\mu) \text{ and } f \in s(\mu)E^1. \tag{3.1}$$

Observe that $\sim$ restricts to an equivalence relation on $(E^{n+1} \times [0, 1]) \cup E^n$ for each $n \geq 0$.

For $\mu \in E^* \setminus E^0$ and $t \in [0, 1]$, we write $[\mu, t]$ for the equivalence-class of (μ, t) under $\sim$, and for $\mu \in E^*$ we write $[\mu]$ for the equivalence class of μ under $\sim$. Putting $\mu = v \in E^0$ in (3.1), we see that $[e, 1] \sim [f, 0]$ whenever $s(e) = r(f)$. For $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, we write

$$\mathcal{S}E^n := \big((E^{n+1} \times [0, 1]) \sqcup E^n\big)/{\sim},$$

and we define

$$\mathcal{S}E^* := \big(((E^* \setminus E^0) \times [0, 1]) \sqcup E^*\big)/{\sim} = \bigsqcup_n \mathcal{S}E^n.$$

It is straightforward to check that there are well-defined maps $r, s : \mathcal{S}E^* \to \mathcal{S}E^0$ such that $r([e\mu, t]) := [e, t]$ and $s([\mu f, t]) := [f, t]$ for $\mu \in E^*$ and $e \in E^1 r(\mu)$ and $f \in s(\mu)E^1$. These maps satisfy $r([\mu]) = [r(\mu)]$ and $s([\mu]) = [s(\mu)]$ for $\mu \in E^*$.

Definition 3.1. Let E be a locally finite graph with no sources. We call the quadruple $\mathcal{S}E := (\mathcal{S}E^0, \mathcal{S}E^1, r, s)$ the *suspension* of E.

The following lemma will make it easier to work with the suspension of a graph.

Lemma 3.2. *Let E be a locally finite graph with no sources. For $\mu, \nu \in E^*$ and $s, t \in [0, 1]$, we have $(\mu, s) \sim (\nu, t)$ if and only if one of the following holds:*

(1) *$\mu = \nu$ and $s = t$;*
(2) *$s = t = 0$ and $\mu_1 \cdots \mu_{|\mu|-1} = \nu_1 \cdots \nu_{|\nu|-1}$, or $s = t = 1$ and $\mu_2 \cdots \mu_{|\mu|} = \nu_2 \cdots \nu_{|\nu|}$; or*
(3) *$s = 1, t = 0$ and $\mu_2 \cdots \mu_{|\mu|} = \nu_1 \cdots \nu_{|\nu|-1}$, or $s = 0, t = 1$ and $\mu_1 \cdots \mu_{|\mu|-1} = \nu_2 \cdots \nu_{|\nu|}$.*

Each $\alpha \in \mathcal{S}E^*$ has a representative of the form (μ, t) where $\mu \in E^*$ and $t \in [0,1)$. If $\alpha \notin \{[\mu] : \mu \in E^*\}$ then this representative is unique. The map $\mu \mapsto [\mu]$ is an injection of E^* into $\mathcal{S}E^*$.

Proof. Consider the relation R_0 on $(E^* \setminus E^0) \times [0,1]$ given by (1)–(3). Then $R_0 \subseteq \sim$. It is clear that R_0 is reflexive and symmetric, and a quick case-by-case check of the possible combinations of (2) and (3) shows that it is transitive. If $(\mu, s) \sim (\nu, t)$, then there is a sequence $(\mu, s) = (\mu_0, s_0) \sim \nu_1 \sim (\mu_1, s_1) \sim \nu_2 \sim \cdots \sim \nu_k \sim (\mu_k, s_k) = (\nu, t)$ where each of the equivalences in the chain is one of the forms appearing in (3.1). It then follows that each equivalence $(\mu_i, s_i) \sim (\mu_{i+1}, s_{i+1})$ is of one of the forms appearing in (2) or (3). Thus $\sim$ is contained in R_0, and the two are equal.

If $\alpha \in \mathcal{S}E^*$ then by definition we have either $\alpha = [\mu]$ for some $\mu \in E^*$ or $\alpha = [\nu, t]$ for some $(\nu, t) \in (E^* \setminus E^0) \times [0,1]$. If $\alpha = [\mu]$ then, since E has no sources, we can find $e \in E^1$ with $r(e) = s(\mu)$, and then $\alpha = [\mu e, 0]$. Likewise if $\alpha = [\nu, 1]$ then we can find $f \in E^1$ with $r(f) = s(\nu)$, and then $\alpha = [\nu f, 0]$. Otherwise $\alpha = [\nu, t]$ already has the desired form. If $\alpha \notin \{[\mu] : \mu \in E^*\}$, then $\alpha = [\nu, t]$ for some $t \in (0,1)$ and $\nu \in E^* \setminus E^0$, and then if we also have $\alpha = [\nu', s]$ then in particular $(\nu, t) \sim (\nu', s)$ with $t \neq \{0, 1\}$. In particular, this equivalence does not appear in (2) or (3), and we deduce that it is of the form (1), so $\nu = \nu'$ and $t = s$.

Suppose that $\mu, \nu \in E^*$ satisfy $\mu \sim \nu$. We can write $[\mu] = [\mu e, 0]$ and $[\nu] = [\nu f, 0]$ for any $e \in s(\mu)E^1$ and $f \in s(\nu)E^1$. We then have $(\mu e, 0) \sim (\nu f, 0)$ which means that the equivalence is of the form (1) or the first of the forms appearing in (2), each of which forces $\mu = \nu$. So $\mu \mapsto [\mu]$ is an injection. □

We call elements of $\mathcal{S}E^*$ *paths* in $\mathcal{S}E$, and elements of

$$\mathcal{S}E^n := \big((E^{n+1} \times [0,1]) \sqcup E^n\big)/\sim$$

paths of length n in $\mathcal{S}E$.

Notation 3.3. Using Lemma 3.2, we regard E^* as a subset of $\mathcal{S}E^*$. In particular, for $n \in \mathbb{N}$, we write

$$\mathcal{S}E^n \setminus E^n = \{[\mu, t] : \mu \in E^{n+1} \text{ and } t \in (0,1)\}.$$

There is a partially defined composition map on $\mathcal{S}E^*$ as follows: the pair $([\mu, t], [\nu, s])$, is composable if and only if $s([\mu, t]) = r([\mu, s])$, which in particular forces $t = s$. If $s([\mu, t]) = r([\nu, t])$, and if these representatives have been chosen with $t \neq 1$ as above, then the composition is defined as

$$[\mu, t][\nu, t] = [\mu_1 \cdots \mu_{|\mu|-1} \nu_1 \cdots \nu_{|\nu|}, t].$$

It then follows that for $\mu, \nu \in E^*$ with $s(\mu) = r(\nu)$, we have $[\mu][\nu] = [\mu\nu]$. We also have $r([\mu, t][\nu, t]) = r([\mu, t])$ and $s([\mu, t][\nu, t]) = s([\nu, t])$ whenever $[\mu, t][\nu, t]$ makes sense. Under this concatenation, $\mathcal{S}E^*$ is a small category with objects $\mathcal{S}E^0$.

Given $\alpha = [\mu, t] \in \mathcal{S}E^*$ with $t \in [0,1)$, we can write $\mu = \mu_1 \cdots \mu_{n+1}$ with each $\mu_i \in E^1$. Defining $\alpha_i := [\mu_i \mu_{i+1}, t] \in \mathcal{S}E^1$ for $i \leq n$, we obtain a factorisation $\alpha = \alpha_1 \cdots \alpha_n$. This is the unique factorisation of α as a composition of edges of $\mathcal{S}E$.

As with directed graphs, given subsets $U, V \subseteq \mathcal{S}E^*$, we write $UV := \{\alpha\beta : \alpha \in U, \beta \in V \text{ and } s(\alpha) = r(\beta)\}$. If U is a singleton $U = \{\alpha\}$, then we write αV and $V\alpha$ in place of $\{\alpha\}V$ and $V\{\alpha\}$. In particular, for $\omega \in \mathcal{S}E^0$ and a subset $U \subseteq \mathcal{S}E^*$ we have $\omega U = U \cap r^{-1}(\omega)$ and $U\omega = U \cap s^{-1}(\omega)$.

Throughout the paper, we write $\mathbb{S}$ for the circle $\mathbb{R}/\mathbb{Z}$. Each element of $\mathbb{S}$ has a unique representative in $[0, 1)$. We often abuse notation slightly and regard elements of $[0, 1]$ as elements of $\mathbb{S}$ (so 1 and 0 are equal as elements of $\mathbb{S}$).

Lemma 3.4. *Let E be a locally finite graph with no sources. There is a continuous map $\varpi : \mathcal{S}E^* \to \mathbb{S}$ such that $\varpi([\mu, t]) = t$ for $\mu \in E^* \setminus E^0$ and $t \in [0, 1)$.*

Proof. Lemma 3.2 shows that $[\mu, t] \mapsto t$ is well-defined from $\mathcal{S}E^*$ to $\mathbb{S}$. To see that it is continuous, let $q : \big((E^* \setminus E^0) \times [0, 1]\big) \to \mathcal{S}E^0$ be the quotient map. For $0 < a < b < 1$, we have $q^{-1}\big(\varpi^{-1}((a, b))\big) = (E^* \setminus E^0) \times (a, b)$, which is open. So $\varpi^{-1}((a, b))$ is open. Likewise, for $\varepsilon < 1/2$, we have $q^{-1}\big(\varpi^{-1}((-\varepsilon, \varepsilon))\big) = (E^* \setminus E^0) \times ([0, \varepsilon) \cup (1 - \varepsilon, 1])$, which again is open. So $\varpi^{-1}((-\varepsilon, \varepsilon))$ is open. □

Observe that the map ϖ of Lemma 3.4 satisfies $\varpi(\alpha) = \varpi(s(\alpha)) = \varpi(r(\alpha))$ for all $\alpha \in \mathcal{S}E^*$.

Notation 3.5. For $t \in \mathbb{S}$, we define

$$\mathcal{S}E^*_t = \varpi^{-1}(t) = \{[\mu, t] : \mu \in E^*\},$$

and for $n \in \mathbb{N}$, we define

$$\mathcal{S}E^n_t := \mathcal{S}E^n \cap \mathcal{S}E^*_t = \{[\mu, t] : \mu \in E^{n+1}\}.$$

We then have $\mathcal{S}E^n_t = (\mathcal{S}E^n)(\mathcal{S}E^0_t) = (\mathcal{S}E^0_t)(\mathcal{S}E^n)$ for all n, t. With this notation, $\mathcal{S}E^*_1 = \mathcal{S}E^*_0 = \{[\mu] : \mu \in E^*\}$.

We aim to construct a C^*-algebra from $\mathcal{S}E$. The following lemma shows that we cannot employ Katsura's theory of topological-graph C^*-algebras, or Muhly and Tomforde's theory of topological-quiver C^*-algebras: to get off the ground, both theories require at least that the source map s is an open map.

Lemma 3.6. *Let E be a finite graph with no sources. The maps $s, r : \mathcal{S}E^1 \to \mathcal{S}E^0$ are continuous maps, and restrict to local homeomorphisms from $\mathcal{S}E^1 \setminus E^1$ to $\mathcal{S}E^0 \setminus E^0$. For $e \in E^1$, the map s is open at $[e] \in \mathcal{S}E^1$ if and only if $|E^1 s(e)| = 1$, and the map r is open at $[e]$ if and only if $|r(E)E^1| = 1$. In particular, s is an open map if and only if $|E^1 v| = 1$ for all $v \in E^0$, and r is an open map if and only if $|vE^1| = 1$ for all $v \in E^0$.*

Proof. The range and source maps are continuous by construction. If $0 < t < 1$ and $ef \in E^2$, then for any ε such that $(t - \varepsilon, t + \varepsilon) \subseteq (0, 1)$, the restrictions of s, r to $\{[ef, s] : |t - s| < \varepsilon\}$ are homeomorphisms onto open sets.

Fix $e \in E^1$. To see that s is open at $[e]$ if and only if $|E^1 s(e)| = 1$, first suppose that $|E^1 s(e)| = 1$, so $E^1 s(e) = \{e\}$. Then the sets

$$U_\varepsilon := \{[fe, t] : f \in E^1 r(e), t > 1 - \varepsilon\} \cup \{[ef, t] : f \in s(e)E^1, t < \varepsilon\}$$

indexed by $\varepsilon \in (0, 1/2)$ form a neighbourhood base at $[e]$ and we have $s(U_\varepsilon) = \{[e, t] : t > 1 - \varepsilon\} \cup \{[f, t] : f \in s(e)E^1, t < \varepsilon\}$. Since $E^1 s(e) = \{e\}$, we deduce that, writing $v = s(e)$,

$$s(U_\varepsilon) = \{[f, t] : f \in E^1 v, t > 1 - \varepsilon\} \cup \{[f, t] : f \in vE^1, t < \varepsilon\},$$

which is a basic open neighbourhood of $[v]$. So s is open at $[e]$.

Now suppose that $|E^1 s(e)| \geq 2$, say $f \in E^1 s(e) \setminus \{e\}$, and write $v := s(e)$. Consider the set

$$U := \{[he, t] : h \in E^1 r(e), 1/2 < t \leq 1\} \cup \{[eh, t] : h \in vE^1, 0 \leq t < 1/2\}.$$

This set is open in the quotient topology. We have

$$s(U) = \{[e,t] : 1/2 < t \leq 1\} \cup \{[h,t] : h \in vE^1, 0 \leq t < 1/2\}.$$

In particular, we have $[v] = [e,1] \in s(U)$, but the sequence $([f,(n-1)/n])_{n=1}^\infty$ is contained in the complement of $s(U)$ and converges to $[v]$. So s is not open at $[e]$.

This completes the proof that s is open if and only if each $|E^1 v| = 1$. A symmetric argument (or the same argument applied to the opposite graph) shows that r is open at $[e]$ if and only if $r(e)E^1$ is a singleton.

The first statement implies in particular that r, s are open at α for every $\alpha \in \mathcal{S}E^1 \setminus E^1$, so the final statement follows immediately. □

Example 3.7. Consider the simplest example of a finite graph with no sources: $E^0 = \{v\}$ and $E^1 = \{e\}$, so $r(e) = s(e) = v$. Then the map ϖ of Lemma 3.4 restricts to a homeomorphism $\varpi : [e,t] \mapsto t$ from $\mathcal{S}E^0$ to $\mathbb{S}$. For each $w \in \mathcal{S}E^0$ there is a unique $f_w \in \mathcal{S}E^1$ with $r(f_w) = s(f_w) = w$, and then $\mathcal{S}E^1 = \{f_w : w \in \mathcal{S}E^0\} \cong \mathbb{S}$, and s and r are homeomorphisms.

Example 3.8. Now consider the finite graph such that $E^0 = \{v\}$ and $E^1 = \{e, f\}$, so $r(e) = s(e) = r(f) = s(f) = v$. So $\mathcal{S}E^0$ is equal to the union $\{e,f\} \times \mathbb{S}$ of two circles glued at a point by gluing $(e,0)$ to $(f,0)$. For $g \in E^1$ and $t \in \mathbb{S}$, consider the vertex $[g,t] \in \mathcal{S}E^0$. We have $\mathcal{S}E^1[g,t] = \{[eg,t],[fg,t]\}$, and the ranges of these edges are $[e,t]$ and $[f,t]$ respectively. So as a set, we have $\mathcal{S}E^1 \cong \{e,f\} \times \mathcal{S}E^0$. The sequences $([ee, 1-1/n])_{n=1}^\infty$ and $([fe, 1-1/n])_{n=1}^\infty$ converge in $\mathcal{S}E^1$ to $[ee,1] = [fe,1]$, and $s([ee,1-1/n]) = [e,1-1/n] = s([fe,1-1/n])$ for all n, so s is not a local homeomorphism at $[ee,1]$.

4. The infinite-path space of the suspension of a graph

In this section we describe the infinite-path space of the suspension of E, and we show that if E is finite, then $\mathcal{S}E^\infty$ is homeomorphic to the one-sided suspension flow of the shift space of the graph. In Section 5 we will define the Toeplitz algebra $\mathcal{T}C^*(\mathcal{S}E)$ and the Cuntz–Krieger algebra $C^*(\mathcal{S}E)$ of the suspension of a graph E using analogues of the path-space representation and Calkin representation of a graph C^*-algebra. We will then link this to symbolic dynamics by showing that $C^*(\mathcal{S}E)$ has a natural representation on $\ell^2(\mathcal{S}E^\infty)$.

An *infinite path* in $\mathcal{S}E$ is a sequence $\alpha_1\alpha_2\alpha_3\cdots$ of edges $\alpha_i \in \mathcal{S}E^1$ such that $r(\alpha_{i+1}) = s(\alpha_i)$ for all i. We write $\mathcal{S}E^\infty$ for the set of all infinite paths in $\mathcal{S}E$.

Lemma 4.1. *Let E be a locally finite graph with no sources. There is a bijection $\theta^\infty : E^\infty \times [0,1) \to \mathcal{S}E^\infty$ such that $\theta^\infty(e_1e_2e_3\cdots, t) = [e_1e_2,t][e_2e_3,t][e_3e_4,t]\cdots$ for all $e_1e_2\cdots \in E^\infty$ and $t \in [0,1)$.*

Proof. To see that θ^∞ is surjective, fix $\xi = \alpha_1\alpha_2\cdots$ in $\mathcal{S}E^\infty$. Write each $\alpha_i = (e_if_i, t_i)$ with $t_i \in [0,1)$ as in Lemma 3.2. Then $t_i = t_1$ for all i. Let $t := t_1$. If $t \neq 0$, then $s(\alpha_i) = r(\alpha_{i+1})$ forces $[f_i, t] = [e_{i+1}, t]$, and hence Lemma 3.2 gives $f_i = e_{i+1}$, for all i. Thus $x = e_1e_2e_3\cdots \in E^\infty$ and we have $\xi = \theta^\infty(x,t)$. If $t = 0$, then $s(\alpha_i) = r(\alpha_{i+1})$ forces $s(e_i) = r(e_{i+1})$ for all i, and then since $t = 0$ we have $\alpha_i = [e_ie_{i+1}, 0]$ for each i. So again, $x = e_1e_2\cdots$ belongs to E^∞ and $\xi = \theta^\infty(x,t)$. For injectivity, suppose that $\theta^\infty(x,s) = \theta^\infty(y,t) = \alpha_1\alpha_2\cdots$. Put write $x = e_1e_2\cdots$ and $y = f_1f_2\cdots$. Then $[e_1e_2, s] = \alpha_1 = [f_1f_2, t]$, and since $s, t \neq 1$, Lemma 3.2 forces $s = t$. The definition of $\approx$

shows that if $s, t \neq 1$ and $[ef, s] = [gh, t]$ then $e = g$. Since $[e_i e_{i+1}, s] = \alpha_i = [f_i f_{i+1}, s]$ for all i, we deduce that $e_i = f_i$ for all i, and so θ^∞ is injective. □

We next describe a natural topology on $\mathcal{S}E^\infty$.

Lemma 4.2. *Let E be a locally finite graph with no sources. For $\mu \in E^*$ and $0 < a < b < 1$, let*

$$Z(\mu, (a, b)) := \{\theta^\infty(x, t) : x \in \mu E^\infty \text{ and } t \in (a, b)\},$$

and for $\mu \in E^*$ and $0 < \varepsilon < \frac{1}{2}$, let

$$\begin{aligned} Z(\mu, \varepsilon) := \{\theta^\infty(e\mu x, t) : e \in E^1 r(\mu), x \in s(\mu)E^\infty, \text{ and } t \in (1-\varepsilon, 1)\} \\ \cup \{\theta^\infty(\mu x, t) : x \in s(\mu)E^\infty \text{ and } t \in [0, \varepsilon)\} \end{aligned}$$

Then there is a second-countable Hausdorff topology on $\mathcal{S}E^\infty$ with basis

$$\begin{aligned} \mathcal{B} = \{Z(\mu, (a, b)) : \mu \in E^* \text{ and } 0 < a < b < 1\} \\ \cup \{Z(\mu, \varepsilon) : \mu \in E^* \text{ and } 0 < \varepsilon < \frac{1}{2}\}. \end{aligned}$$

Proof. To see that $\mathcal{B}$ is a basis, first observe that for $t \neq 0$ any element of the form $\theta^\infty(x, t)$ belongs to $Z(r(x), (a, b))$ for any $0 < a < t < b < 1$; and any element of the form $\theta^\infty(x, 0)$ belongs to $Z(x_1, \frac{1}{2})$ for any $e \in E^1$ with $s(e) = r(x)$, so $\bigcup \mathcal{B} = \mathcal{S}E^\infty$.

Given $\mu, \nu \in E^*$, we define

$$\mu \vee \nu := \begin{cases} \mu & \text{if } \mu = \nu\mu' \\ \nu & \text{if } \nu = \mu\nu' \\ \infty & \text{otherwise,} \end{cases}$$

where ∞ is used here purely as a formal symbol. As a notational convenience, we define $Z(\infty, (a, b)) = \emptyset = Z(\infty, \varepsilon)$ for any a, b, ε. We then have

$$\begin{aligned} Z(\mu, (a, b)) \cap Z(\nu, (c, d)) &= Z(\mu \vee \nu, (\max\{a, c\}, \min\{b, d\})), \\ Z(\mu, \varepsilon) \cap Z(\nu, \delta) &= Z(\mu \vee \nu, \min\{\varepsilon, \delta\}), \quad \text{and} \\ Z(\mu, (a, b)) \cap Z(\nu, \varepsilon) &= \Big(\bigcup_{e \in E^1 r(\nu)} Z\big(\mu \vee e\nu, (a, b) \cap (1-\varepsilon, 1)\big)\Big) \\ &\qquad \cup Z\big(\mu \vee \nu, (a, b) \cap [0, \varepsilon)\big). \end{aligned}$$

So $\mathcal{B}$ is a base for a topology on $\mathcal{S}E^\infty$.

This topology is second countable because restricting the values of a, b, and ε to rational values in the definition of $\mathcal{B}$ yields a countable base for the same topology.

