

THE NUCLEAR DIMENSION OF GRAPH C^* -ALGEBRAS

EFREN RUIZ, AIDAN SIMS, AND MARK TOMFORDE

ABSTRACT. Consider a graph C^* -algebra $C^*(E)$ with a purely infinite ideal I (possibly all of $C^*(E)$) such that I has only finitely many ideals and $C^*(E)/I$ is approximately finite dimensional. We prove that the nuclear dimension of $C^*(E)$ is 1. If I has infinitely many ideals, then the nuclear dimension of $C^*(E)$ is either 1 or 2.

1. INTRODUCTION

The point of view that regards C^* -algebras as noncommutative topological spaces has led to a number of notions of topological dimension for C^* -algebras (see, for example, [5, 18, 25, 29]). Each of these captures important C^* -algebraic properties, and many of them have played an important role in the program of classification of C^* -algebras by K -theoretic data pioneered by Elliott (see, for example, [9, 10, 11]). In 2010, Winter and Zacharias introduced the nuclear dimension of a C^* -algebra as a noncommutative analogue of topological covering dimension [31]. Finite nuclear dimension is closely related to \mathcal{Z} -stability [30] where \mathcal{Z} is the Jiang-Su algebra (Toms and Winter have conjectured that the two are equivalent for the class of simple, separable, infinite-dimensional, nuclear C^* -algebras), and so has important implications for classification theory.

Roughly speaking, the nuclear dimension of a C^* -algebra A is the minimum number d for which the identity map on A can be approximately factored, on any finite set of elements, through a finite-dimensional C^* -algebra by a composition $\text{id} \sim \phi \circ \psi$ where ψ is a completely positive contraction and ϕ decomposes as a direct sum of $d + 1$ completely positive orthogonality-preserving contractions. To see how this relates to covering dimension¹, consider a compact Hausdorff space X and the commutative C^* -algebra $C(X)$: given positive $f_1, \dots, f_n \in C(X)$, cover the union of their supports with open sets U_1, \dots, U_h so that each f_i is approximately constant on each U_j . Fix a point x_i in each U_i and define $\psi : C(X) \rightarrow \mathbb{C}^h$ by $\psi(f)_i = f(x_i)$ for all $1 \leq i \leq h$. Define $\phi : \mathbb{C}^h \rightarrow C(X)$ by $\phi(a_1, \dots, a_h) = \sum_{i=1}^h a_i u_i$ for some partition of unity $\{u_i\}_{i=1}^h$ subordinate to $\mathcal{U} = \{U_1, \dots, U_h\}$. So each $\phi \circ \psi(f_i)$ is close to f_i by construction. Partitioning \mathcal{U} into subcollections $\mathcal{U} = \mathcal{U}^0 \sqcup \dots \sqcup \mathcal{U}^d$ so that distinct elements of any given \mathcal{U}^i are disjoint gives a decomposition of \mathbb{C}^h into $d + 1$ direct summands on which ϕ preserves

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orthogonality. The smallest d for which we can always do this (to a refinement of \mathcal{U}) is the covering dimension of X .