To see that this topology is Hausdorff, fix distinct elements $\theta^\infty(x, s)$ and $\theta^\infty(y, t)$ of $\mathcal{S}E^\infty$. First suppose that $s = t$. Then $x \neq y$ so we can find $\mu, \nu \in E^* \setminus E^0$ such that $x \in \mu E^\infty$, $y \in \nu E^\infty$ and $\mu \vee \nu = \infty$. If $s \neq 0$ then for any $0 < a < s < b < t$, the sets $Z(\mu, (a, b))$ and $Z(\nu, (a, b))$ are disjoint neighbourhoods of $\theta^\infty(x, s)$ and $\theta^\infty(y, t)$. If $s = 0$, then that $\mu, \nu \notin E^0$ implies that $e\mu \vee f\nu = \infty$ for any $e \in E^1 r(\mu)$ and $f \in E^1 r(\nu)$. We already have $\mu \vee \nu = \infty$, so we deduce that $Z(\mu, \frac{1}{2})$ and $Z(\nu, \frac{1}{2})$ are disjoint neighbourhoods of $\theta^\infty(x, s)$ and $\theta^\infty(y, t)$. Now suppose that $s \neq t$; without loss of generality, $s \neq 0$. Fix μ with $x \in \mu E^\infty$ and $y \in \nu E^\infty$. Choose $0 < a < s < b < 1$ such that $t \notin (a, b)$. If $t = 0$, then for any $\varepsilon < \min\{a, 1-b\}$ and any $f \in E^1$ with $s(f) = r(\nu)$, the sets $Z(\mu, (a, b))$ and $Z(f\nu, \varepsilon)$ are disjoint neighbourhoods of $\theta^\infty(x, s)$ and $\theta^\infty(y, t)$; and if $t \neq 0$ then for any

$0 < c < t < d < 1$ such that $(a,b) \cap (c,d) = \emptyset$, the sets $Z(\mu,(a,b))$ and $Z(\nu,(c,d))$ are disjoint neighbourhoods of $\theta^\infty(x,s)$ and $\theta^\infty(y,t)$. □

Let $\sim_\sigma$ be the equivalence relation on $E^\infty \times [0,1]$ defined by $(x,s) \sim_\sigma (y,t)$ if and only if either $x = y$ and $s = t$ or $y = \sigma(x)$, $s = 1$ and $t = 0$ or $x = \sigma(y)$, $s = 0$ and $t = 1$; that is, the smallest equivalence relation such that $(x,1) \sim_\sigma (\sigma(x),0)$ for all $x \in E^\infty$. The suspension of (E^∞,σ) is the topological quotient space

$$M(\sigma) := (E^\infty \times [0,1])/{\sim_\sigma}.$$

For $t \in [0,1]$, we write

$$M(\sigma)_t := \{[x,t] : x \in E^\infty\} \subseteq M(\sigma).$$

So $M(\sigma)_1 = M(\sigma)_0 \cong E^\infty$.

Remark 4.3. We can identify $M(\sigma)$ with the quotient space $(E^\infty \times [0,\infty))/{\sim_\sigma}$ where $(x,s) \sim_\sigma (y,t)$ if and only if $s - \lfloor s\rfloor = t - \lfloor t\rfloor$ and $\sigma^{\lfloor s\rfloor}(x) = \sigma^{\lfloor t\rfloor}(y)$. The identification sends $[x,t] \in M_\sigma$ to the corresponding class $[\![x,t]\!]$ in $(E^\infty \times \mathbb{R})/{\sim_\sigma}$; the inverse sends $[\![x,t]\!]$ to $[\sigma^{\lfloor t\rfloor}(x), t - \lfloor t\rfloor]$.

Proposition 4.4. *Let E be a finite graph with no sources. The suspension $M(\sigma)$ is a compact Hausdorff space, and the map $\theta^\infty : E^\infty \times [0,1) \to \mathcal{S}E^\infty$ described in Lemma 4.1 induces a homeomorphism of $M(\sigma)$ onto $\mathcal{S}E^\infty$.*

Proof. The equivalence classes for $\sim_\sigma$ in $E^\infty \times [0,1]$ are finite: if $t \notin \{0,1\}$ then the equivalence class of (x,t) is a singleton; if $t = 0$ then the equivalence class of (x,t) is $\{(x,0)\} \cup \{(ex,1) : e \in E^1 r(x)\}$, and if $t = 1$ then the equivalence class of (x,t) is $\{(\sigma(x),0)\} \cup \{(e\sigma(x),1) : e \in E^1 r(\sigma(x))\}$. In particular, the $\sim_\sigma$-equivalence classes in $E^\infty \times [0,1]$ are discrete, and so the quotient topology on $M(\sigma)$ is Hausdorff. Since $E^\infty \times [0,1]$ is compact, so is $M(\sigma)$.

The quotient map q from $E^\infty \times [0,1]$ to $M(\sigma)$ restricts to a bijection from $E^\infty \times [0,1)$ to $M(\sigma)$, and so θ^∞ induces a bijection τ from $M(\sigma)$ to $\mathcal{S}E^\infty$ such that $\tau(q(x,t)) = \theta^\infty(x,t)$ for $x \in E^\infty$ and $t \in [0,1)$. Since $\mathcal{S}E^\infty$ is Hausdorff and $M(\sigma)$ is compact, to see that τ is a homeomorphism, it suffices to show that it is continuous.

For this, observe that for $\mu \in E^*$ and $0 < a < b < 1$, we have $q^{-1}\big(\tau^{-1}\big(Z(\mu,(a,b))\big)\big) = \mu E^\infty \times (a,b)$, and so $\tau^{-1}\big(Z(\mu,(a,b))\big)$ is open by definition of the quotient topology. Similarly, $q^{-1}\big(\tau^{-1}(Z(e\mu,\varepsilon))\big) = \big(e\mu E^\infty \times (1-\varepsilon,1]\big) \cup \big(\mu E^\infty \times [0,\varepsilon)\big)$, which again is open in $E^\infty \times [0,1]$; so again by definition of the quotient topology, $\tau^{-1}(Z(e\mu,\varepsilon))$ is open, and hence τ is continuous. □

5. The C^*-algebras of the suspension of a graph

In this section we define two C^*-algebras associated to the suspension of a graph E. We define the first of these algebras in terms of a concrete representation on a non-separable Hilbert space, for which we use the following notational convention.

Notation 5.1. Throughout the rest of the paper, given any set X, we write

$$\ell^2(X) := \{f : X \to \mathbb{C} : f^{-1}(\mathbb{C} \setminus \{0\}) \text{ is countable, and } \sum_{x\in X} |v(x)|^2 < \infty\},$$

which is a Hilbert space with inner product given by $\langle f,g\rangle = \sum_{x\in X} v(x)\overline{w(x)}$. We denote the canonical basis elements of $\ell^2(X)$ by $\{h_x : x \in X\}$; so $h_x(y) = \delta_{x,y}$ for $x,y \in X$.

Remark 5.2. Using Notation 3.5, the set $\mathcal{S}E_0^n = \mathcal{S}E_1^n$ is the canonical copy $\{[\mu] : \mu \in E^n\}$ of E^n in $\mathcal{S}E^*$. In particular, there is a unitary $U_0 : \ell^2(E^*) \to \ell^2(\mathcal{S}E_0^*)$ satisfying

$$U_0 h_\mu = h_{[\mu]}.$$

For $t \in (0,1)$, and for each $n \in \mathbb{N}$, there is a bijection $\mathcal{S}E_t^n \to \widehat{E}^n$ given by $[\mu, t] \mapsto (\mu_1\mu_2)(\mu_2\mu_3)\cdots(\mu_n\mu_{n+1})$. In particular, for each $t \in (0,1)$ there is a unitary $U_t : \ell^2(\widehat{E}^*) \to \ell^2(\mathcal{S}E_t^*)$ given by

$$U_t h_{(\mu_1\mu_2)(\mu_2\mu_3)\cdots(\mu_{|\mu|-1}\mu_{|\mu|})} = h_{[\mu,t]}.$$

Lemma 5.3. *Let E be a locally finite graph with no sources. There is an injective nondegenerate representation $\rho : C_0(\mathcal{S}E^0) \to \mathcal{B}(\ell^2(\mathcal{S}E^*)$ such that*

$$\rho(a)h_\alpha = a(r(\alpha))h_\alpha \quad \textit{for all } a \in C_0(\mathcal{S}E^0) \textit{ and } \alpha \in \mathcal{S}E^*.$$

There is a linear map $\psi : C_c(\mathcal{S}E^1) \to \mathcal{B}(\ell^2(\mathcal{S}E^*))$ such that

$$\psi(\xi)h_\alpha = \sum_{\beta \in \mathcal{S}E^1 r(\alpha)} \xi(\beta)h_{\beta\alpha} \quad \textit{for all } \xi \in C_c(\mathcal{S}E^1) \textit{ and } \alpha \in \mathcal{S}E^*. \tag{5.1}$$

We have $\|\psi(\xi)\| \le \|\xi\|_\infty \cdot |\{ef \in E^2 : \xi([ef,t]) \ne 0$ for some $t \in [0,1]\}|$.

Proof. The representation ρ is the direct-sum of the representations $a \mapsto a(\omega)\,\mathrm{Id}_{\ell^2(\omega\mathcal{S}E^*)}$ indexed by $\omega \in \mathcal{S}E^0$. It is nondegenerate because for each $\alpha \in \mathcal{S}E^*$ and any $a \in C_0(\mathcal{S}E^0)$ with $a(r(\alpha)) = 1$ we have $\rho(a)h_\alpha = h_\alpha$. To see that it is injective, note that $\|\rho(a)\| \ge \sup_{\omega\in\mathcal{S}E^0} \|\rho(a)h_\omega\| = \|a\|_\infty$.

To see that there is a linear map ψ as claimed, let π be the path-space representation of $\mathcal{T}C^*(E)$. For $t \in (0,1)$, the operator

$$A_t := \sum_{ef\in\widehat{E}^1} \xi([ef,t])\pi(t_e t_f t_f^*) \in \pi(\mathcal{T}C^*(E)) \subseteq \mathcal{B}(\ell^2(E^*))$$

satisfies $\|A_t\| \le \sum_{ef\in\widehat{E}^1} |\xi([ef,t])| \le |\xi|_\infty \cdot |\{ef \in E^2 : \xi([ef,t]) \ne 0 \text{ for some } t\}|$ because the $\pi(t_e t_f t_f^*)$ are partial isometries. Similarly, the operator

$$A_0 := \sum_{e\in E^1} \xi([e])\pi(t_e)$$

satisfies $\|A_0\| \le \sum_{e\in E^1} |\xi([e])| \le |\xi|_\infty \cdot |\{ef \in E^2 : \xi([ef,t]) \ne 0 \text{ for some } t\}|$. Let $U_t : \ell^2(\mathcal{S}E_t^*) \to \ell^2(\widehat{E}^*)$, $0 < 1 < t$, and $U_0 : \ell^2(\mathcal{S}E_0^*) \to \ell^2(E^*)$ be the unitaries of Remark 5.2. Let $U := \bigoplus U_t : \ell^2(\mathcal{S}E^*) \to \bigoplus_{t\in[0,1)} \ell^2(E^\infty)$. Then

$$\psi(\xi) := U^*\Big(\bigoplus_{t\in[0,1)} A_t\Big)U \in \mathcal{B}(\ell^2(\mathcal{S}E^*))$$

satisfies (5.1). The map ψ_∞ thus defined is clearly linear, and satisfies the desired norm estimate because the A_t all do. □

We are now ready to define the Toeplitz algebra of $\mathcal{S}E$.

Definition 5.4. Let E be a locally finite graph with no sources. We define $\mathcal{T}C^*(\mathcal{S}E)$ to be the C^*-subalgebra of $\mathcal{B}(\ell^2(\mathcal{S}E^*))$ generated by $\rho(C_0(\mathcal{S}E^0))$ and $\psi(C_c(\mathcal{S}E^1))$.

To define $C^*(\mathcal{S}E)$ we first need to observe that the operators $\rho(a)$ and $\psi(\xi)$ above respect the fibration of $\ell^2(\mathcal{S}E^*)$ over $\mathbb{S}$.

Lemma 5.5. *Let E be a locally finite graph with no sources. For each $\omega \in \mathcal{S}E^0$, the subspace $\ell^2(\mathcal{S}E^*\omega) \subseteq \ell^2(\mathcal{S}E^*)$ is invariant for $\mathcal{T}C^*(\mathcal{S}E)$. In particular, each $\ell^2(\mathcal{S}E^*_t)$ is invariant for $\mathcal{T}C^*(\mathcal{S}E)$. For each $t \in \mathbb{S}$ and each $\omega \in \mathcal{S}E^0_t$, we have $\mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*\omega)} \cap \mathcal{K}(\ell^2(\mathcal{S}E^*_t)) = \mathcal{K}(\ell^2(\mathcal{S}E^*\omega))$.*

Proof. For $a \in C_0(\mathcal{S}E^0)$, $\xi \in C_c(\mathcal{S}E^1)$ and $\alpha \in \mathcal{S}E^*$, the element $\rho(a)h_\alpha$ is a scalar multiple of h_α. We have $\psi(\xi)h_\alpha = \sum_{\beta \in \mathcal{S}E^1 r(\alpha)} \xi(\beta)h_{\beta\alpha}$, and a quick calculation using inner-products shows that

$$\psi(\xi)^* h_\alpha = \begin{cases} \xi(\alpha_1)h_{\alpha_2\cdots\alpha_{|\alpha|}} & \text{if } |\alpha| \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\rho(a)h_\alpha$, $\psi(\xi)h_\alpha$ and $\psi(\xi)^*h_\alpha$ all belong to $\ell^2(\mathcal{S}E^*s(\alpha))$. Since the elements $\rho(a)$ and $\psi(\xi)$ generate $\mathcal{T}C^*(\mathcal{S}E)$, it follows that each $\ell^2(\mathcal{S}E^*\omega)$ is invariant. Consequently each $\ell^2(\mathcal{S}E^*_t) = \bigoplus_{\omega \in \mathcal{S}E^0_t} \ell^2(\mathcal{S}E^*_\omega)$ is invariant as well.

To prove the final statement, we first show that $\theta_{\omega,\omega} \in \mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*_t)}$ for each $t \in \mathbb{S}$ and $\omega \in \mathcal{S}E^0_t$. For this, fix $\omega \in \mathcal{S}E^0$. Fix $a \in C_0(\mathcal{S}E^0)$ such that $a|_{\mathcal{S}E^0_t} = \delta_\omega$ (this is possible since $\mathcal{S}E^0_t$ is a discrete subset of $\mathcal{S}E^0$). Likewise, the set $\mathcal{S}E^1_t$ is discrete in $\mathcal{S}E^1$, so for each $\alpha \in \omega\mathcal{S}E^1$ we can choose $\xi_\alpha \in C_c(\mathcal{S}E^1)$ such that $\xi_\alpha|_{\mathcal{S}E^1_t} = \delta_\alpha$. Using the formulas described in the preceding paragraph for the actions of $\rho(a)$ and the $\psi(\xi_\alpha)$ and their adjoints on basis elements, we see that

$$\theta_{h_\omega,h_\omega} = \Big(\rho(a) - \sum_{\alpha \in \omega\mathcal{S}E^1} \psi(\xi_\alpha)\psi(\xi_\alpha)^*\Big)\Big|_{\ell^2(\mathcal{S}E^*_t)} \in \mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*_t)}.$$

Now fix $t \in \mathbb{S}$, $\omega \in \mathcal{S}E^0_t$, and $\alpha \in \mathcal{S}E^*\omega \setminus \{\omega\}$. Factor $\alpha = \alpha_1 \cdots \alpha_m$ where each $\alpha_i \in \mathcal{S}E^1$. Using that $\mathcal{S}E^1_t$ is discrete, we choose $\xi_1, \dots, \xi_m \in C_c(\mathcal{S}E^1)$ such that $\xi_i|_{\mathcal{S}E^1_t} = \delta_{\alpha_i}$. Again calculating with basis vectors, we see that $\theta_{h_\alpha,h_\omega} = \psi(\xi_1)\cdots\psi(\xi_m)\theta_{h_\omega,h_\omega} \in \mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*_t)}$. Since each $\ell^2(\mathcal{S}E^*\omega)$ is invariant for $\mathcal{T}C^*(\mathcal{S}E)$, we deduce that $\theta_{h_\omega,h_\alpha} = \theta_{h_\omega,h_\omega}(\psi(\xi_1)\cdots\psi(\xi_m))^* \in \mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*_t)}$. Now for arbitrary $\alpha, \beta \in \mathcal{S}E^*\omega$ we have $\theta_{h_\alpha,h_\beta} = \theta_{h_\alpha,h_\omega}\theta_{h_\omega,h_\beta} \in \mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*_t)}$.

It follows that $\mathcal{K}(\ell^2(\mathcal{S}E^*_\omega)) \subseteq \mathcal{T}C^*(\mathcal{S}E)|_{\ell^2(\mathcal{S}E^*_t)} \cap \mathcal{K}(\ell^2(\mathcal{S}E^*_t))$ for each $t \in \mathbb{S}$ and $\omega \in \mathcal{S}E^0_t$. Since we also know that each $\ell^2(\mathcal{S}E^*\omega) \subseteq \ell^2(\mathcal{S}E^*_t)$ is invariant for $\mathcal{T}C^*(\mathcal{S}E)$, we have the reverse containment as well. □

Given a Hilbert space $\mathcal{H}$, we write $\mathcal{Q}(\mathcal{H})$ for the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Definition 5.6. Let E be a locally finite graph with no sources. We define $\tilde{\rho} : C_0(\mathcal{S}E^0) \to \bigoplus_{t \in \mathbb{S}} \mathcal{Q}(\ell^2(\mathcal{S}E^*_t))$ by

$$\tilde{\rho}(a) = \bigoplus_{t \in \mathbb{S}} \rho(a)|_{\ell^2(\mathcal{S}E^*)_t} + \mathcal{K}(\ell^2(\mathcal{S}E^*_t)),$$

and for $\xi \in C_c(\mathcal{S}E^1)$, we define

$$\tilde{\psi}(\xi) = \bigoplus_{t \in \mathbb{S}} \psi(\xi)|_{\ell^2(\mathcal{S}E^*)_t} + \mathcal{K}(\ell^2(\mathcal{S}E^*_t)).$$

We define $C^*(\mathcal{S}E)$ to be the C^*-subalgebra of $\bigoplus_{t \in \mathbb{S}} \mathcal{Q}(\ell^2(\mathcal{S}E^*_t))$ generated by $\tilde{\rho}(C_0(\mathcal{S}E^0))$ and $\tilde{\psi}(C_c(\mathcal{S}E^1))$.

Remark 5.7. For each $t \in \mathbb{S}$ and each $\omega \in \mathcal{S}E^0_t$, the subspace $\ell^2(\mathcal{S}E^*\omega) \subseteq \ell^2(\mathcal{S}E^*_t)$ is invariant for $\mathcal{T}C^*(\mathcal{S}E)$. It follows that there is an injective homomorphism from $C^*(\mathcal{S}E)$ to $\bigoplus_{\omega\in\mathcal{S}E^0} \mathcal{Q}(\ell^2(\mathcal{S}E^*\omega))$ that carries $\bigoplus_{t\in\mathbb{S}} \big(a|_{\ell^2(\mathcal{S}E^*_t)} + \mathcal{K}(\ell^2(\mathcal{S}E^*_t))\big)$ to $\bigoplus_{\omega\in\mathcal{S}E^0} \big(a|_{\ell^2(\mathcal{S}E^*\omega)} + \mathcal{K}(\ell^2(\mathcal{S}E^*\omega))\big)$ for all $a \in \mathcal{T}C^*(\mathcal{S}E)$.

We link our definition of $C^*(\mathcal{S}E)$ to the suspension of one-sided shift of E using the infinite-path space of $\mathcal{S}E$ described in the preceding section.

Proposition 5.8. *Let E be a locally finite graph with no sources, and suppose that every cycle in E has an entrance. Then there is a faithful representation $\Theta : C^*(\mathcal{S}E) \to \ell^2(M(\sigma))$ such that for $x \in E^\infty$ and $t \in [0, 1)$*

$$\Theta(\tilde{\rho}(a))h_{[x,t]} = a([x_1, t])h_{[x,t]} \text{ for all } a \in C_0(\mathcal{S}E^0)$$

and

$$\Theta(\tilde{\psi}(\xi))h_{[x,t]} = \sum_{e\in E^1 r(x)} \xi([e, t])h_{[ex,t]} \text{ for all } \xi \in C_c(\mathcal{S}E^1).$$

Proof. Since every cycle in E has an entrance, the Cuntz–Krieger uniqueness theorem [19, Theorem 3.7] shows that the infinite-path-space representations $\pi_\infty : C^*(E) \to \mathcal{B}(\ell^2(E^\infty))$ and $\hat{\pi}_\infty : C^*(\widehat{E}) \to \mathcal{B}(\ell^2(\widehat{E}^\infty))$ are both faithful. As discussed in Section 2.3, the Calkin representations $\pi_\mathcal{Q} : C^*(E) \to \mathcal{Q}(\ell^2(E^*))$ and $\hat{\pi}_\mathcal{Q} : C^*(\widehat{E}) \to \mathcal{Q}(\ell^2(\widehat{E}^*))$ are also faithful. Hence $\theta := \pi_\infty \circ \pi_\mathcal{Q}^{-1} : \pi_\mathcal{Q}(C^*(E)) \to \pi_\infty(C^*(E))$ is an isomorphism that carries $Q_v + \mathcal{K}(\ell^2(E^*))$ to $P_v := \operatorname{proj}_{\ell^2(vE^\infty)}$ and carries $T_e + \mathcal{K}(\ell^2(E^*))$ to $S_e : h_x \mapsto \delta_{s(e),r(x)}h_{ex}$. Similarly, $\hat{\theta} := \hat{\pi}_\infty \circ \hat{\pi}_\mathcal{Q}^{-1} : \hat{\pi}_\mathcal{Q}(C^*(\widehat{E})) \to \hat{\pi}_\infty(C^*(\widehat{E}))$ is an isomorphism carrying $Q_e + \mathcal{K}(\ell^2(\widehat{E}^*))$ to $P_e := \operatorname{proj}_{\ell^2(e\widehat{E}^\infty)}$ and carrying $T_{ef} + \mathcal{K}(\ell^2(\widehat{E}^*))$ to $S_{ef} : h_x \mapsto \delta_{f,r(x)}h_{(ef)x}$. It follows that $\theta \oplus (\bigoplus_{t\in\mathbb{S}\setminus\{0\}} \hat{\theta}) : C^*(\mathcal{S}E) \to \ell^2(E^\infty) \oplus \big(\bigoplus_{t\in\mathbb{S}\setminus\{0\}} \ell^2(\widehat{E}^\infty)\big)$ is a faithful representation.

As in the proof of Lemma 5.3, let $U_t : \ell^2(\mathcal{S}E^*_t) \to \ell^2(\widehat{E}^*)$, $0 < 1 < t$, and $U_0 : \ell^2(\mathcal{S}E^*_0) \to \ell^2(E^*)$ be the unitaries of Remark 5.2, and let $U := \bigoplus U_t : \ell^2(\mathcal{S}E^*) \to \bigoplus_{t\in[0,1)} \ell^2(E^\infty)$. Then $\Theta(x) := U^*\big(\theta \oplus (\bigoplus_{t\in\mathbb{S}\setminus\{0\}} \widehat{\theta})\big)(x)U$ defines a faithful representation of $C^*(\mathcal{S}E)$ on $\bigoplus_{t\in\mathbb{S}} \ell^2(\mathcal{S}E^\infty_t) = \ell^2(\mathcal{S}E^\infty)$. Direct calculation using the definitions of $\tilde{\rho}$, $\tilde{\psi}$, θ and $\hat{\theta}$ shows that Θ satisfies the prescribed formulae. □

6. Fractional higher-power graphs and their C^*-algebras

In this section, given a graph E, we generalise the construction of Section 3 by constructing, for each real number l a quiver $\mathcal{S}^lE$ in such a way that $\mathcal{S}^1E = \mathcal{S}E$ and $\mathcal{S}^{-1}E = \mathcal{S}E^{\mathrm{op}}$. To each $\mathcal{S}^lE$ we associate a Toeplitz algebra $\mathcal{T}C^*(\mathcal{S}^lE)$ and a C^*-algebra $C^*(\mathcal{S}^lE)$ so that $\mathcal{T}C^*(\mathcal{S}^1E) \cong \mathcal{T}C^*(\mathcal{S}E)$ and $\mathcal{T}C^*(\mathcal{S}^{-1}E) \cong \mathcal{T}C^*(\mathcal{S}E^{\mathrm{op}})$, and similarly at the level of Cuntz–Krieger algebras. We show that $\mathcal{T}C^*(\mathcal{S}^0E) \cong C_0(\mathcal{S}E^0) \otimes \mathcal{T}$ and $C^*(\mathcal{S}^0E) \cong C_0(\mathcal{S}E^0) \otimes C(\mathbb{T})$.

We then show that if $l = \frac{m}{n}$ is rational, then there is a graph F, closely related to E, such that $\mathcal{T}C^*(\mathcal{S}^lE) \cong \mathcal{T}C^*(\mathcal{S}^{|m|}F)$, and this isomorphism descends to an isomorphism $C^*(\mathcal{S}^lE) \cong C^*(\mathcal{S}^{|m|}F)$. This motivates the next section, in which we give a concrete description of each of $\mathcal{T}C^*(\mathcal{S}^mE)$ and $C^*(\mathcal{S}^mE)$ for a large class of graphs E and all positive integers m. Combining this result with those of the current section yields an explicit description of $\mathcal{T}C^*(\mathcal{S}^lE)$ and $C^*(\mathcal{S}^lE)$ for all rational l and for a large class of graphs E.

We use the following notation: given integers $0 \le m \le n \le L$, if $\mu = \mu_1\mu_2 \dots \mu_L \in E^L$, then

$$\mu(m, n) = \begin{cases} \mu_{m+1} \cdots \mu_n & \text{if } n > m \\ r(\mu_{m+1}) & \text{if } n = m. \end{cases}$$

It will be convenient in this section to use the following alternative description of $\mathcal{S}E^0$: it is homeomorphic to the quotient of $\bigsqcup_{n\ge 0} E^n \times [0, n]$ by the equivalence relation $(\mu, s) \sim (\nu, t)$ if $s - \lfloor s\rfloor = t - \lfloor t\rfloor$ and $\mu(\lfloor s\rfloor, \lceil s\rceil) = \nu(\lfloor t\rfloor, \lceil t\rceil)$ (see [12]).

Definition 6.1. Let E be a locally finite graph with no sources. Fix $l \in \mathbb{R}$. Let $\approx_l$ be the equivalence relation on $\{(\mu, t) : \mu \in E^*$ and $t, t + l \in [0, |\mu|]\}$ given by $(\mu, s) \approx_l (\nu, t)$ if and only if $s - \lfloor s\rfloor = t - \lfloor t\rfloor$ and

$$\begin{cases} \mu(\lfloor s\rfloor, \lceil s + l\rceil) = \nu(\lfloor t\rfloor, \lceil t + l\rceil) & \text{if } l \ge 0, \\ \mu(\lfloor s + l\rfloor, \lceil s\rceil) = \nu(\lfloor t + l\rfloor, \lceil t\rceil) & \text{if } l \le 0. \end{cases}$$

We define

$$\mathcal{S}^lE^1 := \{(\mu, t) : \mu \in E^* \text{ and } t, t + l \in [0, |\mu|]\}/\approx_l.$$

We write $[\mu, t]_l$ for the equivalence class of (μ, t) under $\approx_l$. Define $r_l, s_l : \mathcal{S}E^l \to \mathcal{S}E^0$ by $r_l([\mu, t]_l) := [\mu, t]_l$ and $s_l([\mu, t]_l) = [\mu, t + l]_l$. The quadruple $(\mathcal{S}E^0, \mathcal{S}^lE^1, r, s)$ is denoted $\mathcal{S}^lE$.

Remark 6.2. The special case $l = 1$ coincides with the space $\mathcal{S}E^1$ and range and source maps $r, s : \mathcal{S}E^1 \to \mathcal{S}E^0$ of Section 3. When $l = -1$ there is a homeomorphism from the space $\mathcal{S}^{-1}E^1$ to the space $(\mathcal{S}E^{\mathrm{op}})^1$ associated to the opposite graph of E satisfying $[ef, t] \mapsto [f^{\mathrm{op}}e^{\mathrm{op}}, 1 - t]$, and this homeomorphism intertwines r_{-1} with $r : (\mathcal{S}E^{\mathrm{op}})^1 \to (\mathcal{S}E^{\mathrm{op}})^0$ and intertwines s_{-1} with $s : (\mathcal{S}E^{\mathrm{op}})^1 \to (\mathcal{S}E^{\mathrm{op}})^0$. When $l = 0$, we see that $\mathcal{S}^0E^1$ is a copy of the vertex space, and r_0, s_0 are both just the identity map $\mathcal{S}E^0 \to \mathcal{S}E^0$.

Notation 6.3. We will frequently just write r, s in place of r_l, s_l, and $[\mu, t]$ in place of $[\mu, t]_l$, when the parameter l is clear from context.

As with $\mathcal{S}E$, the quivers $\mathcal{S}^lE$ are not usually topological graphs, except if $l = 0$.

Lemma 6.4. *Let E be a locally finite graph with no sources. The range and source maps s_0 and r_0 on $\mathcal{S}^0E$ are homeomorphisms. Fix $l \in \mathbb{R} \setminus \{0\}$. Then r_l and s_l are continuous maps. If $l > 0$, then the map s_l is an open map if and only if each $|E^1v| = 1$ and r_l is an open map if and only if each $|vE^1| = 1$. If $l < 0$, then the map s_l is open if and only if each $|vE^1| = 1$, and r_l is open if and only if each $|E^1v| = 1$.*

Proof. The final statement of Remark 6.2 shows that s_0 and r_0 are homeomorphisms. The proof of the remaining statements is very similar to that of Lemma 3.6. □

A *path* in $\mathcal{S}^lE$ is a sequence $[\mu_1, t_1][\mu_2, t_2][\mu_3, t_3] \cdots [\mu_k, t_k]$ of elements of $\mathcal{S}^lE^1$ such that $s_l([\mu_i, t_i]) = r_l([\mu_{i+1}, t_{i+1}])$ for all $1 \le i < k$. We write $\mathcal{S}^lE^*$ for the space $\bigsqcup_{i=0}^\infty \mathcal{S}^lE^i$ of all paths in $\mathcal{S}^lE$, including the vertices, which are regarded as paths of length 0. So $\mathcal{S}^lE^0 = \mathcal{S}E^0$. We define $r(v) = s(v) = v$ for each $v \in \mathcal{S}E^0$, and for

$$\alpha = [\mu_1, t_1][\mu_2, t_2][\mu_3, t_3] \cdots [\mu_k, t_k] \in \mathcal{S}^lE^k$$

we define $r(\alpha) = r([\mu_1, t_1])$ and $s(\alpha) = s([\mu_k, t_k])$.

Remark 6.5. For each $t \in \mathbb{S}$ we write $\mathcal{S}^l E^n_t$ for $\{\alpha \in \mathcal{S}^l E^n : \varpi(s(\alpha)) = t\} = (\mathcal{S}^l E^n)(\mathcal{S}E^0_t)$. We have $r(\mathcal{S}^l E^n_t) = \mathcal{S}E^0_{t-l}$, where addition on the subscript is modulo $\mathbb{Z}$. If $l = m \in \mathbb{Z}$ *is* an integer, then as in Remark 5.2, the space $\mathcal{S}^m E^n_0 = \mathcal{S}^m E^n_1$ is the canonical copy $\{[\mu] : \mu \in E^{nm}\}$ of E^{nm} in $\mathcal{S}^m E^*$. In particular there is a unitary $U_0 : \ell^2(E(0,m)^*) \to \ell^2(\mathcal{S}^m E^*_0)$ satisfying

$$U_0 h_\mu = h_{[\mu]}.$$

For $m \geq 0$, $t \in (0,1)$, and each $n \in \mathbb{N}$, there is a bijection $\mathcal{S}^m E^n_t \to E(1, m+1)^n$ such that $[\mu, t] \mapsto (\mu_1 \cdots \mu_{m+1})(\mu_m \cdots \mu_{2m+1}) \cdots (\mu_{(n-1)m} \cdots \mu_{nm+1})$. Thus for each $t \in (0,1)$ there is a unitary $U_t : \ell^2(E(1, m+1)^*) \to \ell^2(\mathcal{S}^m E^*_t)$ given by

$$U_t h_{(\mu_1 \cdots \mu_{m+1})(\mu_m \cdots \mu_{2m+1}) \cdots (\mu_{(n-1)m} \cdots \mu_{nm+1})} = h_{[\mu,t]}$$

The space $\mathcal{S}^l E^1$ is endowed with the quotient topology inherited from $\mathcal{S}E^{\lceil l \rceil + 1} \times [0,1]$.

Lemma 6.6. *Let E be a locally finite graph with no sources and fix $l \in \mathbb{R}$. Let $\{h_\alpha : \alpha \in \mathcal{S}^l E^*\}$ denote the canonical orthonormal basis for $\ell^2(\mathcal{S}^l E^*)$. There is an injective nondegenerate representation $\rho_l : C_0(\mathcal{S}E^0) \to \mathcal{B}(\ell^2(\mathcal{S}^l E^*)$ such that*

$$\rho_l(a) h_\alpha = a(r(\alpha)) h_\alpha \quad \text{for all } a \in C_0(\mathcal{S}E^0) \text{ and } \alpha \in \mathcal{S}^l E^*,$$

and there is a linear map $\psi_l : C_c(\mathcal{S}^l E^1) \to \mathcal{B}(\ell^2(\mathcal{S}^l E^*))$ such that

$$\psi_l(\xi) h_\alpha = \sum_{\beta \in \mathcal{S}^l E^1 r(\alpha)} \xi(\beta) h_{\beta\alpha} \quad \text{for all } \xi \in C_c(\mathcal{S}^l E^1) \text{ and } \alpha \in \mathcal{S}^l E^*.$$

We have

$$\|\psi_l(\xi)\| \leq \|\xi\|_\infty \cdot |\{\mu \in E^{\lceil |l| \rceil + 1} : \xi([\mu, t]) \neq 0 \text{ for some } t \in [0,1]\}|. \tag{6.1}$$

For each $\omega \in \mathcal{S}E^0$, the subspace $\ell^2(\mathcal{S}^l E^* \omega) \subseteq \ell^2(\mathcal{S}^l E^*)$ is invariant for the C^*-subalgebra of $\mathcal{B}(\ell^2(\mathcal{S}^l E^*))$ generated by the images of ρ_l and ψ_l.

Proof. The first two statements follow from an argument almost identical to that of Lemma 5.3. The final statement follows from an argument nearly identical to the proof of the first statement of Lemma 5.5. □

Definition 6.7. Let E be a locally finite graph with no sources.

(1) We define $\mathcal{T}C^*(\mathcal{S}^l E) := C^*\big(\rho_l(C_0(\mathcal{S}E^0)) \cup \psi_l(C_c(\mathcal{S}^l E^1))\big)$, the C^*-subalgebra of $\mathcal{B}(\ell^2(\mathcal{S}^l E^*))$ generated by the images of ρ_l and ψ_l.
(2) We define $\tilde{\rho}_l : C_0(\mathcal{S}E^0) \to \oplus_{\omega \in \mathcal{S}E^0} \mathcal{Q}(\ell^2(\mathcal{S}^l E^* \omega))$ by

$$\tilde{\rho}_l(a) = \bigoplus_{\omega \in \mathcal{S}E^0} \Big(\rho(a)|_{\ell^2(\mathcal{S}^l E^* \omega)} + \mathcal{K}(\ell^2(\mathcal{S}^l E^* \omega))\Big)$$

and we define $\tilde{\psi}_l : C_c(\mathcal{S}^l E^1) \to \oplus_{\omega \in \mathcal{S}E^0} \mathcal{Q}(\ell^2(\mathcal{S}^l E^* \omega))$ by

$$\tilde{\psi}_l(\xi) = \bigoplus_{\omega \in \mathcal{S}E^0} \Big(\psi(\xi)|_{\ell^2(\mathcal{S}^l E^* \omega)} + \mathcal{K}(\ell^2(\mathcal{S}^l E^* \omega))\Big).$$

(3) We define $C^*(\mathcal{S}^l E)$ to be the subalgebra of $\bigoplus_{\omega \in \mathcal{S}E^0} \mathcal{Q}(\ell^2(\mathcal{S}^l E^* \omega))$ generated by $\tilde{\rho}_l(C_0(\mathcal{S}E^0))$ and $\tilde{\psi}_l(C_c(\mathcal{S}^l E^1))$.

Example 6.8. If E consists of a single vertex v and a single edge e, then each $\mathcal{S}^l E$ is a copy of the topological graph $F_l := (\mathbb{S}, \mathbb{S}, t \mapsto t-l, \mathrm{id})$ determined by the rotation homeomorphism $t \mapsto t - l$ of $\mathbb{S}$. We have $C_c(\mathcal{S}^l E^1) = C(\mathcal{S}^l E^1) = C(F_l^1) = X(F_l)$, the topological-graph bimodule of F_l. It is routine to verify that (ρ_l, ψ_l) is a representation of $X(F_l)$ in the sense of Section 2.5. So there is a surjective homomorphism $\psi_l \times \rho_l : \mathcal{T}C^*(F_l) \to \mathcal{T}C^*(\mathcal{S}^l E)$ that carries $i_{X(F_l)}(\xi)$ to $\psi_l(\xi)$ and carries $i_{C(\mathbb{S})}(a)$ to $\rho_l(a)$. The space $\mathcal{S}^l E^*$ can be identified with $\mathbb{S} \times \mathbb{N}$ by the map that sends $\alpha \in \mathcal{S}^l E^n$ to $(s(\alpha), n)$. Under this identification, the map $a \mapsto \rho_l(a)(1 - \psi_l(1)\psi_l(1)^*)$ is the canonical faithful representation of $C(\mathbb{S})$ on $\ell^2(\mathbb{S} \times \{1\}) \cong \ell^2(\mathbb{S})$, so the uniqueness theorem [9, Theorem 2.1] shows that $\psi_l \times \rho_l$ is injective. Therefore $\mathcal{T}C^*(\mathcal{S}^l E)$ is isomorphic to the Toeplitz algebra $\mathcal{T}C^*(F_l)$. Since $1 - \psi_l(1)\psi_l(1)^*$ belongs to $\bigoplus_{t \in \mathbb{S}} \mathcal{K}(\mathcal{S}^l E^* t)$, we see that $\tilde{\psi}_l(1)\tilde{\psi}_l(1)^* = 1$ in $C^*(\mathcal{S}^l E)$. For any $a \in C(\mathcal{S}^l E^0)$ the left action of a on F_l is given by $a \cdot \xi = \theta_{a,1}(\xi)$, and so we see that if $\phi : C(\mathcal{S}^l E^0) \to \mathcal{L}(F_l)$ denotes the homomorphism implementing the left action, we have

$$\tilde{\rho}_l(a) - \tilde{\psi}_l^{(1)}(\phi(a)) = \tilde{\rho}_l(a) - \tilde{\psi}_l(a)\tilde{\psi}_l(1)^* = \tilde{\rho}_l(a)(\tilde{\rho}_l(1) - \tilde{\psi}_l(1)\tilde{\psi}_l(1)^*) = 0.$$

Hence $(\tilde{\psi}_l, \tilde{\rho}_l)$ is a covariant representation of F_l, and therefore induces a homomorphism $\tilde{\psi}_l \times \tilde{\rho}_l : C^*(F_l) \to C^*(\mathcal{S}^l E)$. The action of $\mathbb{T}$ on $\mathcal{B}(\ell^2(\mathcal{S}^l E^*))$ determined by conjugation by the unitaries $\{W_z : z \in \mathbb{T}\}$ given by $W_z(h_{t,n}) = z^n h_{t,n}$ induces an action on $\bigoplus_{t \in \mathbb{S}} \mathcal{Q}(\ell^2(\mathcal{S}^l E^* t))$, and it is routine to check that $\tilde{\psi}_l \times \tilde{\rho}_l$ is equivariant for this action and the gauge action on $C^*(F_l)$. Since $\tilde{\rho}_l : a \mapsto \bigoplus_\omega a(\omega) 1_{\mathcal{Q}(\ell^2(\mathcal{S}^l E^* \omega))}$ is injective, it follows from the gauge-invariant uniqueness theorem [14, Theorem 4.5] that $\tilde{\psi} \times \tilde{\rho}$ is injective. So $C^*(\mathcal{S}^l E) \cong C^*(F_l)$. By [15, Proposition 10.5] there is an isomorphism of the rotation algebra A_l—the universal C^*-algebra generated by unitaries U, V such that $UV = e^{2\pi i l} VU$—onto $C^*(F_l)$ that carries U to $i_{C(\mathbb{S})}(t \mapsto e^{2\pi i t})$ and carries V to $i_{X(F_1)}(1)$. So we deduce that there is an isomorphism $A_l \cong C^*(\mathcal{S}^l E)$ that carries U to $\tilde{\rho}_l(t \mapsto e^{2\pi i t})$ and carries V to $\tilde{\psi}_l(1_{\mathcal{S}^l E^1})$.

Remark 6.9. More generally, if E is a row-finite graph with no sources and satisfying $|E^1 v| = 1$ for all v, then for each $l > 0$ the quadruple $F := (\mathcal{S}E^0, \mathcal{S}E^l, r, s)$ is a topological graph in the sense of Katsura, and an analysis like that of Example 6.8 shows that $\mathcal{T}C^*(\mathcal{S}^l E)$ and $C^*(\mathcal{S}^l E)$ coincide with the topological-graph C^*-algebras $\mathcal{T}C^*(F)$ and $C^*(F)$ respectively.

In the next few sections we will give a recipe for describing both $\mathcal{T}C^*(\mathcal{S}^l E)$ and $C^*(\mathcal{S}^l E)$ for rational values of l provided that sufficiently many vertices of E both emit and receive at least two edges. We do not yet have a concrete description of $C^*(\mathcal{S}^l E)$ for arbitrary $l \in \mathbb{R}$ and an arbitrary graph E.

The following theorem relates the constructions described in this section with those of Section 3 and Section 5. For the following result, we denote by $\mathcal{T}$ the classical Toeplitz algebra generated by a non-unitary isometry S.

Theorem 6.10. *Let E be a locally finite graph with no sources.*

(1) *There is an isomorphism $\mathcal{T}C^*(\mathcal{S}^1 E) \cong \mathcal{T}C^*(\mathcal{S}E)$ that carries $\rho_1(a)$ to $\rho(a)$ for $a \in C_0(\mathcal{S}E^0)$ and carries $\psi_1(\xi)$ to $\psi(\xi)$ for $\xi \in C_c(\mathcal{S}^1 E^1)$. This isomorphism descends to an isomorphism $C^*(\mathcal{S}^1 E) \cong C^*(\mathcal{S}E)$.*

(2) *There is an isomorphism $\mathcal{T}C^*(\mathcal{S}^0 E) \cong C_0(\mathcal{S}E^0) \otimes \mathcal{T}$ that carries $\rho_0(a)$ to $a \otimes 1$ and carries $\psi_0(\xi)$ to $(\xi \circ r_0^{-1}) \otimes S$ for $\xi \in C_c(\mathcal{S}^0 E^1) = C_c(\mathcal{S}E^0)$, and this isomorphism descends to an isomorphism $C^*(\mathcal{S}E^0) \cong C_0(\mathcal{S}E^0) \otimes C(\mathbb{T})$.*

(3) *There is an isomorphism* $\mathcal{T}C^*(\mathcal{S}^{-1}E) \cong \mathcal{T}C^*(\mathcal{S}E^{\mathrm{op}})$ *that carries* $\rho_{-1}(a)$ *to the element* $\rho([e^{\mathrm{op}}, t] \mapsto a([e, 1-t]))$ *for* $a \in C_0(\mathcal{S}E^0)$ *and carries* $\psi_{-1}(\xi)$ *to* $\psi\big([ef, t] \mapsto \xi([f^{\mathrm{op}}e^{\mathrm{op}}, (1-t)])\big)$ *for* $\xi \in C_c(\mathcal{S}^{-1}E^1)$. *This isomorphism descends to an isomorphism* $C^*(\mathcal{S}^{-1}E) \cong C^*(\mathcal{S}E^{\mathrm{op}})$.