In [31], Winter and Zacharias established a number of fundamental properties of nuclear dimension. They showed that nuclear dimension behaves well with respect to stabilization, direct sums, tensor products, hereditary subalgebras, direct limits, and extensions. They also showed that for $2 \leq n < \infty$ the Cuntz algebras \mathcal{O}_n have nuclear dimension 1. To do this, they used a construction for graph C^* -algebras described by Kribs and Solel in [19] to construct pairs ϕ_m and ψ_m so that each $\psi_m : \mathcal{O}_n \rightarrow F$ is a completely positive contraction onto a finite-dimensional C^* -algebra, each $\phi_m : F \rightarrow \mathcal{O}_n \otimes M_{d_m}$ is a direct sum of two orthogonality-preserving completely positive contractions, and, as $m \rightarrow \infty$, we have $\phi_m \circ \psi_m \rightarrow \kappa_m$ pointwise, where $\kappa_m : \mathcal{O}_n \rightarrow \mathcal{O}_n \otimes M_{d_m}$ is a homomorphism that induces multiplication by m in K -theory. Since $K_*(\mathcal{O}_n) = (\mathbb{Z}_{n-1}, 0)$, choosing the m appropriately ensures that the κ_m induce the identity in K -theory, and then Kirchberg-Phillips' classification results show that the $\phi_m \circ \psi_m$ are asymptotically approximately conjugate to the identity map. Winter and Zacharias' results about extensions then show that the Toeplitz-Cuntz algebras \mathcal{TO}_n have nuclear dimension at most 2, and it follows that \mathcal{O}_∞ , being a direct limit of the \mathcal{TO}_n , also has nuclear dimension at most 2. Finally, using direct-limit decompositions of Kirchberg algebras and the behavior of nuclear dimension with respect to tensor products, Winter and Zacharias deduce that every Kirchberg algebra has nuclear dimension at most 5.

More recently, Enders [8] developed a technique for showing that \mathcal{O}_∞ in fact has nuclear dimension 1. The rough idea of his argument is to proceed as in Winter and Zacharias' proof for \mathcal{O}_n up to the construction of the $\phi_m \circ \psi_m$ that approximate homomorphisms κ_m inducing multiplication by m in K -theory. At this point, Enders uses that multiplication by -1 in $K_*(\mathcal{O}_\infty)$ is an isomorphism, and so, by Kirchberg-Phillips' results, is induced by an automorphism λ of \mathcal{O}_∞ . He then constructs approximating pairs Φ_m, Ψ_m from the $\phi_{m+1} \circ \psi_{m+1}$ and $\lambda \circ \phi_m \circ \psi_m$ such that the $\Phi_m \circ \Psi_m$ approximate homomorphisms that induce the identity map in K -theory. He then argues using Kirchberg-Phillips' results again that the $\Phi_m \circ \Psi_m$ are approximately conjugate to the identity. Enders actually shows that all Kirchberg algebras in the Rosenberg-Schochet bootstrap category \mathcal{N} with torsion-free K_1 have nuclear dimension 1, and that those with arbitrary K_1 have nuclear dimension at most 2. More recently, Matui-Sato [20] have shown that all Kirchberg algebras have nuclear dimension at most 3.

Here we address the nuclear dimension of nonsimple graph C^* -algebras; our cleanest results are for the purely infinite situation. Winter and Zacharias' results imply that if $C^*(E)$ has at most m ideals, then the nuclear dimension of $C^*(E)$ is at most $6m - 1$, and Enders' result improves this bound to $3m - 1$. We prove that in fact if E is a directed graph and $C^*(E)$ is purely infinite and has finitely many ideals, then $\dim_{\text{nuc}}(C^*(E)) = 1$. In particular, every Cuntz-Krieger algebra with finitely many ideals has nuclear dimension 1. Using an adaptation of a direct-limit decomposition of graph C^* -algebras due to Jeong and Park [14], we deduce that every purely infinite graph C^* -algebra has nuclear dimension at most 2, as does its Toeplitz extension. We then consider graph C^* -algebras of "mixed" type. We show that if I is an ideal of $C^*(E)$ for which the quotient is AF, then the extension is quasidiagonal in the sense that I contains an approximate identity of projections that is asymptotically central in $C^*(E)$. Using this we deduce that the nuclear dimension of $C^*(E)$ is at most that of I . So if I is purely infinite, then the nuclear

dimension of $C^*(E)$ is at most 2, and if I is purely infinite and has finitely many ideals, then the nuclear dimension of $C^*(E)$ is 1.