Proof. The proofs of the first and third statements are almost identical. For the first statement, observe that the identification $\mathcal{S}^1E^1 \cong \mathcal{S}E^1$ of Remark 6.2 intertwines r, s with r_1 and s_1, and so induces a homeomorphism $\mathcal{S}^1E^* \cong \mathcal{S}E^*$, which induces a unitary $U_1 : \ell^2(\mathcal{S}^1E^*) \cong \ell^2(\mathcal{S}E^*)$. This unitary intertwines ρ and ρ_1 and intertwines ψ and ψ_1, and so Ad_{U_1} restricts to the desired isomorphism $\mathcal{T}C^*(\mathcal{S}^1E) \cong \mathcal{T}C^*(\mathcal{S}E)$. Since U_1 carries each $\ell^2(\mathcal{S}^1E^*\omega)$ to $\ell^2(\mathcal{S}E^*\omega)$, the map Ad_{U_1} carries each $\mathcal{K}(\ell^2(\mathcal{S}^1E^*\omega))$ to $\mathcal{K}(\ell^2(\mathcal{S}E^*\omega))$, and so descends to an isomorphism $C^*(\mathcal{S}^1E) \cong C^*(\mathcal{S}E)$ as claimed. For the third statement, we argue exactly the same way, using the homeomorphism $\mathcal{S}^{-1}E^1 \cong (\mathcal{S}E^{\mathrm{op}})^1$ of Remark 6.2 to induce a unitary $U_{-1} : \ell^2(\mathcal{S}^{-1}E^*) \cong \ell^2((\mathcal{S}E^{\mathrm{op}})^*)$.

For the second statement, first identify $\ell^2(\mathcal{S}^0E^*)$ with $\ell^2(\mathcal{S}E^0) \otimes \ell^2(\mathbb{N})$ by the unitary U_0 that carries $h_{[\mu,t]}$ to $h_{[\mu_1,t]} \otimes h_{|\mu|-1}$. Let $\pi : C_0(\mathcal{S}E^0) \to \mathcal{B}(\ell^2(\mathcal{S}E^0))$ be the canonical faithful representation $\pi(a)h_\omega = a(\omega)h_\omega$. Direct calculation shows that $U_0\rho_0(a)U_0^* = \pi(a) \otimes \mathrm{id}$, and $U_0\psi_0(\xi)U_0^* = \pi(\xi \circ r_0^{-1}) \otimes S$ for $a \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^0E^1)$. So Ad_{U_0} carries $\mathcal{T}C^*(\mathcal{S}^0E)$ onto the subalgebra of $\mathcal{B}(\ell^2(\mathcal{S}E^0) \otimes \ell^2(\mathbb{N}))$ generated by products of the form $(\pi(a) \otimes \mathrm{id})(\mathrm{id} \otimes S)$. This is precisely the tensor product $C_0(\mathcal{S}E^0) \otimes \mathcal{T}$. Moreover, Ad_{U_0} carries each $\mathcal{K}(\ell^2(\mathcal{S}^0E^*\omega))$ to $\mathcal{K}(\mathbb{C}h_\omega \otimes \ell^2(\mathbb{N}))$, and so it carries the kernel of the quotient map $\mathcal{T}C^*(\mathcal{S}^0E) \to C^*(\mathcal{S}^0E)$ to $C_0(\mathcal{S}E^0) \otimes \mathcal{K}(\ell^2(\mathbb{N})) \lhd C_0(\mathcal{S}E^0) \otimes \mathcal{T}$. It therefore descends to an isomorphism

$$C^*(\mathcal{S}^0E) \cong (C_0(\mathcal{S}E^0) \otimes \mathcal{T})/(C_0(\mathcal{S}E^0) \otimes \mathcal{K}) \cong C_0(\mathcal{S}E^0) \otimes (\mathcal{T}/\mathcal{K}) \cong C_0(\mathcal{S}E^0) \otimes C(\mathbb{T}). \quad \square$$

The remainder of the section is devoted to reducing the study of the C^*-algebras $\mathcal{T}C^*(\mathcal{S}^lE)$ and $C^*(\mathcal{S}^lE)$ for rational values of l to the study of the C^*-algebras $\mathcal{T}C^*(\mathcal{S}^mF)$ and $C^*(\mathcal{S}^mF)$ for nonnegative integers m and appropriate graphs F. We will analyse these latter in the next two sections.

Our first step is to show that we need only consider $l \geq 0$ by showing that $\mathcal{T}C^*(\mathcal{S}^lE) \cong \mathcal{T}C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$ for $l \in (-\infty, 0)$.

Lemma 6.11. *Let E be a locally finite graph with no sources. Fix $l \in (-\infty, 0)$. There is an isomorphism $\mathcal{T}C^*(\mathcal{S}^lE) \cong \mathcal{T}C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$ that carries $\rho_l(a)$ to $\rho_{|l|}\big([e^{\mathrm{op}}, t] \mapsto a([e, 1-t])\big)$ for $a \in C_0(\mathcal{S}E^0)$ and carries $\psi_l(\xi)$ to $\psi_{|l|}\big([\mu^{\mathrm{op}}, t] \mapsto \xi([\mu, |\mu|-t])\big)$ for $\xi \in C_c(\mathcal{S}^{|l|}E^1)$. This isomorphism descends to an isomorphism $C^*(\mathcal{S}^lE) \cong C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$.*

Proof. This follows the argument of Theorem 6.10(3): The map $[\mu, t] \mapsto [\mu^{\mathrm{op}}, |\mu| - t]$ defines homeomorphism τ_l of $\mathcal{S}^lE^*$ onto $(\mathcal{S}^{|l|}E^{\mathrm{op}})^*$ that intertwines the range and source maps. This homeomorphism determines a unitary $U : \ell^2(\mathcal{S}^lE^*) \to \ell^2(\mathcal{S}^{|l|}E^{\mathrm{op}}$, conjugation by which implements an isomorphism $\mathcal{T}C^*(\mathcal{S}^lE) \cong \mathcal{T}C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$ that satisfies the desired formulae. Since $U(\ell^2(\mathcal{S}^lE^*\omega)) = \ell^2(\mathcal{S}^{|l|}E^{\mathrm{op}}\tau_l(\omega))$ for each $\omega \in \mathcal{S}E^0$, we have

$$U\mathcal{K}(\ell^2(\mathcal{S}^lE^*\omega))U^* = \mathcal{K}(\ell^2(\mathcal{S}^{|l|}E^{\mathrm{op}}\tau_l(\omega)))$$

for each ω. Hence Ad_U descends to the desired isomorphism $C^*(\mathcal{S}^lE) \cong C^*(\mathcal{S}^{|l|}E^{\mathrm{op}})$. $\square$

In the remainder of the section we must show that if $m \in \mathbb{N}$ and $n \in \mathbb{N}\backslash\{0\}$, then there is a graph F such that $\mathcal{T}C^*(\mathcal{S}^{\frac{m}{n}}E) \cong \mathcal{T}C^*(\mathcal{S}^mF)$ and similarly at the level of Cuntz–Krieger algebras.

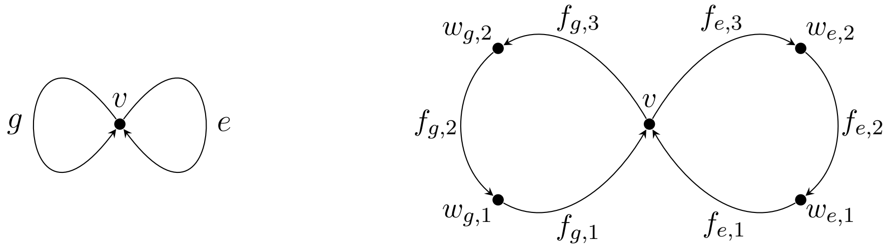
Given a graph E and an integer $n \geq 1$, the n^{th} *delay* of E is the graph $D_n(E)$ described as follows. We set

$$D_n(E)^0 := E^0 \sqcup \{w_{e,j} : e \in E^1, 1 \leq j \leq n-1\} \qquad \text{and}$$
$$D_n(E)^1 := \{f_{e,j} : e \in E^1, 1 \leq j \leq n\}.$$

The range and source maps are given by

$$r(f_{e,j}) = \begin{cases} w_{e,j-1} & \text{if } j \geq 2 \\ r(e) & \text{if } j = 1 \end{cases} \qquad \text{and} \qquad s(f_{e,j}) = \begin{cases} w_{e,j} & \text{if } j < n \\ s(e) & \text{if } j = n. \end{cases}$$

In words, $D_n(E)$ is the graph obtained by inserting $n-1$ new vertices along each edge of E. The example below pictures a graph E on the left and the delayed graph $D_3(E)$ on the right.



E $\qquad$ $D_3(E)$

We will prove that for $m \geq 0$ and $n > 0$, the graph $\mathcal{S}^{\frac{m}{n}}E$ is isomorphic to $\mathcal{S}^m(D_n(E))$.

Observe that there is a range and source preserving map $D_n^* : E^* \to \bigcup_{k=0}^\infty D_n(E)^{kn}$ given by

$$D_n^*(e_1 \dots e_k) = f_{e_1,1} \dots f_{e_1,n} f_{e_2,1} \cdots f_{e_2,n} \cdots f_{e_k,1} \cdots f_{e_k,n}.$$

Lemma 6.12. *Let E be a locally finite graph with no sources, and fix integers $n \geq 1$ and $m \geq 0$. There are homeomorphisms $\mathcal{S}D_n^0 : \mathcal{S}^{\frac{m}{n}}E^0 \to \mathcal{S}^m D_n(E)^0$ and $\mathcal{S}D_n^1 : \mathcal{S}^{\frac{m}{n}}E^1 \to \mathcal{S}^m D_n(E)^1$ such that for $j \in \{1, \dots, n\}$ and $t \in [0,1]$,*

$$\mathcal{S}D_n^0\Big(\Big[e, \frac{j-1+t}{n}\Big]\Big) = [f_{e,j}, t]$$

for all $e \in E^1$, and

$$\mathcal{S}D_n^1\Big(\Big[\mu, \frac{j-1+t}{n}\Big]\Big) = [D_n^*(\mu), j-1+t]$$

for all $\mu \in E^*$ such that $0 \leq \frac{j-1+t}{n}, \frac{j-1+m+t}{n} \leq |\mu|$. We have $\mathcal{S}D_n^0(s(\alpha)) = s(\mathcal{S}D_n^1(\alpha))$ and $\mathcal{S}D_n^0(r(\alpha)) = r(\mathcal{S}D_n^1(\alpha))$ for all $\alpha \in \mathcal{S}^m D_n(E)^1$.

Proof. Define $\overline{\mathcal{S}D_n}^0 : E^0 \sqcup (E^1 \times [0,1]) \to \mathcal{S}D_n(E)^0 = \mathcal{S}^m D_n(E)^0$ by $\overline{\mathcal{S}D_n}^0(v) := [v]$ for $v \in E^0$, and $\overline{\mathcal{S}D_n}^0\Big(\Big(e, \frac{j-1+t}{n}\Big)\Big) = [f_{e,j}, t]$ for $e \in E^1$, $1 \leq j \leq n$ and $t \in [0,1]$. Then

$$\overline{\mathcal{S}D_n}^0((e,0)) = [f_{e,0}, 0] = [r(f_{e,0})] = [r(e)] = \overline{\mathcal{S}D_n}^0(r(e)),$$

and similarly, $\overline{\mathcal{S}D_n}^0((e,1)) = \overline{\mathcal{S}D_n}^0(s(e))$, so $\overline{\mathcal{S}D_n}^0$ descends to a map $\mathcal{S}D_n^0 : \mathcal{S}^{\frac{m}{n}}E^0 \to \mathcal{S}^m D_n(E)^0$. Since $\overline{\mathcal{S}D_n}^0$ is continuous, so is $\mathcal{S}D_n^0$. It is routine using Lemma 3.2 to see that $\mathcal{S}D_n^0$ is injective, and it is clearly surjective. To see that it is open, fix $\omega \in \mathcal{S}E^0$. If $\omega = [e, \frac{j-1+t}{n}]$ with $t \in (0,1)$, then the sets $\{[e, \frac{j-1+t+s}{n}] : s \in (-\delta, \delta)\}$ indexed by sufficiently small δ form a neighbourhood basis at ω and are carried to the open sets $\{[f_{e,j}, t+s] : s \in (-\delta,\delta)\}$. If $\omega = [e, \frac{j-1}{n}]$ for some $1 < j \le n$, then the sets $\{[e, \frac{j-1+s}{n}] : s \in (-\delta,\delta)\}$ indexed by $\delta \in (0,\frac{1}{2})$ form a neighbourhood base at ω and are carried to the open sets $\{[f_{e,j-1}, t] : t > 1-\delta\} \cup \{[f_{e,j}, t] : t < \delta\}$. Finally, if $\omega = [v]$ for $v \in E^0$, then the sets $\{[e,s] : e \in E^1 v : s > 1-\delta\} \cup \{[e,s] : e \in vE^1 : s < \delta\}$ indexed by $\delta < \frac{1}{n}$ are a neighbourhood base at E, and are carried to the open sets $\{[f_{e,n}, t] : e \in E^1 v, t > 1-n\delta\} \cup \{[f_{e,1}, t] : e \in vE^1, t < n\delta\}$. Therefore $\mathcal{S}D^0_n$ is an open map, and therefore a homeomorphism.

A similar argument shows that $\mathcal{S}D^1_n$ exists and is a homeomorphism, and a simple comparison of formulas shows that $\mathcal{S}D^0_n$ and $\mathcal{S}D^1_n$ are compatible with the range and source maps as claimed. □

Corollary 6.13. *Let E be a locally finite graph with no sources. Fix integers $n \ge 1$ and $m \ge 0$. There is a unitary $U_{m,n} : \ell^2(\mathcal{S}^{\frac{m}{n}}E^*) \to \ell^2(\mathcal{S}^m D_n(E)^*)$ such that*

$$U_{m,n}\rho_{m/n}(a)U^*_{m,n} = \rho_m(a \circ \mathcal{S}D^0_n) \quad \text{for } a \in C_0(\mathcal{S}^m D_n(E)^0)$$

and

$$U_{m,n}\psi_{m/n}(\xi)U^*_{m,n} = \psi_m(\xi \circ \mathcal{S}D^0_n) \quad \text{for } \xi \in C_c(\mathcal{S}^m D_n(E)^1).$$

Conjugation by $U_{m,n}$ restricts to an isomorphism $\Theta_{m,n} : \mathcal{T}C^*(\mathcal{S}^{\frac{m}{n}}E) \to \mathcal{T}C^*(\mathcal{S}^m D_n(E))$. This $\Theta_{m,n}$ induces an isomorphism $\widetilde{\Theta}_{m,n} : C^*(\mathcal{S}^{\frac{m}{n}}E) \to C^*(\mathcal{S}^m D_n(E))$ such that

$$\widetilde{\Theta}_{m,n}(\tilde{\rho}_{m/n}(a)) = \tilde{\rho}_m(a \circ \mathcal{S}D^0_n) \text{ for } a \in C_0(\mathcal{S}^m D_n(E)^0)$$

and

$$\widetilde{\Theta}_{m,n}(\tilde{\psi}_{m/n}(\xi)) = \tilde{\psi}_m(\xi \circ \mathcal{S}D^0_n) \text{ for } \xi \in C_c(\mathcal{S}^m D_n(E)^1).$$

Proof. For $k \ge 1$ there is a bijection $\mathcal{S}D^k_n : \mathcal{S}^{\frac{m}{n}}E^k \to \mathcal{S}^m D_n(E)^k$ given by $\mathcal{S}D^k_n(\alpha_1 \cdots \alpha_k) = \mathcal{S}D^1_n(\alpha_1) \cdots \mathcal{S}D^1_n(\alpha_k)$. Combining these bijections $\mathcal{S}D^k_n$ for $k > 0$ with the bijection $\mathcal{S}D^0_n$ we obtain a length-preserving bijection $\mathcal{S}D^*_n : \mathcal{S}^{\frac{m}{n}}E^* \to \mathcal{S}^m D_n(E)^*$, which induces a unitary

$$U_{m,n} : \ell^2(\mathcal{S}^{\frac{m}{n}}E^*) \to \ell^2(\mathcal{S}^m D_n(E)^*).$$

This $U_{m,n}$ intertwines $\rho_{m/n}$ and $a \mapsto \rho_m(a \circ \mathcal{S}D^0_n)$ and intertwines $\psi_{m,n}$ with $\xi \mapsto \psi_m(\xi \circ \mathcal{S}D^1_n)$. This proves the first statement. Since the unitary $U_{m,n}$ carries $\ell^2(\mathcal{S}^{\frac{m}{n}}E^*\omega)$ to $\ell^2(\mathcal{S}^m D_n(E)^* \mathcal{S}D^0_n(\omega))$ for each $\omega \in \mathcal{S}^{\frac{m}{n}}E^0$, we have

$$\operatorname{Ad}_{U_{m,n}}\Big(\bigoplus_{\omega \in \mathcal{S}^{\frac{m}{n}}E^0} \mathcal{K}(\ell^2(\mathcal{S}^{\frac{m}{n}}E^*\omega)\Big) = \bigoplus_{\omega \in \mathcal{S}^m D_n(E)^0} \mathcal{K}(\ell^2(\mathcal{S}^m D_n(E)^*\omega).$$

Hence $\Theta_{m,n}$ carries the kernel of the quotient map $\mathcal{T}C^*(\mathcal{S}^{\frac{m}{n}}E) \to C^*(\mathcal{S}^{\frac{m}{n}}E)$ to the kernel of the quotient map $\mathcal{T}C^*(\mathcal{S}^m D_n(E)) \to C^*(\mathcal{S}^m D_n(E))$. It follows that $\Theta_{m,n}$ descends to the desired isomorphism $\widetilde{\Theta}_{m,n}$. □

7. Analysis of $\mathcal{T}C^*(\mathcal{S}^m E)$

We now analyse the C^*-algebras $\mathcal{T}C^*(\mathcal{S}^m E)$ and $C^*(\mathcal{S}^m E)$ for integers $m \ge 1$ (both $\mathcal{T}C^*(\mathcal{S}^0 E)$ and $C^*(\mathcal{S}^0 E)$ are described by part (2) of Theorem 6.10). This will complete our analysis of the C^*-algebras $\mathcal{T}C^*(\mathcal{S}^l E)$ and $C^*(\mathcal{S}^l E)$ for rational l.

To analyse $\mathcal{T}C^*(\mathcal{S}^m E)$ we will first establish that the ideal of $\mathcal{T}C^*(\mathcal{S}^m E)$ generated by the image of $C_0(\mathcal{S}E^0 \setminus E^0)$ is isomorphic to $\mathcal{T}C^*(E(1, m+1)) \otimes C_0((0,1))$, and show that $\mathcal{T}C^*(\mathcal{S}^m E)$ itself is a $C(\mathbb{S})$-algebra. We begin with some preliminary structural results.

Given a locally compact Hausdorff space X, we write $C_b(X)$ for the algebra of bounded continuous complex-valued functions on X.

Lemma 7.1. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. The space $C_c(\mathcal{S}^m E^1)$ is a $C_b(\mathcal{S}E^0)$-bimodule with respect to the actions $(a \cdot \xi)(\alpha) = a(r(\alpha))\xi(\alpha)$ and $(\xi \cdot a)(\alpha) = \xi(\alpha)a(s(\alpha))$. For each $\xi \in C_c(\mathcal{S}^m E^1)$ there exists $a \in C_c(\mathcal{S}E^0)$ such that $a \cdot \xi = \xi = \xi \cdot a$.*

Proof. For $a \in C_b(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^m E^1)$, the function $a \cdot \xi$ is the pointwise product of $a \circ r$ and ξ and therefore a continuous function. Since its support is contained in that of ξ it belongs to $C_c(\mathcal{S}^m E^1)$. Similarly $\xi \cdot a \in C_c(\mathcal{S}^m E^1)$. It is routine that these actions make $C_c(\mathcal{S}^m E^1)$ into a $C_b(\mathcal{S}E^0)$-bimodule. For the final assertion, fix $\xi \in C_c(\mathcal{S}^m E^1)$. Since $r, s : \mathcal{S}^m E^1 \to \mathcal{S}E^0$ are continuous, $K := r(\operatorname{supp}(\xi)) \cup s(\operatorname{supp}(\xi)) \subseteq \mathcal{S}E^0$ is compact, so Tietze's theorem yields a function $a \in C_c(\mathcal{S}E^0)$ such that $a|_K \equiv 1$. We then have $a \cdot \xi = \xi = \xi \cdot a$ by definition of the actions of $C_0(\mathcal{S}E^0)$ on $C_c(\mathcal{S}^m E^1)$. □

To analyse the ideal of $\mathcal{T}C^*(\mathcal{S}^m E)$ generated by $C_0(\mathcal{S}E^0 \setminus E^0)$, we first observe that the subgraph of $\mathcal{S}^m E$ with vertex set $\mathcal{S}E^0 \setminus E^0$ is a topological graph in the sense of Katsura.

Lemma 7.2. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Then $(\mathcal{S}E^0 \setminus E^0, \mathcal{S}^m E^1 \setminus E^1, r, s)$ is a topological graph isomorphic to the product $E(1, m+1) \times (0, 1)$.*

Proof. Lemma 3.2 shows that the quotient maps from $E^1 \times [0, 1]$ to $\mathcal{S}E^0$ and from $E^{m+1} \times [0, 1]$ to $\mathcal{S}^m E^1$ restrict range and source preserving homeomorphisms from $E(1, m+1)^0 \times (0, 1)$ to $\mathcal{S}E^0 \setminus E^0$ and from $E(1, m+1)^1 \times (0, 1)$ to $\mathcal{S}^m E^1 \setminus E^1$. □

We can now describe the ideal of $\mathcal{T}C^*(\mathcal{S}^m E)$ generated by $C_0(\mathcal{S}E^0 \setminus E^0)$. In the following proof, given $e \in E^1$ and $g \in C_0((0, 1))$, we denote by $1_e \times g$ the element of $C_0(\mathcal{S}E^0 \setminus E^0) \subseteq C_0(\mathcal{S}E^0)$ given by

$$(1_e \times g)([f, t]) := \delta_{e,f} g(t) \quad \text{for all } f \in E^1 \text{ and } t \in (0, 1),$$

and likewise for $\mu \in E^{m+1}$ and $g \in C_0((0, 1))$, we write $1_\mu \times g$ for the element of $C_0(\mathcal{S}^m E^1 \setminus E^m) \subseteq C_0(\mathcal{S}^m E^1)$ given by

$$(1_\mu \times g)([\nu, t]) := \delta_{\mu,\nu} g(t) \quad \text{for all } \nu \in E^{m+1} \text{ and } t \in (0, 1).$$

Lemma 7.3. *Let E be a row-finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Let J be the ideal of $\mathcal{T}C^*(\mathcal{S}^m E)$ generated by $\rho_m(C_0(\mathcal{S}E^0))$. There is an isomorphism $\kappa_0 : C_0((0, 1)) \otimes \mathcal{T}C^*(E(1, m+1)) \to J$ such that*

$$\kappa_0(g \otimes Q_e) = \pi(1_e \times g) \qquad \text{and} \qquad \kappa_0(g \otimes T_\mu) = \psi(1_\mu \times g)$$

for all $g \in C_0((0, 1))$, all $e \in E(1, m+1)^1 = E^1$ and all $\mu \in E(1, m+1)^1 = E^{m+1}$.