Let A be a separable, nuclear, purely infinite, tight C^* -algebra over an accordion space X . Suppose that $K_1(A(x))$ is free and that $A(x)$ belongs to the Rosenberg-Schochet bootstrap category \mathcal{N} for each $x \in X$. Recent results of Arklint, Bentmann, and Katsura ([2] and [1]), show that A is stably isomorphic to $C^*(E)$ for some row-finite graph E . Since nuclear dimension is preserved by stabilization, these results imply that A has nuclear dimension 1. Evidence suggests that, more generally, if A is a separable, nuclear, purely infinite, tight C^* -algebra over any finite topological space X and if $K_1(A(x))$ is free and $A(x)$ is in \mathcal{N} for all $x \in X$, then A is a purely infinite graph C^* -algebra. If so, then our results would imply that if A is a separable, nuclear, purely infinite C^* -algebra such that every simple subquotient of A belongs to \mathcal{N} and has a free abelian K_1 -group, then A has nuclear dimension 1.

2. APPROXIMATION BY ORDER-ZERO MAPS

In this section we construct homomorphisms from a graph algebra $C^*(E)$ into related graph algebras $C^*(E(m))$ that can be approximately factored through finite-dimensional C^* -algebras as sums of two order-zero maps. Our approach closely follows the technique developed by Winter and Zacharias in [31, Section 7] to compute the nuclear dimension of Cuntz algebras.

For a directed graph E , let E^* denote the set of all finite paths in E , let E^n denote the set of all paths in E of length n , and let $E^{<n}$ denote the set of all paths in E of length strictly less than n . We regard vertices as paths of length zero. For a path μ in E , we define

$$\begin{aligned} E^n \mu &:= \{\alpha \mu : \alpha \in E^n \text{ and } r(\alpha) = s(\mu)\} \\ \mu E^n &:= \{\mu \alpha : \alpha \in E^n \text{ and } s(\alpha) = r(\mu)\}, \end{aligned}$$

and we define μE^* , $E^* \mu$, $\mu E^{<n}$, and $E^{<n} \mu$ similarly.

Recall that if E is a row-finite directed graph, then its Toeplitz algebra $\mathcal{TC}^*(E)$ is the universal C^* -algebra generated by mutually orthogonal projections $\{q_v : v \in E^0\}$ and elements $\{t_e : e \in E^1\}$ such that

- (TCK1) $t_e^* t_e = q_{r(e)}$ for all $e \in E^1$, and
(TCK2) $q_v \geq \sum_{e \in v E^1} t_e t_e^*$ for each $v \in E^0$.

There is a Toeplitz-Cuntz-Krieger E -family in $\ell^2(E^*)$ given by $T_e \xi_\mu = \delta_{r(e), s(\mu)} \xi_{e\mu}$ and $Q_v \xi_\mu = \delta_{v, s(\mu)} \xi_\mu$. Since $(Q_v - \sum_{e \in v E^1} T_e T_e^*) \xi_v = \xi_v$, we see that the inequality in (TCK2) is strict in $\mathcal{TC}^*(E)$.

We recall some background from [31]. A completely positive map $\varphi : A \rightarrow B$ between C^* -algebras has *order zero* if, whenever a, b are positive elements of A with $ab = ba = 0$, we also have $\varphi(a)\varphi(b) = \varphi(b)\varphi(a) = 0$. A C^* -algebra A has nuclear dimension at most $n \geq 0$ if there is a net $(F_\lambda, \psi_\lambda, \varphi_\lambda)$ such that the F_λ are finite-dimensional C^* -algebras, and $\psi_\lambda : A \rightarrow F_\lambda$ and $\varphi_\lambda : F_\lambda \rightarrow A$ are completely positive maps satisfying

- (1) $\varphi_\lambda \circ \psi_\lambda(a) \rightarrow a$ for each $a \in A$;
- (2) $\|\psi_\lambda\| \leq 1$ for all λ ; and
- (3) each F_λ decomposes as $F_\lambda = \bigoplus_{i=0}^n F_\lambda^{(i)}$ with each $\varphi_\lambda|_{F_\lambda^{(i)}}$ a completely positive contraction of order zero.