Proof. By Lemma 7.2, we have $(\mathcal{S}E^0 \setminus E^0, \mathcal{S}^m E^1 \setminus E^m, r, s) \cong E(1, m+1) \times (0, 1)$ as topological graphs. Recall from [22] that an s-section in a topological graph F is an open set $U \subseteq F^1$ such that $s : U \to s(U)$ is a homeomorphism onto an open subset of F^0. The collection

$$\mathcal{B} := \big\{\{[\nu, t] : t \in (a, b)\} : \nu \in E(1, m)^1 \text{ and } 0 < a < b < 1\big\}$$

is a basis of open s-sections for the topology on $\mathcal{S}^m E^1 \setminus E^m$. Let $\mathcal{F} := \{1_\nu \times g : \nu \in E(1, m+1)^1 \text{ and } g \in C_0((0, 1))\}$. Then this $\mathcal{F}$ and $\mathcal{B}$ satisfy [22, Equation (4.4)]. Direct computation with basis elements on each $\ell^2(\mathcal{S}^m E_t)$ show that $(\rho_m, \psi_m|_{\operatorname{span} \mathcal{F}})$ satisfy

[22, Equation (4.5)] and [22, Equation (4.6)]. Thus [22, Proposition 4.12] shows that $(\rho_m|_{C_0(\mathcal{S}E^0\setminus E^0)}, \psi_m|_{C_c(\mathcal{S}^mE^1\setminus E^1)})$ is a representation of the topological graph $E(1, m+1) \times (0, 1)$.

The range of $\rho_m|_{C_0(\mathcal{S}E^0\setminus E^0)}$ belongs to J by definition. The image of $\psi_m|_{C_c(\mathcal{S}^mE^1\setminus E^m)}$ belongs to J by the argument of Lemma 7.1. These elements generate J because $C_0(\mathcal{S}E^0 \setminus E^0) \cdot C_c(\mathcal{S}^mE^1)$ is contained in the space $C_d(\mathcal{S}^mE^1 \setminus E^m)$ described at (2.3). Thus [22, Theorem 2.4] shows that there is a surjective homomorphism $(\rho| \times \psi|) : \mathcal{T}C^*(E(1, m + 1) \times (0, 1)) \to J$ such that $(\rho| \times \psi|) \circ i_{E(1,m+1)^0\times(0,1)} = \rho_m|_{C_0(\mathcal{S}E^0\setminus E^0)}$ and $(\rho| \times \psi|) \circ i_{E(1,m+1)^1\times(0,1)} = \psi_m|_{C_c(\mathcal{S}^mE^1\setminus E^m)}$. To see that this homomorphism is injective, recall that J is a subalgebra of $\mathcal{B}(\ell^2(\mathcal{S}^mE^* \setminus E(0, m)^*))$, and observe that

$$\ell^2(\mathcal{S}E^0 \setminus E^0) \subseteq \overline{\{\psi_m(\xi)v : \xi \in C_c(\mathcal{S}^mE^1 \setminus E^m), v \in \ell^2(\mathcal{S}^mE^*)\}}^{\perp}.$$

For $a \in C_0(\mathcal{S}E^0 \setminus E^0)$, the restriction of $\rho_m(a)$ to $\ell^2(\mathcal{S}E^0 \setminus E^0)$ is given by $\rho_m(a)h_\omega = a(\omega)h_\omega$, and so the reduction of ρ_m to this subspace is faithful. Hence [9, Theorem 2.1] shows that $(\rho| \times \psi|)$ is injective.

The argument of [15, Proposition 7.7] shows that $\mathcal{T}C^*(E(1, m+1)\times(0, 1)) \cong C_0((0, 1))\otimes \mathcal{T}C^*(E(1, m+1))$, so we obtain a surjective representation of $C_0((0, 1))\otimes\mathcal{T}C^*(E(1, m+1))$ in J that carries $g \otimes Q_e$ to $\pi(1_e \times g)$ and $g \otimes T_\nu$ to $\psi(1_\nu \times g)$. □

We observe next that the actions of $C_b(\mathcal{S}E^0)$ on $C_c(\mathcal{S}^mE^1)$ induce a central action of $C(\mathbb{S})$ via the surjection $\mathcal{S}E^0 \to \mathbb{S}$ of Lemma 3.4.

Corollary 7.4. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus 0$. There are left and right actions of $C(\mathbb{S})$ on $C_c(\mathcal{S}^mE^1)$ given by $(g \cdot \xi)([\mu, t]) = g(t)\xi([\mu, t]) = (\xi \cdot g)([\mu, t])$ for $g \in C(\mathbb{S})$ and $\xi \in C_c(\mathcal{S}^mE^1)$.*

Proof. The surjection $\varpi : \mathcal{S}E^0 \to \mathbb{S}$ of Lemma 3.4 induces an injection $\varpi^* : C(\mathbb{S}) \to C_b(\mathcal{S}E^0)$ given by $\varpi^*(g)([e, t]) = g(\varpi([e, t])) = g(t)$. So $g \cdot \xi := \varpi^*(g) \cdot \xi$ and $\xi \cdot g := \xi \cdot \varpi^*(g)$ satisfy the formulae given for the desired action. The definition of ϖ shows that $g \cdot \xi = \xi \cdot g$. □

Proposition 7.5. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus 0$. The pair (ρ_m, ψ_m) is a bimodule homomorphism in the sense that $\rho_m(a)\psi_m(\xi) = \psi_m(a \cdot \xi)$ and $\psi_m(\xi)\rho_m(a) = \psi_m(\xi \cdot a)$ for all $a \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^mE^1)$. Writing $\bar\rho_m$ for the extension of ρ_m to $C_b(\mathcal{S}E^0) = \mathcal{M}(C_0(\mathcal{S}E^0))$ and $\varpi_* : C(\mathbb{S}) \to C_b(\mathcal{S}E^0)$ for the homomorphism induced by the map $\varpi : \mathcal{S}E^0 \to \mathbb{S}$ of Lemma 3.4, and writing $\iota_m := \bar\rho_m\circ\varpi_*$, the action of $C(\mathbb{S})$ on $C_c(\mathcal{S}E^1)$ of Corollary 7.4 satisfies*

$$\iota_m(g)\psi_m(\xi) = \psi_m(g \cdot \xi) = \psi_m(\xi)\iota_m(g) \quad \text{for all } g \in C(\mathbb{S}) \text{ and } \xi \in C_c(\mathcal{S}^mE^1).$$

In particular, ι_m is an injective unital inclusion of $C(\mathbb{S})$ in $\mathcal{ZM}(\mathcal{T}C^*(\mathcal{S}^mE))$.

Proof. We just calculate with basis vectors: for $a, b \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^mE^1)$, and for $\alpha \in \mathcal{S}^mE^*$, we have

$$\begin{aligned}\rho_m(a)\psi_m(\xi)\rho_m(b)h_\alpha &= \sum_{\beta\in\mathcal{S}^mE^1r(\alpha)} a(r(\beta))\xi(\beta)b(r(\alpha))h_{\beta\alpha}\\ &= \sum_{\beta\in\mathcal{S}^mE^1r(\alpha)} (a \cdot \xi \cdot b)(\beta)h_{\beta\alpha} = \psi_m(a \cdot \xi \cdot b)h_\alpha.\end{aligned}$$

Taking a such that $a \cdot \xi = \xi$ as in Lemma 7.1 gives $\psi_m(\xi)\rho_m(b) = \psi_m(\xi \cdot b)$. Likewise, taking b such that $\xi \cdot b = \xi$ gives $\rho_m(a)\psi_m(\xi) = \psi_m(a \cdot \xi)$.

Now fix $\xi \in C_c(\mathcal{S}^m E^1)$ and $g \in C(\mathbb{S})$. Choose $a \in C_c(\mathcal{S}E^0)$ such that $a \cdot \xi = \xi = \xi \cdot a$. Then by definition of $\bar{\rho}$,

$$\begin{aligned}\iota_m(g)\psi_m(\xi) &= \iota_m(g)\psi_m(a \cdot \xi) = \bar{\rho}_m(\varpi^*(g))\rho_m(a)\psi_m(\xi)\\ &= \rho_m(\varpi_*(g)a)\psi_m(\xi) = \psi_m((\varpi_*(g)a) \cdot \xi) = \psi_m(g \cdot (a \cdot \xi)) = \psi_m(g \cdot \xi).\end{aligned}$$

Likewise $\psi_m(\xi)\iota_m(g) = \psi_m(\xi \cdot g)$. Corollary 7.4 shows that $g \cdot \xi = \xi \cdot g$, so we obtain $\iota_m(g)\psi_m(\xi) = \psi_m(g \cdot \xi) = \psi_m(\xi)\iota_m(g)$ as claimed.

Since $\psi_m(C_c(\mathcal{S}E^1)$ and $\rho_m(C_0(\mathcal{S}E^0)$ generate $\mathcal{T}C^*(\mathcal{S}^m E)$, we deduce that

$$\iota_m(C(\mathbb{S}))\mathcal{T}C^*(\mathcal{S}^m E) \subseteq \mathcal{T}C^*(\mathcal{S}^m E),$$

and taking adjoints gives $\mathcal{T}C^*(\mathcal{S}^m E)\iota_m(C(\mathbb{S})) \subseteq \mathcal{T}C^*(\mathcal{S}^m E)$ as well. So we can regard ι_m as a homomorphism of $C(\mathbb{S})$ into $\mathcal{M}(\mathcal{T}C^*(\mathcal{S}^m E))$. Since $C(\mathcal{S}E^0)$ is abelian, the elements of $\iota_m(C(\mathbb{S}))$ commute with the elements $\rho_m(a)$, and we have just established that they commute with the elements of $\psi_m(C_c(\mathcal{S}^m E^1)$. Again, since the $\rho_m(a)$ and the $\psi_m(\xi)$ generate $\mathcal{T}C^*(\mathcal{S}^m E)$ we see that ι takes values in $\mathcal{ZM}(\mathcal{T}C^*(\mathcal{S}^m E))$. Finally, the identity function $1 \in C(\mathbb{S}))$ satisfies $1 \cdot \xi = \xi = \xi \cdot 1$ for all $\xi \in C_c(\mathcal{S}^m E^1)$. So we obtain $\iota_m(1)\psi_m(\xi) = \psi_m(1 \cdot \xi) = \psi_m(\xi)$ for all ξ, and clearly $\iota_m(1)\rho_m(a) = \rho_m(a)$ for all a by definition of ι_m and ϖ^*. Hence ι_m is unital. □

The general theory of $C(X)$-algebras (see Section 2.6) now implies that $\mathcal{T}C^*(\mathcal{S}^m E)$ is isomorphic to the algebra of continuous sections of an upper-semicontinuous bundle of C^*-algebras over $\mathbb{S}$.

Notation 7.6. Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. For each $t \in \mathbb{S}$ we write J_t for the ideal of $\mathcal{T}C^*(\mathcal{S}^m E)$ generated by $\iota(\{g \in C(\mathbb{S}) : g(t) = 0\})$. Following the standard conventions for $C(X)$-algebras, we then write $\mathcal{T}C^*(\mathcal{S}^m E)_t$ for the quotient $\mathcal{T}C^*(\mathcal{S}^m E)/J_t$. For each $a \in \mathcal{T}C^*(\mathcal{S}^m E)$, we write $\gamma_a : \mathbb{S} \to \bigsqcup_{t\in\mathbb{S}} \mathcal{T}C^*(\mathcal{S}^m E)_t$ for the section given by $\gamma_a(t) := a + J_t$.

We first show that for $t \in (0, 1)$, the fibre $\mathcal{T}C^*(\mathcal{S}^m E)_t$ is a copy of $\mathcal{T}C^*(E(1, m))$, and describe standard representatives in $\mathcal{T}C^*(\mathcal{S}^m E)$ of its canonical generators.

The following notation will be helpful for the next few results.

Notation 7.7. Let E be a locally finite graph with no sources. By Remark 6.5 the space $\mathcal{S}^m E^*_0$ can be identified with $E(0, m)^*$ via the map $[\mu] \mapsto \mu$. Let π_0 be the representation of $\mathcal{T}C^*(E(0, m))$ on $\ell^2(\mathcal{S}^m E^*_0)$ obtained from the path-space representation of $\mathcal{T}C^*(E(0, m))$ and this identification; so

$$\pi_0(Q_v)h_{[\nu]} = \delta_{v,r(\nu)}h_{[\nu]} \quad \text{and} \quad \pi_0(T_\mu)h_{[\nu]} = \delta_{s(\mu),r(\nu)}h_{[\mu\nu]}.$$

For $t \in \mathbb{S} \setminus \{0\}$, the set $\mathcal{S}^m E^*_t$ can be identified with $E(1, m + 1)^*$ via the map $\mathcal{S}^m E^k_t \ni [\mu, t] \mapsto (\mu_1 \cdots \mu_{m+1})(\mu_{m+1} \cdots \mu_{2m+1}) \cdots (\mu_{(k-1)m}\mu_{km+1}) \in E(1, m + 1)^k$ for $k \in \mathbb{N}$, $\mu \in E^{km+1}$ and $t \in (0, 1)$. We write π_t for the representation of $\mathcal{T}C^*(E(1, m+1))$ on $\ell^2(\mathcal{S}^m E^*_t)$ obtained from this identification and the path-space representation of $\mathcal{T}C^*(E(1, m + 1))$. So

$$\pi_t(Q_e)h_{[\mu,t]} = \delta_{e,\mu_1}h_{[\mu,t]} \quad \text{and} \quad \pi_t(T_\mu)h_{[\nu,t]} = \delta_{\mu_{|\mu|},\nu_1}h_{[\mu\nu_2\cdots\nu_{|\nu|},t]}.$$

Lemma 7.8. *Let E be a locally finite graph with no sources, fix $m \in \mathbb{N} \setminus \{0\}$, and take $t \in [0, 1)$. If $a, a' \in C_0(\mathcal{S}E^0)$ satisfy $a|_{\mathcal{S}E^0_t} = a'|_{\mathcal{S}E^0_t}$, then $\pi_m(a) + J_t = \pi_m(a') + J_t$ in $\mathcal{T}C^*(\mathcal{S}^m E)_t$. If $\xi, \xi' \in C_c(\mathcal{S}^m E^1)$ satisfy $\xi|_{C_c(\mathcal{S}^m E^1_t)} = \xi'|_{C_c(\mathcal{S}^m E^1_t)}$, then $\psi_m(\xi) + J_t = \psi_m(\xi') + J_t$ in $\mathcal{T}C^*(\mathcal{S}^m E)_t$.*

Proof. Let d be the quotient metric on $\mathbb{S}$ induced by the usual metric on $\mathbb{R}$. For each n, fix a function $f_n \in C_0(\mathbb{S} \setminus \{t\})$ such that $0 \le f_n \le 1$ and $f_n(s) = 1$ whenever $d(s, t) \ge 1/n$.

For the first statement, note that $C_0(\mathcal{S}E^0 \setminus \{[e, t] : e \in E^1\})$ belongs to the ideal generated by the $\varpi_*(f_n)$, and so $\rho(C_0(\mathcal{S}E^0 \setminus \mathcal{S}E^0_t)) \subseteq J_t$. Since $a - a' \in C_0(\mathcal{S}E^0 \setminus \mathcal{S}E^0_t)$ this proves the first statement.

For the second statement, let

$$N := |\{\mu \in E(1, m+1)^1 : \max\{|\xi([\mu, s])|, |\xi'([\mu, s])|\} > 0 \text{ for some } s \in [0, 1)\}|.$$

Since ξ and ξ' have compact support, N is finite. Fix $\varepsilon > 0$. The set $X_\varepsilon := r(\{\alpha \in \mathcal{S}^m E^1 : |(\xi - \xi')(\alpha)| \ge \varepsilon/N\})$ is a compact subset of $\mathcal{S}E^0 \setminus \mathcal{S}E^0_t$ and so there exists $n > 0$ such that $f_n|_{X_\varepsilon} \equiv 1$. For this n, we have

$$\big\|(\xi - \xi') - f_n \cdot (\xi - \xi')\|\infty \le \varepsilon/N,$$

and $\operatorname{supp}\big((\xi - \xi') - f_n \cdot (\xi - \xi')\big) \subseteq \operatorname{supp}(\xi) \cup \operatorname{supp}(\xi')$.

For any $\eta \in C_c(\mathcal{S}^m E^1)$, we have, using the representations π_t of Notation 7.7,

$$\begin{aligned}
\|\psi_m(\eta)\| &= \sup_{s \in \mathbb{S}} \|\psi_m(\eta)|_{\ell^2(\mathcal{S}^m E^*_s)}\| \\
&\le \max\Big\{ \sup_{s \in (0,1)} \Big\| \sum_{\mu \in E^{m+1}} \eta([\mu, s]) \pi_s(T_\mu) \Big\|, \Big\| \sum_{\nu \in E^m} \eta([\nu]) \pi_0(T_\nu) \Big\| \Big\} \\
&\le \sup_{s \in \mathbb{S}} \sum_{\alpha \in \mathcal{S}^m E^1_s} |\eta(\alpha)|.
\end{aligned}$$

Applying this to $\eta = (\xi - \xi') - f_n \cdot (\xi - \xi')$ and using the definition of N, we deduce that $\big\|\psi_m\big((\xi - \xi') - f_n \cdot (\xi - \xi')\big)\big\| \le \varepsilon$. Since the f_n all vanish at t, it follows that $\xi - \xi' \in J_t$ as claimed. □

We can now prove that for each $t \in (0, 1)$, the corresponding fibre $\mathcal{T}C^*(\mathcal{S}^m E)_t$ is isomorphic to $\mathcal{T}C^*(E(1, m+1))$.

Proposition 7.9. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Take $t \in (0, 1)$. For each $e \in E^1$, fix a function $a_{e,t} \in C_0(\mathcal{S}E^0, [0, 1])$ such that $\operatorname{supp}(a_{e,t}) \subseteq \{[e, s] : 0 < s < 1\}$ and $a_{e,t}([e, t]) = 1$. For each $\mu \in E^{m+1}$, fix a function $\xi_{\mu,t} \in C_c(\mathcal{S}^m E^1)$ such that $\operatorname{supp}(\xi_{\mu,t}) \subseteq \{[\mu, s] : 0 < s < 1\}$ and $\xi_{\mu,t}([\mu, t]) = 1$. There is an isomorphism $\theta_t : \mathcal{T}C^*(E(1, m+1)) \to \mathcal{T}C^*(\mathcal{S}^m E)_t$ such that*

$$\theta_t(Q_e) = \rho_m(a_{e,t}) + J_t \quad \text{for all } e \in E(1, m+1)^0 = E^1,$$

and such that

$$\theta_t(T_\mu) = \psi_m(\xi_{\mu,t}) + J_t \quad \text{for all } \mu \in E(1, m+1)^1 = E^{m+1}.$$

Proof. Lemma 7.8 shows that the elements $\rho_m(a_{e,t}) + J_t$ and $\psi_m(\xi_{ef,t}) + J_t$ generate $\mathcal{T}C^*(\mathcal{S}^m E)_t$, so it suffices to construct an injective homomorphism θ_t satisfying the given formulae.

For this, define $q_e := \rho_m(a_{e,t}) + J_t$ for each $e \in E^1$ and $t_\mu := \psi_m(\xi_{\mu,t}) + J_t$ for each $\mu \in E^{m+1}$. We will show that (q, t) is a Toeplitz–Cuntz–Krieger $E(1, m+1)$-family.

Since $a_{e,t}^2([f,t]) = \overline{a_{e,t}}([f,t]) = a_{e,t}([f,t])$ for all f, Lemma 7.8 shows that the q_e are projections. We have $a_{e,t}a_{f,t} = 0$ in $C_0(\mathcal{S}E^0)$ for $e \neq f$, so the q_e are mutually orthogonal.

Fix $\mu \in E^{m+1}$. Define $a'_{\mu,t} : \mathcal{S}E^0 \to \mathbb{C}$ by

$$a'_{\mu,t}([e,s]) = \begin{cases} |\xi_{\mu,t}(s)|^2 & \text{if } \mu_{m+1} = e \\ 0 & \text{otherwise.} \end{cases}$$

Since $\operatorname{supp}(\xi_{\mu,t})$ is a compact subset of $\{[\mu,s] : 0 < s < 1\}$, we have $a'_{\mu,t} \in C_c(\mathcal{S}E^0)$, and by construction, $a'_{\mu,t}$ and $a_{\mu_{m+1},t}$ agree at $[g,t]$ for every $g \in E^1$. So Lemma 7.8 implies that $q_{\mu_{m+1}} = \rho_m(a'_{\mu,t}) + J_t$. Using the representations π_t of Notation 7.7, we have

$$\begin{aligned}\psi_m(\xi_{\mu,t})^*\psi(\xi_{\mu,t}) &= \Big(\sum_{\alpha,\beta\in E^m} \overline{\xi_{\mu,t}([\alpha])}\xi_{\mu,t}([\beta])\pi_0(T_\alpha^* T_\beta)\Big) \\ &\qquad \oplus \bigoplus_{0<s<1} \sum_{\eta,\zeta\in E^{m+1}} \overline{\xi_{\mu,t}}([\eta,s])\xi_{\mu,t}([\zeta,s])\pi_s(T_\eta^* T_\zeta) \\ &= 0 \oplus \bigoplus_{0<s<1} |\xi_{\mu,t}([\mu,s])|^2 \tilde{\pi}^\infty(Q_{\mu_{m+1}}) = \rho_m(a'_{\mu,t}).\end{aligned}$$

Thus $t_\mu^* t_\mu = \rho(a'_{\mu,t}) + J_t = \rho(a_{\mu_{m+1},t}) + J_t = q_{s_{1,m+1}(\mu)}$.