Given a countable set S , the compact operators on $\ell^2(S)$ will be denoted by \mathcal{K}_S . Equivalently, \mathcal{K}_S is the unique nonzero C^* -algebra generated by matrix units indexed by S ; that is, elements $\{\theta_{s,t} : s, t \in S\}$ such that $\theta_{s,t}^* = \theta_{t,s}$ and $\theta_{s,t}\theta_{u,v} = \delta_{t,u}\theta_{s,v}$.

Lemma 2.1. *Let E be a row-finite directed graph. For $\mu \in E^*$, let $\Delta_\mu := t_\mu t_\mu^* - \sum_{\mu e \in \mu E^1} t_\mu e t_\mu^* \in \mathcal{TC}^*(E)$. Then $\Delta_\mu = t_\mu \Delta_{r(\mu)} t_\mu^*$ for each $\mu \in E^*$, and there is an isomorphism from $\overline{\text{span}}\{t_\mu \Delta_v t_\nu^* : v \in E^0 \text{ and } \mu, \nu \in E^* v\}$ onto $\bigoplus_{v \in E^0} \mathcal{K}_{E^* v}$ that carries each $t_\mu \Delta_v t_\nu^*$ to $\theta_{\mu,\nu}$. For each $m \in \mathbb{N}$, the series $\sum_{\mu \in E^{< m}} \Delta_\mu$ converges strictly to a projection $\Phi_m \in \mathcal{M}(C^*(E))$. Moreover,*

$$\Phi_m t_\alpha t_\beta^* \Phi_m = \sum_{\substack{\tau \in r(\alpha) E^*, \\ |\alpha\tau|, |\beta\tau| < m}} t_{\alpha\tau} \Delta_{r(\tau)} t_{\beta\tau}^*,$$

and

$$\Phi_m C^*(E) \Phi_m = \overline{\text{span}}\{t_\mu \Delta_{r(\mu)} t_\nu^* : |\mu|, |\nu| < m\}.$$

Proof. We have $t_\mu \Delta_{r(\mu)} t_\mu^* = t_\mu q_{r(\mu)} t_\mu^* - t_\mu \sum_{e \in r(\mu) E^1} t_e t_e^* t_\mu^* = \Delta_\mu$, proving the first assertion. We also have

$$\|t_\mu \Delta_v t_\nu^*\| \geq \|t_\mu^* t_\mu \Delta_v t_\nu^* t_\nu\| = \|\Delta_v\| = 1$$

as discussed immediately after the definition of a Toeplitz-Cuntz-Krieger family above. Hence to establish the desired isomorphism from $\overline{\text{span}}\{t_\mu \Delta_v t_\nu^* : v \in E^0 \text{ and } \mu, \nu \in E^* v\}$ onto $\bigoplus_{v \in E^0} \mathcal{K}_{E^* v}$, it suffices to show that the $t_\mu \Delta_v t_\nu^*$ are matrix units. Certainly $(t_\mu \Delta_v t_\nu^*)^* = t_\nu \Delta_v t_\mu^*$ because Δ_v is a projection. If $\mu, \nu \in E^* v$ and $\alpha, \beta \in E^* w$, then

$$t_\mu \Delta_v t_\nu^* t_\alpha \Delta_w t_\beta^* = \begin{cases} t_\mu \Delta_v t_{\alpha'} \Delta_w t_\beta^* & \text{if } \alpha = \nu \alpha' \\ t_\mu \Delta_v t_{\nu'} \Delta_w t_\beta^* & \text{if } \nu = \alpha \nu' \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha = \nu \alpha'$ and $|\alpha'| > 0$, then $\alpha' = e \alpha''$ for some $e \in v E^1$, and since $\Delta_v \leq q_v - t_e t_e^*$, we have $\Delta_v t_{\alpha'} = 0$. Symmetrically, $t_{\nu'} \Delta_w = 0$ if $\nu = \alpha \nu'$ and $|\nu'| > 0$. So if $t_\mu \Delta_v t_\nu^* t_\alpha \Delta_w t_\beta^* \neq 0$, then $\nu = \alpha$, and then $v = w$, and we obtain $t_\mu \Delta_v t_\nu^* t_\alpha \Delta_w t_\beta^* = t_\mu \Delta_v q_v \Delta_w t_\beta^*$. Hence $t_\mu \Delta_v t_\nu^* t_\alpha \Delta_w t_\beta^* = \delta_{\nu,\alpha} t_\mu \Delta_v t_\beta^*$ as required.