Now fix $e \in E^1$. For each $\mu \in s(e)E^m$, define $\xi'_{e\mu,t}$ by $\xi'_{e\mu,t}([\nu,s]) = \delta_{e\mu,\nu}\sqrt{a_{e,t}([e,s])}$ for $\nu \in E^{m+1}$ and $s \in [0,1)$. Each $\xi'_{e\mu,t} \in C_c(\mathcal{S}^m E^1)$ because $a_{e,t}$ is supported on $\{[e,s] : 0 < s < 1\}$. Lemma 7.8 shows that $t_{e\mu} = \psi_m(\xi'_{e\mu,t})$ for each $\mu \in E(1,m+1)^1$. Arguing as above, we see that

$$\begin{aligned}\sum_{e\mu\in eE(1,m)^1} t_{e\mu}t_{e\mu}^* &= \sum_{\mu\in s(e)E^m} \psi(\xi'_{e\mu,t})\psi(\xi'_{e\mu,t})^* \\ &= 0 \oplus \bigoplus_{0<s<1} \sum_{\mu\in s(e)E^m} |\xi'_{e\mu,t}(e\mu,s)|^2 \pi_s(T_{e\mu}T_{e\mu}^*) \\ &= 0 \oplus \bigoplus_{0<s<1} a_{e,t}(e,s) 1_{\overline{\operatorname{span}}\{h_{[\mu,s]}:\mu\in eE(1,m+1)^*\setminus\{e\}\}}. \qquad (7.1)\end{aligned}$$

Also,

$$\rho_m(a_{e,t}) = 0 \oplus \bigoplus_{0<s<1} a_{e,t}(e,s) 1_{\overline{\operatorname{span}}\{h_{[\mu,s]}:\mu\in eE(1,m+1)^*\}}. \qquad (7.2)$$

We deduce that $\rho_m(a_{e,t}) > \sum_{e\mu\in eE(1,m+1)^1} \psi_m(\xi'_{e\mu,t})\psi_m(\xi'_{e\mu,t})^*$. In particular, in the quotient, $q_e \geq \sum_{e\mu\in eE(1,m+1)^1} t_{e\mu}t_{e\mu}^*$.

So (q,t) is a Toeplitz–Cuntz–Krieger $E(1,m+1)$-family as claimed. The universal property of $\mathcal{T}C^*(E(1,m+1))$ therefore yields a homomorphism $\theta_t : \mathcal{T}C^*(E(1,m+1)) \to \mathcal{T}C^*(\mathcal{S}^m E)_t$ such that $\theta_t(Q_e) = \rho_m(a_{e,t}) + J_t$ and $\theta_t(T_\mu) = \psi_m(\xi_{\mu,t}) + J_t$.

It remains to prove that θ_t is injective. Since J_t is contained in the kernel of the restriction map $x \mapsto x|_{\ell^2(\mathcal{S}^m E_t^*)}$ on $\mathcal{T}C^*(\mathcal{S}^m E)_t$, we see that (with the functions ξ'_{ef} used in the calculation (7.1) above), each

$$\|q_e - \sum_{e\mu\in eE(1,m+1)^1} t_{e\mu}t_{e\mu}^*\| \geq \Big\|\Big(\rho_m(a_{e,t}) - \sum_{e\mu\in eE(1,m+1)^1} \psi_m(\xi'_{e\mu,t})\psi_m(\xi'_{e\mu,t})\Big)\Big|_{\ell^2(\mathcal{S}^m E_t^*)}\Big\|.$$

The calculations (7.1) and (7.2) therefore show that

$$\Big\| q_e - \sum_{e\mu\in eE(1,m+1)^1} t_{e\mu}t_{e\mu}^* \Big\| \geq \|1_{\mathbb{C}h_e}\| = 1.$$

So each $q_e - \sum_{e\mu\in eE(1,m+1)^1} t_{e\mu}t_{e\mu}^* \neq 0$, and the uniqueness theorem [9, Theorem 4.1] shows that θ_t is injective. □

Corollary 7.10. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N}\setminus\{0\}$. Take $t \in (0,1)$. Let $U_t : \ell^2(E(1,m+1)^*) \to \ell^2(\mathcal{S}^m E_t^*)$ be the unitary of Remark 6.5. Let $\pi : \mathcal{T}C^*(E(1,m+1)) \to \mathcal{B}(\ell^2(E(1,m+1)^*))$ be the path-space representation. Then the map $a+J_t \mapsto \pi^{-1}(U_t^* a|_{\ell^2\mathcal{S}^m E_t^*} U_t)$ is an isomorphism of $\mathcal{T}C^*(\mathcal{S}^m E)_t$ onto $\mathcal{T}C^*(E(1,m+1))$.*

Proof. Consider the inverse θ_t^{-1} of the isomorphism described in Proposition 7.9. It is straightforward to check that for $a \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^m E^1)$, we have

$$\theta_t^{-1}(\rho_m(a) + J_t) = \sum_{e\in E^1} a([e,t])Q_e = \pi^{-1}(U_t^*\rho_m(a)|_{\ell^2(\mathcal{S}^m E_t^*)}U_t)$$

and

$$\theta_t^{-1}(\psi_m(\xi) + J_t) = \sum_{\mu\in E(1,m+1)^1} \xi([\mu,t])T_{ef} = \pi^{-1}(U_t^*\psi_m(\xi)|_{\ell^2(\mathcal{S}E_t^*)}U_t).$$

Since the elements $\rho_m(a) + J_t$ and $\psi_m(\xi) + J_t$ generate $\mathcal{T}C^*(\mathcal{S}^m E)_t$, it follows that $x \mapsto \pi^{-1}(U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)}U_t)$ agrees with θ_t^{-1}, so is an isomorphism as claimed. □

We must now describe the fibre $\mathcal{T}C^*(\mathcal{S}^m E)_0$. The idea is that for $a \in \mathcal{T}C^*(\mathcal{S}^m E^0)$, using the unitaries $U_t : \ell^2(E(1,m+1)^*) \to \ell^2(\mathcal{S}^m E_t^*)$ of Remark 6.5, the function $t \mapsto U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)}U_t$ from $(0,1)$ to $\mathcal{B}(\ell^2(E(1,m+1)^*))$ converges in norm as $t \to 0$ and as $t \to 1$, and the limits $\varepsilon_0(a)$ and $\varepsilon_1(a)$ belong to the image of $\mathcal{T}C^*(E(1,m+1))$ in its path-space representation. We use these limits to construct an injective homomorphism of $\mathcal{T}C^*(\mathcal{S}^m E)_0$ into $\mathcal{T}C^*(E(1,m+1)) \oplus \mathcal{T}C^*(E(1,m+1))$.

Lemma 7.11. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N}\setminus\{0\}$. For $t \in (0,1)$, let $U_t : \ell^2(E(1,m+1)^*) \to \ell^2(\mathcal{S}^m E_t^*)$ be the unitary of Remark 6.5. Let $\pi : \mathcal{T}C^*(E(1,m+1)) \to \mathcal{B}(\ell^2(E(1,m+1)^*))$ be the path-space representation. For $a \in C_0(\mathcal{S}E^0)$ we have*

$$\lim_{t\searrow 0} U_t^*\rho_m(a)|_{\ell^2(\mathcal{S}^m E_t^*)}U_t = \sum_{e\in E^1} a([r(e)])\pi(Q_e).$$

For $\xi \in C_c(\mathcal{S}^m E^1)$, we have

$$\lim_{t\searrow 0} U_t^*\psi_m(\xi)|_{\ell^2(\mathcal{S}^m E_t^*)}U_t = \sum_{\mu\in E^m, e\in s(\mu)E^1} \xi[\mu]\pi(T_{\mu e}).$$

For $a \in \mathcal{T}C^*(\mathcal{S}^m E)$, the limit $\lim_{t\searrow 0} U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)}U_t$ exists and belongs to $\pi(\mathcal{T}C^*(E(1,m+1)))$, and $\varepsilon_0 : a \mapsto \pi^{-1}(\lim_{t\searrow 0} U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)}U_t)$ is a homomorphism from $\mathcal{T}C^*(\mathcal{S}^m E)$ to $\mathcal{T}C^*(E(1,m+1))$.

Proof. Fix $a \in C_0(\mathcal{S}E^0)$. Fix $\varepsilon > 0$, and let

$$F := \{e \in E^1 : |a([e,t])| \geq \varepsilon/2 \text{ for some } t \in [0,1]\}.$$

Then F is finite, and for each $e \in F$ there exists $\delta_e > 0$ such that $0 < t < \delta_e \implies |a([e,t]) - a([e,0])| < \varepsilon$. Let $\delta = \min_{e\in F}\delta_e$. Then for $e \in E^1$ and $0 < t < \delta$, if

$e \in F$ then $|a([e,t]) - a([e,0])| < \varepsilon$ by choice of δ and if $e \notin F$, then $|a([e,t]) - a([e,0])| \leq |a([e,t])| + |a([e,0])| < \varepsilon$ by choice of F. By definition of ρ_m, we have $U_t^* \rho_m(a)|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \sum_{e \in E^1} a([e,t])\pi(Q_e)$. Since $[e,0] = [r(e)]$ for each e and since each $\|Q_e\| = 1$ it follows that

$$\lim_{t \searrow 0} U_t^* \rho_m(a)|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \sum_{e \in E^1} a([r(e)])\pi(Q_e). \tag{7.3}$$

Now fix $\xi \in C_c(\mathcal{S}^m E^1)$. Let $F = \{\mu \in E^{m+1} : \xi([\mu,t]) \neq 0 \text{ for some } t \in [0,1]\}$. Fix $\varepsilon > 0$. For each $\mu \in F$ there exists $\delta_\mu > 0$ such that $0 < t < \delta_\mu$ implies $|\xi([\mu,t]) - \xi([\mu,0])| < \varepsilon/|F|$. Let $\delta := \min_{\mu \in F} \delta_\mu$. Fix $t \in (0,\delta)$. We have

$$\begin{aligned}
\Big\| U_t^* \psi_m(\xi)|_{\ell^2(\mathcal{S}^m E_t^*)} U_t &- \sum_{\mu \in E^m, e \in s(\mu)E^1} \xi[\mu]\pi(T_{\mu e}) \Big\| \\
&= \Big\| \sum_{\mu e \in F} (\xi[\mu e, t] - \xi([\mu]))\pi(T_{\mu e}) \Big\| \\
&\leq \sum_{\mu e \in F} |\xi[\mu e, t] - \xi([\mu])| \|\pi(T_\mu)\| < \varepsilon
\end{aligned}$$

since each $\|\pi(T_\mu)\| = 1$. Hence

$$\lim_{t \searrow 0} U_t^* \psi_m(\xi)|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \sum_{\mu \in E^m, e \in s(\mu)E^1} \xi[\mu]\pi(T_{\mu e}). \tag{7.4}$$

For the final statement, first consider a finite linear combination $x = \sum_i \alpha_{i,1}\alpha_{i,2} \cdots \alpha_{i,k_i}$ where each $\alpha_{i,j} \in \rho_m(C_0(\mathcal{S}E^0)) \cup \psi_m(C_c(\mathcal{S}^m E^1)) \cup \psi_m(\mathcal{S}^m E^1)^*$. By the first two statements, for each i,j we have $\lim_{t \searrow 0} U_t^* \alpha_{i,j}|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \pi(\beta_{i,j})$ for some $\beta_{i,j} \in \mathcal{T}C^*(E(1,m+1))$, and it follows that $\lim_{t \searrow 0} U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \pi\big(\sum_i \beta_{i,1} \dots \beta_{i,k_j}\big) \in \pi(\mathcal{T}C^*(E(1,m+1)))$.

Now fix $x \in \mathcal{T}C^*(\mathcal{S}^m E)$. Fix $\varepsilon > 0$. Fix a linear combination $a = \sum_i \alpha_{i,1}\alpha_{i,2} \cdots \alpha_{i,k_i}$ where each $\alpha_{i,j} \in \rho_m(C_0(\mathcal{S}E^0)) \cup \psi_m(C_c(\mathcal{S}^m E^1)) \cup \psi_m(\mathcal{S}^m E^1)^*$ such that $\|a - x\| < \varepsilon/4$. Then in particular, $\|(a-x)|_{\ell^2(\mathcal{S}^m E_t^*)}\| < \varepsilon/4$ for all $t \in (0,1)$. By the preceding paragraph, $a_0 := \lim_{t \searrow 0} U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)} U_t$ exists and belongs to $\mathcal{T}C^(E(1,m+1))$, so there exists $\delta > 0$ such that $\|U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)} U_t - a_0\| < \varepsilon/4$ for all $t < \delta$. In particular, there exists $\delta > 0$ such that $0 < s,t < \delta$ implies

$$\begin{aligned}
&\|U_s^* x|_{\ell^2(\mathcal{S}^m E_s^*)} U_s - U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t\| \\
&\quad \leq \|U_s^*(x|_{\ell^2(\mathcal{S}^m E_s^*)} - a|_{\ell^2(\mathcal{S}^m E_s^*)} U_s)\| + \|U_s^* a|_{\ell^2(\mathcal{S}^m E_s^*)} U_s - a_0\| \\
&\qquad + \|a_0 - U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)} U_t\| + \|U_t^*(a|_{\ell^2(\mathcal{S}^m E_t^*)} - x|_{\ell^2(\mathcal{S}^m E_t^*)}) U_t\| < \varepsilon.
\end{aligned}$$

Hence $\big(U_{1/n}^* x|_{\ell^2(\mathcal{S}^m E_{1/n}^*)} U_{1/n}\big)_{n=1}^\infty$ is a Cauchy sequence, and therefore converges to some $x_0 \in \pi(\mathcal{T}C^*(E(1,m+1)))$.

We show that $\lim_{t \searrow 0} U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = x_0$. Fix $\varepsilon > 0$. Using that $U_{1/n}^* x|_{\ell^2(\mathcal{S}^m E_{1/n}^*)} U_{1/n} \to x_0$ and also the preceding paragraph, we can choose $\delta > 0$ such that $\|U_{1/n}^* x|_{\ell^2(\mathcal{S}^m E_{1/n}^*)} U_{1/n} - x_0\| < \varepsilon/2$ whenever $n > \delta^{-1}$, and such that $\|U_s^* x|_{\ell^2(\mathcal{S}^m E_s^*)} U_s - U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t\| < \varepsilon/2$ whenever $s,t < \delta$. In particular, for $t < \delta$, and any choice of $n > \delta^{-1}$, we have

$$\begin{aligned}
&\|U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t - x_0\| \\
&\quad \leq \|U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t - U_{1/n}^* x|_{\ell^2(\mathcal{S}^m E_{1/n}^*)} U_{1/n}\| + \|U_{1/n}^* x|_{\ell^2(\mathcal{S}^m E_{1/n}^*)} U_{1/n} - x_0\| < \varepsilon.
\end{aligned}$$

Hence $U_t^* x|_{\ell^2(\mathcal{S}^m E_t^*)} U_t \to x_0 \in \pi(\mathcal{T}C^*(E(1, m+1)))$ as claimed.

Since π is injective, we deduce that the map ε_0 exists. It is a homomorphism because each $a \mapsto a|_{\ell^2(\mathcal{S}^m E_t^*)}$ is a homomorphism and the algebraic operations in $\mathcal{T}C^*(\mathcal{S}^m E)$ are continuous. □

Lemma 7.12. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. For $t \in (0,1)$, let $U_t : \ell^2(E(1, m+1)^*) \to \ell^2(\mathcal{S}^m E_t^*)$ be the unitary of Remark 6.5. Let $\pi : \mathcal{T}C^*(E(1, m+1)) \to \mathcal{B}(\ell^2(E(1, m+1)^*))$ be the path-space representation. For $a \in C_0(\mathcal{S}E^0)$ we have*

$$\lim_{t \nearrow 1} U_t^* \rho_m(a)|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \sum_{e \in E^1} a([s(e)])\pi(Q_e).$$

For $\xi \in C_c(\mathcal{S}^m E^1)$, we have

$$\lim_{t \nearrow 1} U_t^* \psi_m(\xi)|_{\ell^2(\mathcal{S}^m E_t^*)} U_t = \sum_{\mu \in E^m, e \in E^1 r(\mu)} \xi[\mu]\pi(T_{e\mu}).$$

For $a \in \mathcal{T}C^*(\mathcal{S}^m E)$, the limit $\lim_{t \nearrow 1} U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)} U_t$ exists and belongs to $\pi(\mathcal{T}C^*(E(1, m+1)))$, and $\varepsilon_1 : a \mapsto \pi^{-1}(\lim_{t \nearrow 1} U_t^* a|_{\ell^2(\mathcal{S}^m E_t^*)} U_t)$ is a homomorphism from $\mathcal{T}C^*(\mathcal{S}^m E)$ to $\mathcal{T}C^*(E(1, m+1))$.

Proof. The proof is essentially identical to that of Lemma 7.11. □

Proposition 7.13. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. There is an injective homomorphism $\eta : \mathcal{T}C^*(\mathcal{S}^m E)_0 \to \mathcal{T}C^*(E(1, m+1)) \oplus \mathcal{T}C^*(E(1, m+1))$ such that, for any $a \in C_0(\mathcal{S}E^0)$ we have*

$$\eta(\rho_m(a)_0) = \sum_{v \in E^0} a([v])\Big(\Big(\sum_{e \in E^1 v} Q_e\Big) \oplus \Big(\sum_{f \in vE^1} Q_f\Big)\Big),$$

and such that for any $\xi \in C_c(\mathcal{S}^m E^1)$ we have

$$\eta(\psi_m(\xi)_0) = \sum_{\mu \in E^m} \xi([\mu])\Big(\Big(\sum_{e \in E^1 r(\mu)} T_{e\mu}\Big) \oplus \Big(\sum_{f \in s(\mu)E^1} T_{\mu f}\Big)\Big).$$

Proof. Let $\varepsilon_0, \varepsilon_1 : \mathcal{T}C^*(\mathcal{S}^m E) \to \mathcal{T}C^*(E(1, m+1))$ be the homomorphisms of Lemmas 7.11 and 7.12. Since $\varepsilon_0, \varepsilon_1$ vanish on $\rho_m(C_0(\mathcal{S}E^0 \setminus E^0))$, they descend to homomorphisms $\tilde{\varepsilon}_0, \tilde{\varepsilon}_1 : \mathcal{T}C^*(\mathcal{S}^m E)_0 \to \mathcal{T}C^*(E(1, m+1))$. The homomorphism $\eta := \tilde{\varepsilon}_0 \oplus \tilde{\varepsilon}_1$ satisfies the formulae above, so it suffices to show that this homomorphism η is injective.

For this, fix $x \in \mathcal{T}C^*(\mathcal{S}^m E)$ such that $\eta(x_0) = 0$; so $\varepsilon_0(x) = \varepsilon_1(x) = 0$. We must show that $x_0 = 0$. We have

$$(7.5) \qquad 0 = \|\varepsilon_0(x)\| = \lim_{t \searrow 0} \|x|_{\ell^2(\mathcal{S}^m E_t^*)}\|, \qquad \text{and} \qquad 0 = \|\varepsilon_1(x)\| = \lim_{t \nearrow 1} \|x|_{\ell^2(\mathcal{S}^m E_t^*)}\|.$$

Corollary 7.10 implies that $\|x|_{\ell^2(\mathcal{S}^m E_t^*)}\| = \|x_t\|$ for $x \in \mathcal{T}C^*(\mathcal{S}^m E)$ and $t \in (0,1)$, and so (7.5) implies that $\lim_{t \to 0} \|x_t\| = 0$. It now follows from the properties of upper semi-continuous C^*-bundles—see equation 2.4—that $x_t \to 0 \in C^*(\mathcal{S}E)_0$. Since $t \mapsto x_t$ is a continuous section, we deduce that $x_0 = 0$. □

We will show that the image of η is isomorphic to $\mathcal{T}C^*(E(1, m+1)) \oplus \mathcal{T}C^*(E(0, m))$ provided that enough vertices in E admit at least two edges.

Proposition 7.14. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Suppose that for every $v \in E^0$ there exist $n \geq 1$ and $\mu \in E^{nm}v$ such that $|E^1 r(\mu)| \geq 2$. Let $\eta : \mathcal{T}C^*(\mathcal{S}^m E)_0 \to \mathcal{T}C^*(E(1, m+1)) \oplus \mathcal{T}C^*(E(1, m+1))$ be the homomorphism of Proposition 7.13. Then the range of η is $\mathcal{T}C^*(E(1, m+1)) \oplus \jmath_{1,m+1}(\mathcal{T}C^*(E(0, m)))$, and $(\mathrm{id} \oplus \jmath_{1,m+1}^{-1}) \circ \eta$ is an isomorphism of $\mathcal{T}C^*(\mathcal{S}^m E)_0$ onto $\mathcal{T}C^*(E(1, m+1)) \oplus \mathcal{T}C^*(E(0, m))$.*

Proof. By Proposition 7.13, we just need to show that the range of η is $\mathcal{T}C^*(E(1, m+1)) \oplus \jmath(\mathcal{T}C^*(E(0, m)))$.

For each $v \in E^0$, let $w_v := \sum_{e \in E^1 v} Q_e \in \mathcal{T}C^*(E(1, m+1))$, and let $x_v := \sum_{f \in vE^1} Q_f \in \mathcal{T}C^*(E(1, m+1))$. For each $\mu \in E^m$, let $y_\mu := \sum_{e \in E^1 r(\mu)} T_{e\mu} \in \mathcal{T}C^*(E(1, m+1))$ and let $z_\mu := \sum_{f \in s(\mu)E^1} T_{\mu f} \in \mathcal{T}C^*(E(1, m+1))$.

We first show that the elements w_v and y_μ generate $\mathcal{T}C^*(E(1, m+1))$. For this, let

$$A := C^*(\{w_v : v \in E^0\} \cup \{y_\mu : \mu \in E^m\}) \subseteq \mathcal{T}C^*(E(1, m+1)).$$

We must show that $\mathcal{T}C^*(E(1, m+1)) \subseteq A$. Fix $\mu \in E^m$. For $e, f \in E^1 r(\mu)$, we have $T^*_{e\mu} T_{f\mu} = \delta_{e,f} Q_{\mu_m}$. Therefore

$$y^*_\mu y_\mu = \sum_{e,f \in E^1 r(\mu)} T^*_{e\mu} T_{f\mu} = |E^1 r(\mu)| Q_{\mu_m}. \tag{7.6}$$

Since E has no sinks, $|E^1 r(\mu)| \neq 0$, so $Q_{\mu_m} \in A$. Again since E has no sinks, for each $e \in E^1$, the set $E^{m-1} r(e)$ is nonempty, so for any $e \in E^1$, we have $Q_e = y^*_{\lambda e} y_{\lambda e} \in A$ for any $\lambda \in E^{m-1} r(e)$. Since the Q_e are mutually orthogonal projections, for $e \in E^1$ and $\mu \in s(e)E^m$ we have

$$T_{e\mu} = Q_e T_{e\mu} = Q_e \sum_{f \in E^1 r(\mu)} T_{f\mu} = Q_e y_\mu \in A.$$

We have now established that all the generators of $\mathcal{T}C^*(E(1, m+1))$ belong to A, and so $\mathcal{T}C^*(E(1, m+1)) \subseteq A$.

Next note that we have $x_v = \jmath_{1,m+1}(Q_v)$ for $v \in E^0$ and $z_\mu = \jmath_{1,m+1}(T_\mu)$ for $\mu \in E^m$, so (x, z) is a Toeplitz–Cuntz–Krieger $E(0, m)$-family, and $C^*(\{x_v : v \in E^0\} \cup \{z_\mu : \mu \in E^m\}) = \jmath_{1,m+1}(\mathcal{T}C^*(E(0, m)))$.

We show next that $(0, x_v) \in \eta(\mathcal{T}C^*(\mathcal{S}E)_0)$ for each $v \in E^0$. First, fix $v \in E^0$ and $\mu \in E^m$ satisfying $|E^1 r(\mu)| \geq 2$. Equation (7.6) and that (x, z) is a Toeplitz–Cuntz–Krieger family show that

$$\begin{aligned}\eta(\mathcal{T}C^*(\mathcal{S}E)_0) &\ni ((y_\mu, z_\mu)^*(y_\mu, z_\mu))^2 - (y_\mu, z_\mu)^*(y_\mu, z_\mu) \\ &= (|E^1 r(\mu)|^2 Q_{\mu_m}, x_v) - (|E^1 r(\mu)| Q_{\mu_m}, x_v) \\ &= \big((|E^1 r(\mu)|^2 - |E^1 r(\mu)|) Q_{\mu_m}, 0\big).\end{aligned}$$

Since $|E^1 r(\mu) \geq 2|$, we have $|E^1 r(\mu)|^2 - |E^1 r(\mu)| > 0$, and so $(Q_{\mu_m}, 0) \in \eta(\mathcal{T}C^*(\mathcal{S}E)_0)$. We then obtain

$$\eta(\mathcal{T}C^*(\mathcal{S}E)_0) \ni (y_e, z_e)^*(y_e, z_e) - |E^1 r(\mu)|(Q_{\mu_m}, 0) = (0, x_v).$$

Now take any $v \in E^0$. By hypothesis, there exists $n \in \mathbb{N}$ and $\mu \in E^{nm} r(e)$ such that $|E^1 r(\mu)| \geq 2$. Write $\mu = \mu_1 \cdots \mu_n$ where each $\mu_i \in E^m$. We have

$$(0, x_{s(\mu)}) = (y_{\mu_n}, z_{\mu_n})^* \cdots (y_{\mu_2}, z_{\mu_2})^*(0, x_{s(\mu_1)})(y_{\mu_2}, z_{\mu_2}) \cdots (y_{\mu_n}, z_{\mu_n}) \in \eta(\mathcal{T}C^*(\mathcal{S}E)_0).$$

This shows that $(0, x_v) \in B$ for every $v \in E^0$ as claimed.

It follows that $(0, z_e) = (0, x_{r(e)})(y_e, z_e) \in \eta(\mathcal{T}C^*(\mathcal{S}E)_0)$ for each $e \in E^1$. Since each $x_v = \jmath(Q_v)$ and each $z_e = \jmath(T_e)$, we deduce that $0 \oplus \jmath(\mathcal{T}C^*(E(0,m))) \subseteq \eta(\mathcal{T}C^*(\mathcal{S}E)_0)$.

It now suffices to show that $\mathcal{T}C^*(E(1,m+1)) \oplus 0 \subseteq \eta(\mathcal{T}C^*(\mathcal{S}E)_0)$ as well. Since we have already proved that $0 \oplus \jmath_{1,m}(\mathcal{T}C^*(E(0,m))) \subseteq \eta(\mathcal{T}C^*(\mathcal{S}E)_0)$, we know that each $(w_v, 0) = (w_v, x_v) - (0, x_v)$ and each $(y_\mu, 0) = (y_\mu, z_\mu) - (0, z_\mu)$ belongs to $\eta(\mathcal{T}C^*(\mathcal{S}E)_0)$. We saw above that the elements w_v and y_μ generate $\mathcal{T}C^*(E(1,m+1))$. This completes the proof. □

If $x \in \mathcal{T}C^*(\mathcal{S}^mE)$ belongs to the ideal generated by $C_0(\mathbb{S} \setminus \{0\})$, then $x|_{\ell^2(\mathcal{S}^mE_0^*)} = 0$. Consequently, there is a homomorphism $\mathcal{T}C^*(\mathcal{S}^mE)_0 \to \mathcal{B}(\ell^2(\mathcal{S}^mE_0^*))$ such that $x_0 \mapsto x|_{\ell^2\mathcal{S}^mE_0^*}$ for $x \in \mathcal{T}C^*(\mathcal{S}^mE)$. It therefore follows from Proposition 7.13 that $\big\|x|_{\ell^2\mathcal{S}^mE_0^*}\big\| \leq \|\eta(x_0)\|$ for all $x \in \mathcal{T}C^*(\mathcal{S}^mE)$. The following result gives a direct proof of this by showing that in fact we can use the injection $\jmath_{1,m+1}$ to see that the $x \mapsto x_{\ell^2(\mathcal{S}^mE_0^*)}$ can be identified with the map obtained by following η with the second-coordinate projection $\mathcal{T}C^*(E(1,m+1)) \oplus \mathcal{T}C^*(E(1,m+1)) \to 0 \oplus \mathcal{T}C^*(E(1,m+1))$. We will also make use of this identification in our analysis of $C^*(\mathcal{S}^mE)$ in Section 8.

Lemma 7.15. *Let E be a locally finite graph with no sources. Let $U_0 : \ell^2(E(0,m)^*) \to \ell^2(\mathcal{S}^mE_0^*)$ and, for $0 < t < 1$, $U_t : \ell^2(E(1,m+1)^*) \to \ell^2(\mathcal{S}^mE_t^*)$ be the unitaries of Remark 6.5. Let $\pi_{E(0,m)} : \mathcal{T}C^*(E(0,m))) \to \mathcal{B}(\ell^2(E(0,m)^*))$ and $\pi_{E(1,m+1)} : \mathcal{T}C^*(E(1,m+1))) \to \mathcal{B}(\ell^2(E(1,m+1)^*))$ be the path-space representations. For $x \in \mathcal{T}C^*(\mathcal{S}E)$, we have*

$$\lim_{t \searrow 0} U_t^* x|_{\ell^2(\mathcal{S}^mE_t^*)} U_t = \pi_{E(1,m+1)}(\jmath_{1,m+1}(\pi_{E(0,m)}^{-1}(U_0^* x|_{\ell^2(\mathcal{S}^mE_0^*)} U_0))).$$

Proof. For $x \in \rho_m(C_0(\mathcal{S}E^0))$, this follows from Equation 7.3 in the proof of Lemma 7.11, and for $x \in \psi_m(C_c(\mathcal{S}^mE^1))$, it follows from Equation 7.4 in the same proof. Since $\mathcal{T}C^*(\mathcal{S}^mE)$ is generated by $\rho_m(C_0(\mathcal{S}E^0)) \cup \psi_m(C_c(\mathcal{S}^mE^1))$, the result follows. □

We are now able to give an explicit description of $\mathcal{T}C^*(\mathcal{S}^mE)$ provided that enough vertices of E emit at least two edges.

Theorem 7.16. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Suppose that for every $v \in E^0$ there exist $n \geq 1$ and $\mu \in E^{nm}v$ such that $|E^1r(\mu)| \geq 2$. Let $\jmath_{1,m+1} : \mathcal{T}C^*(E(0,m)) \hookrightarrow \mathcal{T}C^*(E(1,m+1))$ be the injective homomorphism of Lemma 2.1. There is an isomorphism*

$$\kappa_m : \mathcal{T}C^*(\mathcal{S}^mE) \to \{f \in C([0,1], \mathcal{T}C^*(E(1,m+1))) : f(0) \in \jmath_{1,m+1}(\mathcal{T}C^*(E(0,m)))\}$$

such that for $a \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^mE^1)$, we have

$$\kappa_m(\rho_m(a))(t) = \begin{cases} \sum_{v \in E^0} a([v]) \jmath_{1,m+1}(Q_v) & \text{if } t = 0 \\ \sum_{e \in E^1} a([e,t]) Q_e & \text{if } t \in (0,1) \\ \sum_{v \in E^0} a([v]) \sum_{e \in E^1 v} Q_e & \text{if } t = 1 \end{cases}$$

and

$$\kappa_m(\psi_m(\xi))(t) = \begin{cases} \sum_{\mu \in E^m} \xi([\mu]) \jmath(T_\mu) & \text{if } t = 0 \\ \sum_{\nu \in E^{m+1}} \xi([\nu,t]) T_\nu & \text{if } t \in [0,1) \\ \sum_{\mu \in E^m} \xi([\mu]) \sum_{e \in E^1 r(\mu)} T_{e\mu} & \text{if } t = 1. \end{cases}$$

Proof. For $t \in \mathbb{S}$ let $q_t : \mathcal{T}C^*(\mathcal{S}^mE) \to \mathcal{T}C^*(\mathcal{S}^mE)_t$ be the quotient map $a \mapsto a + J_t$. For $t \in (0,1)$, let $\theta_t : \mathcal{T}C^*(E(1,m+1)) \to \mathcal{T}C^*(\mathcal{S}^mE)_t$ be the isomorphism of Proposition 7.9, and define $\varepsilon_t := \theta_t^{-1} \circ q_t : \mathcal{T}C^*(\mathcal{S}^m E) \to \mathcal{T}C^*(E(1, m+1))$. Let $\varepsilon_1 : \mathcal{T}C^*(\mathcal{S}^m E) \to \mathcal{T}C^*(E(1, m+1))$ be the map of Lemma 7.12, and let $\varepsilon_0 : \mathcal{T}C^*(\mathcal{S}^m E) \to \mathcal{T}C^*(E(1, m+1))$ be the homomorphism of Lemma 7.11. Proposition 7.14 shows that ε_1 is surjective and that the range of ε_0 is $\jmath_{1,m+1}(\mathcal{T}C^*(E(0, m)))$.

Fix $a \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^m E^1)$. Lemma 7.3 implies that for any $a \in \mathcal{T}C^*(\mathcal{S}^m E)$ the function $t \mapsto \varepsilon_t(a)$ is continuous at each $t \in (0, 1)$; Lemmas 7.11 and 7.12 show that it is continuous at 0 and 1 as well. Hence there is a homomorphism $\kappa_m : \mathcal{T}C^*(\mathcal{S}^m E) \to C([0, 1], \mathcal{T}C^*(E(1, m+1)))$ given by $\kappa_m(a)(t) = \varepsilon_t(a)$ for all $a \in \mathcal{T}C^*(\mathcal{S}^m E)$ and $t \in [0, 1]$.

To see that κ is injective, suppose that $\kappa(a) = 0$. We must show that $a = 0$. Proposition C.10(c) of [32] shows that, $\|a\| = \sup_{t \in \mathbb{S}} \|q_t(a)\|$, so it suffices to show that each $q_t(a) = 0$. Since $\kappa(a) = 0$, we have $\varepsilon_t(a) = 0$ for all t. Since θ_t is an isomorphism for $t \in (0, 1)$, we deduce that $q_t(a) = 0$ for $t \neq 0$, and Proposition 7.14 shows that $\|q_0(a)\| = \max\{\|\varepsilon_0(a)\|, \|\varepsilon_1(a)\|\} = 0$.

It remains to show that

$$\kappa(\mathcal{T}C^*(\mathcal{S}^m E)) = \{f \in C([0, 1], \mathcal{T}C^*(E(1, m+1))) : f(0) \in \jmath(\mathcal{T}C^*(E(0, m)))\}. \tag{7.7}$$

The containment $\subseteq$ follows from Proposition 7.14. For the reverse containment, fix an element f of the right-hand side of (7.7). Proposition 7.14 shows that there exists $a \in \mathcal{T}C^*(\mathcal{S}^m E)$ such that $\eta_0(a) = f(0)$ and $\eta_1(a) = f(1)$. Hence $f - \kappa(a) \in C_0((0, 1), \mathcal{T}C^*(E(1, m+1)))$. Consequently Lemma 7.3 shows that there exists $b \in \mathcal{T}C^*(\mathcal{S}^m E)$ such that $\kappa(b) = f - \kappa(a)$. Hence $f = \kappa(a + b) \in \kappa(\mathcal{T}C^*(\mathcal{S}^m E))$. □

We deduce that under the hypotheses of the preceding theorem, $\mathcal{T}C^*(\mathcal{S}^m E)$ is homotopy equivalent to $\mathcal{T}C^*(E(0, m))$, and hence compute its K-theory.

For this, recall from [28, Definition 3.2.5] that C^*-homomorphisms $\varphi_0, \varphi_1 : A \to B$ are *homotopic* if there is a homomorphism $\varphi : A \to C([0, 1], B)$ such that $\varphi(a)(0) = \varphi_0(a)$ and $\varphi(a)(1) = \varphi_1(a)$ for all $a \in A$. Also recall that C^*-algebras A and B are *homotopy equivalent* if there are homomorphisms $\varphi : A \to B$ and $\psi : B \to A$ such that $\varphi \circ \psi$ is homotopic to id_B and $\psi \circ \varphi$ is homotopic to id_A. We use the following elementary lemma.

Lemma 7.17. *Let A and B be C^*-algebras, and let $\iota : B \to A$ be an injective homomorphism. Then the C^*-algebra $C_\iota := \{f \in C([0, 1], A) : f(0) \in \iota(B)\}$ is homotopy equivalent to B.*

Proof. Define $\varphi : C_\iota \to B$ by $\psi(f) = \iota^{-1}(f(0))$ and define $\psi : B \to C_\iota$ by $\psi(b)(t) = \iota(b)$ for all $t \in [0, 1]$. Then $\varphi \circ \psi$ is equal, and in particular homotopic, to id_B. Define $\rho : C_\iota \to C([0, 1], C_\iota)$ by

$$\big(\rho(f)(s)\big)(t) = \begin{cases} f(0) & \text{if } t \leq s \\ f(t - s) & \text{if } t > s. \end{cases}$$

Then ρ is a homomorphism, and $\rho(f)(0) = f = \mathrm{id}_{C_\iota}(f)$ and $\rho(f)(1) = \psi(\varphi(f))$ for all $f \in C_\iota$. So $\psi \circ \varphi$ is homotopic to id_{C_ι}. □

Corollary 7.18. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Suppose that for every $v \in E^0$ there exist $n \geq 1$ and $\mu \in E^{nm} v$ such that $|E^1 r(\mu)| \geq 2$. Then $\mathcal{T}C^*(\mathcal{S}^m E)$ is homotopy equivalent to $\mathcal{T}C^*(E(0, m))$, and we have $K_0(\mathcal{T}C^*(\mathcal{S}^m E)) \cong \mathbb{Z}E^0$, and $K_1(\mathcal{T}C^*(\mathcal{S}^m E)) = 0$.*

Proof. By Theorem 7.16, for the first statement we just have to show that the algebra $A := \{f \in C([0,1], \mathcal{T}C^*(E(1,m+1))) : f(0) \in \jmath(\mathcal{T}C^*(E(0,m)))\}$ is homotopy equivalent to $\mathcal{T}C^*(E(0,m))$. This follows from Lemma 7.17 applied to $\iota = \jmath_{1,m+1}$.

Now [28, Proposition 3.2.6] shows that $K_0(A) \cong K_0(\mathcal{T}C^*(E(0,m)))$, and [28, Proposition 8.2.2(vi)] shows that $K_1(A) \cong K_1(\mathcal{T}C^*(E(0,m)))$. By [9, Theorem 4.1], the algebra $\mathcal{T}C^*(E(0,m))$ is isomorphic to the Toeplitz algebra of a Hilbert bimodule over $C_0(E(0,m)^0) = C_0(E^0)$. Theorem 4.4 of [24] therefore implies that $\mathcal{T}C^*(E(0,m))$ is KK-equivalent to $C_0(E^0)$, and hence $K_*(\mathcal{T}C^*(E(0,m))) \cong K_*(C_0(E^0)) \cong (\mathbb{Z}E^0, 0)$. □

8. Analysis of $C^*(\mathcal{S}^mE)$

In this section we analyse the quotient $C^*(\mathcal{S}^mE)$, using the analysis of $\mathcal{T}C^*(\mathcal{S}^mE)$ from the previous section. The quotient map $\mathcal{T}C^*(\mathcal{S}^mE) \to C^*(\mathcal{S}^mE)$ induces a homomorphism $\pi_t : \mathcal{T}C^*(\mathcal{S}^mE)_t \to C^*(\mathcal{S}^mE)_t$ on each fibre. We show that under the isomorphisms $\mathcal{T}C^*(\mathcal{S}^mE)_t \cong \mathcal{T}C^*(E(0,m+1))$ (for $t \neq 0$) and $\mathcal{T}C^*(\mathcal{S}^mE)_0 \cong \mathcal{T}C^*(E(1,m+1)) \oplus \mathcal{T}C^*(E(0,m))$ described in the preceding section, these homomorphisms π_t become the canonical quotient maps $\mathcal{T}C^*(E(1,m+1)) \to C^*(E(1,m+1))$ (for $t \neq 0$), and $\mathcal{T}C^*(E(0,m)) \oplus \mathcal{T}C^*(E(1,m+1)) \to C^*(E(0,m)) \oplus C^*(E(1,m+1))$ (for $t = 0$).

Recall that given any graph E, we denote by I_E the ideal of $\mathcal{T}C^*(E)$ generated by the projections $\Delta_v := Q_v - \sum_{e \in vE^1} T_e T_e^*$ indexed by $v \in E^0$. The path-space representation $\pi_E : \mathcal{T}C^*(E) \to \mathcal{B}(\ell^2(E^*))$ restricts to an isomorphism of I_E onto $\oplus_{v \in E^0} \mathcal{K}(\ell^2(E^*v)) \subseteq \mathcal{K}(\ell^2(E^*))$, and in particular satisfies $\pi(T_\mu \Delta_{s(\mu)} T_\nu^*) = \theta_{\mu,\nu} \in \mathcal{K}(\ell^2(E^*s(\mu)))$ for all $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.

Lemma 8.1. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N} \setminus \{0\}$. Let $K \lhd \mathcal{T}C^*(\mathcal{S}^mE)$ be the ideal*

$$K := \{x \in \mathcal{T}C^*(\mathcal{S}^mE) : x|_{\ell^2(\mathcal{S}^mE_t^*)} \in \mathcal{K}(\ell^2(\mathcal{S}^mE_t^*)) \text{ for all } t \in \mathbb{S}\}.$$

Let $\kappa_m : \mathcal{T}C^*(\mathcal{S}^mE) \to \{f \in C([0,1], \mathcal{T}C^*(E(1,m+1))) : f(0) \in \jmath_{1,m+1}(\mathcal{T}C^*(E(0,m)))\}$ be the isomorphism of Theorem 7.16. Then $\kappa_m(K) = \{f \in \kappa_m(\mathcal{T}C^*(\mathcal{S}^mE)) : f(t) \in I_{E(1,m+1)}$ for all $t\}$.

Proof. Fix $x \in K$. Corollary 7.10 shows that for $t \notin \{0,1\}$, we have

$$\kappa_m^{-1}(x)_t = \pi_{E(1,m+1)}^{-1}(U_t^* x|_{\ell^2(\mathcal{S}^mE_t^*)} U_t).$$

So the discussion preceding this lemma shows that $\kappa_m(x)_t \in \mathcal{K}(\ell^2(E(1,m+1)))$ for $t \notin \{0,1\}$. Since $\mathcal{K}(\ell^2(E(1,m+1)))$ is closed, we deduce from the definitions of ε_0 and ε_1 that $\kappa_m^{-1}(x)_0$ and $\kappa_m^{-1}(x)_1$ belong to $\mathcal{K}(\ell^2(E(1,m+1)))$ as well. So $\kappa_m(K) \subseteq \{f \in \kappa_m(\mathcal{T}C^*(\mathcal{S}^mE)) : f(t) \in I_{E(1,m+1)}$ for all $t\}$.

For the reverse inclusion, suppose that $f \in \kappa_m(\mathcal{T}C^*(\mathcal{S}^mE))$ and that $f(t) \in I_{E(1,m+1)}$ for all t. Let $x := \kappa_m^{-1}(f)$. Corollary 7.10 shows that for $t \neq 0$ we have $x|_{\ell^2(\mathcal{S}^mE_t^*)} \in \mathcal{K}(\ell^2(\mathcal{S}^mE_t^*)$, and Lemma 7.15 shows that

$$\pi_{E(1,m+1)}(\jmath_{1,m+1}(\pi_{E(0,m)}^{-1}(U_0^* x|_{\ell^2(\mathcal{S}^mE_0^*)} U_0))) = \lim_{t \searrow 0} U_t^* x|_{\ell^2(\mathcal{S}^mE_t^*)} U_t.$$

Since each $U_t^* x|_{\ell^2(\mathcal{S}^mE_t^*)} U_t \in \mathcal{K}(\ell^2(E(1,m+1)^*))$, we deduce that

$$\pi_{E(1,m+1)}(\jmath_{1,m+1}(\pi_{E(0,m)}^{-1}(U_0^* x|_{\ell^2(\mathcal{S}^mE_0^*)} U_0))) \in \mathcal{K}(\ell^2(E(1,m+1)^*)). \tag{8.1}$$

Lemma 2.1 shows that $\jmath_{1,m+1}^{-1}(I_{E(1,m+1)}) = I_{E(0,m)}$. Since $\pi_{E(1,m+1)}$ carries $I_{E(1,m+1)}$ into $\mathcal{K}(\ell^2(E(1,m+1)^*))$ and $\pi_{E(0,m)}$ carries $I_{E(0,m)}$ to $\mathcal{K}(\ell^2(E(0,m)^*))$, we deduce that

$$\pi_{E(0,m)}(\jmath_{1,m+1}^{-1}(\pi_{E(1,m+1)}^{-1}(\mathcal{K}(\ell^2(E(1,m+1)^*))) \subseteq \mathcal{K}(\ell^2(E(0,m)^*)).$$

So (8.1) gives $U_0^* x|_{\ell^2(\mathcal{S}^m E_0^*)} U_0 \in \mathcal{K}(\ell^2(E(0,m)^*))$ and hence $x|_{\ell^2(\mathcal{S}^m E_0^*)} \in \mathcal{K}(\ell^2(\mathcal{S}^m E_0^*))$. So we have $x|_{\ell^2(\mathcal{S}^m E_t^*)} \in \mathcal{K}(\ell^2(\mathcal{S}^m E_t^*))$ for all $t \in \mathbb{S}$, and therefore $f = \kappa_m(x) \in \kappa_m(K)$. □

Theorem 8.2. *Let E be a locally finite graph with no sources and fix $m \in \mathbb{N}\backslash\{0\}$. Suppose that for every $v \in E^0$ there exist $n \geq 1$ and $\mu \in E^{nm}v$ such that $|E^1 r(\mu)| \geq 2$. There is an isomorphism $\tilde{\kappa}_m : C^*(\mathcal{S}^m E) \to C([0,1], C^*(E(1,m+1)))$ such that for $a \in C_0(\mathcal{S}E^0)$ and $\xi \in C_c(\mathcal{S}^m E^1)$, we have*

$$\tilde{\kappa}_m(\tilde{\rho}_m(a))(t) = \begin{cases} \sum_{v\in E^0} a([v]) \sum_{e\in vE^1} p_e & \text{if } t = 0 \\ \sum_{e\in E^1} a([e,t]) p_e & \text{if } t \in (0,1) \\ \sum_{v\in E^0} a([v]) \sum_{e\in E^1 v} p_e & \text{if } t = 1 \end{cases}$$

and

$$\tilde{\kappa}_m(\tilde{\psi}_m(\xi))(t) = \begin{cases} \sum_{\mu\in E^m} \xi([\mu]) \sum_{e\in s(\mu)E^1} s_{\mu e} & \text{if } t = 0 \\ \sum_{\nu\in E^{m+1}} \xi([\nu,t]) s_\nu & \text{if } t \in [0,1) \\ \sum_{\mu\in E^m} \xi([\mu]) \sum_{e\in E^1 r(\mu)} s_{e\mu} & \text{if } t = 1. \end{cases}$$

Proof. Let

$$A := \{f \in C([0,1], \mathcal{T}C^*(E(1,m+1))) : f(0) \in \jmath_{1,m+1}(\mathcal{T}C^*(E(0,m)))\}, \quad \text{and}$$
$$I := \{f \in A : f(t) \in I_{E(1,m+1)} \text{ for all } t\}.$$

Lemma 8.1 and the definition of $C^*(\mathcal{S}^m E)$ show that the isomorphism

$$\kappa_m : \mathcal{T}C^*(\mathcal{S}^m E) \to \{f \in C([0,1], \mathcal{T}C^*(E(1,m+1))) : f(0) \in \jmath(\mathcal{T}C^*(E(0,m)))\}$$

of Theorem 7.16 descends to an isomorphism $\kappa' : C^*(\mathcal{S}^m E) \to A/I$.

The last statement of Lemma 8.1 shows that $I = \{f : f(t) \in I_{E(1,m+1)} \text{ for } t \neq 0 \text{ and } f(0) \in I_{E(0,m)}\}$. It follows that there is an injective homomorphism $\kappa'' : A/I \to C([0,1], C^*(E(1,m+1)))$ that carries $x + I$ to the function

$$t \mapsto \begin{cases} x(t) + I_{E(1,m+1)} & \text{if } t \neq 0 \\ \jmath_{1,m+1}^{-1}(x(0)) + I_{E(0,m)} & \text{if } t = 0. \end{cases}$$

We claim that κ'' is surjective. For each $t \in (0,1]$, we have $\{\kappa''(x)(t) : x \in A/I\} = C^*(E(1,m+1))$ because $\{x(t) : x \in A\} = \mathcal{T}C^*(E(1,m+1))$. At $t = 0$ we have $\kappa''(\{x(0) : x \in A/I\}) = \tilde{\jmath}_{1,m+1}(C^*(E)) = C^*(E(1,m+1))$ because $\{x(0) : x \in A\} = \jmath_{1,m+1}(\mathcal{T}C^*(E(0,m)))$. The extension $\widetilde{\kappa}''$ of κ'' to $\mathcal{M}(A/I)$ carries the canonical copy of $C([0,1])$ in $\mathcal{M}(A/I)$ to the canonical copy of $C([0,1]) \in \mathcal{M}(C([0,1], C^*(E(1,m+1))))$. Thus for $f \in C([0,1])$ and $x \in A$ we have $f \cdot \kappa''(a+I) = \kappa''((f \cdot a)+I) \in \kappa''(A/I)$. It therefore follows from [32, Proposition C.24] that $\kappa''(A/I)$ is dense in $C([0,1], C^*(E(1,m+1)))$, and therefore all of $C([0,1], C^*(E(1,m+1)))$ because the range of a C^*-homomorphism is closed. □

To finish this section, we use our earlier results to describe, up to Morita equivalence, the C^*-algebras $C^*(\mathcal{S}^l E)$ for rational values of l and for locally finite graphs E with no sources or sinks such that for every v and every m there exist $p, q \geq 1$, $\mu \in vE^{pm}$ and $\nu \in E^{qm}v$

such that $|s(\mu)E^1| \geq 2$ and $|E^1 r(\nu)| \geq 2$. In particular, we show that this applies to any finite, strongly connected graph with period 1 (in the sense of Perron–Frobenius theory).

We first need the following elementary result about the Cuntz–Krieger algebras of the delayed graphs associated to a graph E.

Lemma 8.3. *Let E be a locally finite graph with no sinks or sources, and let m, n be coprime positive integers. Then $C^*(D_n(E)(1, m+1))$ is Morita equivalent to $C^*(E(0, m))$.*

Proof. By [1, Theorem 3.1], we have $C^*(D_n(E)(1, m + 1)) \cong C^*(D_n(E)(0, m))$, so it suffices to show that the latter is Morita equivalent to $C^*(E(0, m))$.

To see this, observe that, by [2, Lemma 1.1], the series $\sum_{v \in E^0} p_v$ converges to a multiplier projection $P \in \mathcal{M}C^*(D_n(E)(0, m))$.

We claim that P is full. For this, first partition $D_n(E)^0$ as $D_n(E)^0 = \bigcup_{j \in \mathbb{Z}/n\mathbb{Z}} V_j$ by setting

$$V_j := \begin{cases} E^0 & \text{if } j = 0 \\ \{w_{e,j} : e \in E^1\} & \text{if } j \neq 0. \end{cases}$$

Then for $\alpha \in D_n(E)^1$, we have $r(\alpha) \in V_j$ if and only if $s(\alpha) \in V_{j+1}$, and it follows that for $\lambda \in D_n(E)^*$, we have $r(\lambda) \in V_j$ if and only if $s(\lambda) \in V_{j+[|\lambda|]_n}$. Now fix $u \in D_n(E)^0$, say $u \in V_j$. Since m, n are coprime, there exists $k \in \mathbb{N}$ such that $km \equiv j \pmod{n}$. Since E has no sinks, there exists $\lambda \in E^{km}u$, and since $s(\lambda) \in V_j$, it follows that $r(\lambda) \in V_{j-[km]} = V_0$, and therefore $p_{r(\lambda)} \leq P$. Hence $p_u = s_\lambda^* p_{r(\lambda)} s_\lambda$ belongs to the ideal generated by P. Now for $\alpha \in D_n(E)^1$, the generator $s_\alpha = s_\alpha p_{s(\alpha)}$ also belongs to the ideal generated by P, and it follows that P is full.

To complete the proof, it suffices to show that $PC^*(D_n(E)(0, m))P \cong C^*(E(0, m))$. We begin by constructing a Cuntz–Krieger $E(0, m)$-family in $PC^*(D_n(E)(0, m))P$. First, for $\mu \in E^m$, we define $\alpha(\mu) \in D_n(E)(0, m)^n$ by

$$\alpha(\mu) = f_{\mu_1,1} \cdots f_{\mu_1,n} f_{\mu_2,1} \cdots f_{\mu_2,n} \cdots f_{\mu_m,1} \cdots f_{\mu_m,n}.$$

For $v \in E(0, m)^0 = E^0$, we define $P_v := p_v \in PC^*(D_n(E)(0, m))P$, and for $\mu \in E(0, m)^1 = E^m$, we define $S_\mu := s_{\alpha(\mu)} \in PC^*(D_n(E)(0, m))P$. It is routine to see that (P, S) is a Cuntz–Krieger $E(0, m)$-family, so the universal property of $C^*(E(0, m))$ implies that there is a homomorphism $\pi : C^*(E(0, m)) \to PC^*(D_n(E)(0, m))P$ such that $\pi(p_v) = P_v$ for all $v \in E^0$ and $\pi(s_\mu) = S_\mu$ for all $\mu \in E(0, m)^1$. We have $\alpha(E^m) = (D_n(E)(0, m)^n)E^0 \subseteq D_n(E)(0, m)^n$. The universal property of $C^*(D_n(E)(0, m))$ shows that there is an action β of $\mathbb{T}$ on $C^*(D_n(E)(0, m))$ such that $\beta_z(s_\nu) = s_\nu$ for all $\nu \in (D_n(E)(0, m)^1) \setminus (D_n(E)(0, m)^1)E^0$, and such that $\beta_z(s_\nu) = zs_\nu$ for all $\nu \in (D_n(E)(0, m)^1)E^0$. Since $\gcd(m, n) = 0$, for each $\mu \in D_n(E)(0, m)^n$, if we factor $\alpha(\mu) = \alpha_1 \cdots \alpha_n$ with each $\alpha_i \in D_n(E)(0, m)^1$, we have $s(\alpha_n) \in E^0$ and $s(\alpha_i) \notin E^0$ for $i < n$. Consequently $\beta_z(s_{\alpha(\mu)}) = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{n-1}}(zs_{\alpha_n}) = zs_{\alpha(\mu)}$. Hence, writing γ for the gauge action on $C^*(E(0, m))$ we have $\pi \circ \gamma_z = \beta_z \circ \pi$. The gauge-invariant uniqueness theorem [2, Theorem 2.1] therefore implies that π is injective.

It now suffices to show that the range of π is $PC^*(D_n(E)(0, m))P$. We have

$$PC^*(D_n(E)(0, m))P = \overline{\operatorname{span}}\{s_\eta s_\zeta^* : \eta, \zeta \in V_0 D_n(E)(0, m)^*, s(\eta) = s(\zeta)\}.$$

Fix $\eta, \zeta \in V_0(D_n(E)(0, m)^*)$ such that $s(\eta) = s(\zeta)$, say $\eta \in D_n(E)(0, m)^k = D_n(E)^{km}$ and $\zeta \in D_n(E)(0, m)^l = D_n(E)^{lm}$. Then $s(\eta) \in V_{[km]}$ and $s(\zeta) \in V_{[lm]}$ forcing $km \equiv$

$lm \pmod n$. Since $\gcd(m,n) = 1$, we deduce that $k \equiv l \pmod n$. Fix p such that $k + p \in n\mathbb{Z}$. Then $l + p \in n\mathbb{Z}$ too, and the Cuntz–Krieger relation forces

$$s_\eta s_\zeta^* = \sum_{\xi \in s(\eta) D_n(E)^{pm}} s_{\eta\xi} s_{\zeta\xi}^* = \sum_{\xi \in s(\eta) D_n(E)(0,m)^p} s_{\eta\xi} s_{\zeta\xi}^*$$

By construction, each $\eta\xi$ has the form $\alpha(\mu_1) \cdots \alpha(\mu_{k+p})$ for some $\mu_i \in E(0,m)^1$, and so each $s_{\eta\xi} \in \pi(C^*(E(0,m)))$, and similarly for each $s_{\zeta\xi}$. So $s_\eta s_\zeta^* \in \pi(C^*(E(0,m)))$. Thus π is an isomorphism of $C^*(E(0,m))$ onto $PC^*(D_n(E)(0,m))P$ as required. □

Corollary 8.4. *Let E be a locally finite graph with no sinks or sources. Suppose that for every $v \in E^0$ and every $m \in \mathbb{Z} \setminus \{0\}$, there exist $n \geq 1$ and $\mu \in E^{nm}v$ such that $|E^1 r(\mu)| \geq 2$. For $n \in \mathbb{N} \setminus \{0\}$ and $m \in \mathbb{Z}$ such that* $\gcd(m,n) = 1$,

$$\begin{aligned} &C^*(\mathcal{S}^{\frac{m}{n}}E) \sim_{\mathrm{Me}} C([0,1], C^*(E(0,m))) && \textit{if } m > 0, \\ (8.2)\qquad &C^*(\mathcal{S}^{\frac{m}{n}}E) \cong C_0(\mathcal{S}E^0) \otimes C(\mathbb{T}) && \textit{if } m = 0, \textit{ and} \\ &C^*(\mathcal{S}^{\frac{m}{n}}E) \sim_{\mathrm{Me}} C([0,1], C^*(E^{\mathrm{op}}(0,-m))) && \textit{if } m < 0. \end{aligned}$$

In particular,

$$K_*(C^*(\mathcal{S}^{\frac{m}{n}}E)) \cong \begin{cases} \big(\operatorname{coker}(1 - (A_E^t)^m), \ker(1 - (A_E^t)^m\big) & \textit{if } m > 0, \\ \big(K^0(\mathcal{S}E^0) \oplus K^1(\mathcal{S}E^0), K^0(\mathcal{S}E^0) \oplus K^1(\mathcal{S}E^0)\big) & \textit{if } m = 0, \textit{ and} \\ \big(\operatorname{coker}(1 - A_E^m), \ker(1 - A_E^m)\big) & \textit{if } m < 0. \end{cases}$$

Proof. For $m > 0$, Theorem 8.2 combined with Lemma 8.3 shows that $C^*(\mathcal{S}^{\frac{m}{n}}) \sim_{\mathrm{Me}} C([0,1], C^*(E(0,m)))$, and then for $m < 0$, it follows from Lemma 6.11 that $C^*(\mathcal{S}^{\frac{m}{n}}) \sim_{\mathrm{Me}} C([0,1], C^*(E^{\mathrm{op}}(0,-m)))$. Combining this with Theorem 6.10(2) proves the first statement.

For the second statement, since $K_*(C([0,1], C^*(E(0,m)))) \cong K_*(C^*(E(0,m)))$ (for example, apply Lemma 7.17 to $\iota = \mathrm{id}_{C^*(E(0,m))}$, and then use [28, Proposition 3.2.6 and Theorem 8.2.2(vi)]), the computation [26, Theorem 3.2] of K-theory for graph C^*-algebras gives

$$K_*(C([0,1], C^*(E(0,m)))) \cong \big(\operatorname{coker}(1 - (A_E^t)^m), \ker(1 - A_E^t)^m\big) \qquad \text{for } m > 0.$$

A similar argument gives

$$K_*(C([0,1], C^*(E(0,m)))) \cong \big(\operatorname{coker}(1 - A_E^m), \ker(1 - A_E^m)\big) \qquad \text{for } m < 0$$

because $A_{E^{\mathrm{op}}}^t = A_E$ and our hypotheses are symmetrical in E and E^{op}. Finally, the Künneth theorem and that operator K-theory agrees with topological K-theory for commutative C^*algebras implies that $K_i(C_0(\mathcal{S}E^0) \otimes C(\mathbb{T})) \cong K^0(C_0(\mathcal{S}E^0)) \oplus K^1(C_0(\mathcal{S}E^0))$ for $i = 1, 2$. □

We finish by applying Corollary 8.4 to strongly connected finite graphs with period 1.

We say that a graph E is strongly connected if for all $v, w \in E^0$ the set $vE^*w \setminus E^0$ is nonempty (we take the convention that a graph consisting of a single vertex and no edges is not strongly connected). If E is a strongly connected finite graph, then the *period* of E is defined as $P(E) := \gcd\{|\mu| : \mu \in E^* \setminus E^0, r(\mu) = s(\mu)\}$.

Also recall that if E is a graph, then there is a map $\partial : \mathbb{Z}E^0 \to \mathbb{Z}E^1$ given by $\partial(a)(e) = a(r(e)) - a(s(e))$. The 0^{th} and 1^{st} homology groups of E are defined by $H_1(E) = \mathbb{Z}E^1/\partial_1(\mathbb{Z}E^0)$, and $H_0(E) = \ker(\partial_1)$. The higher homology groups $H_n(E)$,

$n \geq 2$ are trivial (see, for example, [20, Remark 3.6]). The group $H_0(E)$ is isomorphic to the free abelian group generated by the connected components of E.

We say that a finite graph E is a *simple cycle* if, putting $n = |E^0|$, there are bijections $i \mapsto v_i$ and $i \mapsto e_i$ of $\mathbb{Z}/n\mathbb{Z}$ onto E^0 and E^1 respectively such that $r(e_i) = v_i$ and $s(e_i) = v_{i+1}$ for all i.

Corollary 8.5. *Let E be a finite strongly connected graph that is not a simple cycle, and suppose that $P(E) = 1$. Then for every $v \in E^0$ and every $m \in \mathbb{Z} \setminus \{0\}$, there exist $n \geq 1$ and $\mu \in E^{nm}v$ such that $|E^1 r(\mu)| \geq 2$. For $n > 0$ and $m \in \mathbb{Z}$ with m, n coprime,*

$$\begin{aligned} &C^*(\mathcal{S}^{\frac{m}{n}}E) \sim_{\mathrm{Me}} C([0,1], C^*(E(0,|m|))) && \text{if } m \neq 0, \text{ and} \\ &C^*(\mathcal{S}^{\frac{m}{n}}E) \cong C_0(\mathcal{S}E^0) \otimes C(\mathbb{T}) && \text{if } m = 0. \end{aligned}$$

In particular,

$$K_*(C^*(\mathcal{S}^{\frac{m}{n}}E)) \cong \begin{cases} \big(\operatorname{coker}(1 - A_E^m), \ker(1 - A_E^m)\big) & \text{if } m \neq 0 \\ \big(\mathbb{Z} \oplus H_1(E), \mathbb{Z} \oplus H_1(E)\big) & \text{if } m = 0. \end{cases}$$

Proof. First observe that the opposite graph E^{op} is also strongly connected with $P(E^{\mathrm{op}}) = 1$, and is also not a simple cycle. To prove the first statement, it therefore suffices to consider $m > 0$ (the case $m = 0$ follows from the second line of (8.2)).

So fix $m \geq 1$ and $v \in E^0$. Since E is strongly connected, we have $|E^1 v| \geq 1$ for all v, and since E is not a simple cycle, a counting argument shows that there exists $w \in E^0$ such that $|E^1 w| \geq 2$. Since E is strongly connected, the set wE^*v is nonempty, say $\alpha \in wE^*v$. It is standard (see for example [21, Lemma 6.1] applied with $k = 1$) that $P(E) = \{|\lambda| - |\nu| : \lambda, \nu \in vE^*v\}$. In particular, there are cycles $\lambda, \nu \in vE^*v$ such that $|\lambda| - |\nu| = m - |\alpha|$. It follows that $|\alpha\lambda\nu^{m-1}| = |\alpha| + (|\lambda| - |\nu|) + m|\nu| = m(|\nu| + 1)$. So $n := |\nu| + 1$ and $\mu := \alpha\lambda\nu^{m-1}$ satisfy $\mu \in E^{nm}v$ and $|E^1 r(\mu)| = |E^1 w| \geq 2$.

By Corollary 8.4, it now suffices to show that

$$C([0,1], C^*(E^{\mathrm{op}}(0,m))) \sim_{\mathrm{Me}} C([0,1], C^*(E(0,m))) \quad \text{for all } m \geq 1,$$

and that $K_*(C(\mathcal{S}E^0 \times \mathbb{S})) \cong \big(\mathbb{Z} \oplus H_1(E), \mathbb{Z} \oplus H_1(E)\big)$.

Fix $m > 0$. Since E has period 1, it is easy to see that $E(0,m)$ and $E^{\mathrm{op}}(0,m)$ are strongly connected finite graphs that are not simple cycles. Since

$$\ker(A^t) \cong \ker(A) \quad \text{and} \quad \operatorname{coker}(A^t) \cong \operatorname{coker}(A),$$

(see, for example [13, Section 6.2]), we have $K_*(C^*(E^{\mathrm{op}}(0,m))) \cong K_*(C^*(E(0,m)))$, and so [27, Theorem 6.1] shows that $C^*(E^{\mathrm{op}}(0,m)) \sim_{\mathrm{Me}} C^*(E(0,m))$. Thus the function algebras $C([0,1], C^*(E^{\mathrm{op}}(0,m)))$ and $C([0,1], C^*(E(0,m)))$ are Morita equivalent as well.

Now take $m = 0$. We have $K^*(\mathbb{S}) = (\mathbb{Z}, \mathbb{Z})$. Thus, by the Künneth theorem in K-theory, it suffices to show that $K^0(\mathcal{S}E^0) \cong \mathbb{Z}$ and $K^1(\mathcal{S}E^0) \cong H_1(E)$. Since $\mathcal{S}E^0$ and its suspension are finite CW-complexes of dimension at most 2, [31, Theorem 1] shows that $K^0(\mathcal{S}E^0) \cong \bigoplus_{n \geq 0} H_{2n}(\mathcal{S}E^0) = H_0(\mathcal{S}E^0)$ and that $K^1(\mathcal{S}E^0)$ is isomorphic to the direct sum of the even homology groups of the suspension of $\mathcal{S}E^0$, and hence to the direct sum of the odd homology groups of $\mathcal{S}E^0$. Theorem 6.3 of [20] gives $H_*(\mathcal{S}E^0) \cong H_*(E)$, and we have $H_0(E) \cong \mathbb{Z}$ because E is connected. $\square$

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