Since E is row-finite, for any $v \in E^0$ we have $q_v \Delta_\mu = 0$ for all but finitely many $\mu \in E^{< m}$. Since $\sum_v q_v$ converges strictly to the identity in $\mathcal{M}(\mathcal{TC}^*(E))$ and since $\{\sum_{\mu \in F} \Delta_\mu : F \text{ is a finite subset of } E^{< m}\}$ is bounded, the series $\sum_{\mu \in E^{< m}} \Delta_\mu$ converges strictly to a multiplier Φ_m . This Φ_m is a projection because the preceding paragraph shows that the Δ_μ are mutually orthogonal projections. For $\mu \in E^{< m}$ and $\alpha \in E^*$, we have

$$\Delta_\mu t_\alpha = t_\mu \Delta_{r(\mu)} t_\mu^* t_\alpha = \begin{cases} t_\mu \Delta_{r(\mu)} t_{\alpha'} & \text{if } \alpha = \mu \alpha' \\ t_\mu \Delta_{r(\mu)} t_{\mu'}^* & \text{if } \mu = \alpha \mu' \\ 0 & \text{otherwise.} \end{cases}$$

Arguing as in the preceding paragraph, we see that the right-hand side is zero if $\alpha = \mu \alpha'$ and $|\alpha'| > 0$. So

$$\Delta_\mu t_\alpha = \begin{cases} t_\mu \Delta_{r(\mu)} t_{\mu'}^* & \text{if } \mu = \alpha \mu' \\ 0 & \text{otherwise.} \end{cases}$$

The formula given for $\Phi_m t_\alpha t_\beta^* \Phi_m$ now follows from the preceding paragraph. Since $\mathcal{TC}^*(E) = \overline{\text{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in E^*\}$, this establishes the final assertion of the lemma as well. \square

Notation 2.2. Following [31], for each $m \in \mathbb{N}$, we define $\kappa_m \in M_m([0, \infty))$ to be the matrix such that for $i, j \leq \lceil \frac{m}{2} \rceil$,

$$\begin{aligned} \kappa_m(i, j) &= \kappa_m(m+1-i, j) = \kappa_m(i, m+1-j) \\ &= \kappa_m(m+1-i, m+1-j) = \frac{1}{\lceil \frac{m}{2} \rceil + 1} \min\{i, j\}; \end{aligned}$$

so for $l \in \mathbb{N}$,

$$\kappa_{2l} = \frac{1}{l+1} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & \dots & 2 & 2 & 2 & 2 & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \dots & l-1 & l-1 & l-1 & l-1 & \dots & 2 & 1 \\ 1 & 2 & \dots & l-1 & l & l & l-1 & \dots & 2 & 1 \\ 1 & 2 & \dots & l-1 & l & l & l-1 & \dots & 2 & 1 \\ 1 & 2 & \dots & l-1 & l-1 & l-1 & l-1 & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \dots & 2 & 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix},$$

and

$$\kappa_{2l+1} = \frac{1}{l+2} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & \dots & 2 & 2 & 2 & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \dots & l & l & l & \dots & 2 & 1 \\ 1 & 2 & \dots & l & l+1 & l & \dots & 2 & 1 \\ 1 & 2 & \dots & l & l & l & \dots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \dots & 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

As a notational convenience, we write $\kappa_m(i, j) = 0$ whenever $(i, j) \notin \{1, \dots, m\} \times \{1, \dots, m\}$.

Fix $m \in \mathbb{N}$. Recall that Schur multiplication in $M_n(\mathbb{C})$ is entrywise multiplication of matrices. For $l \leq \lceil \frac{m}{2} \rceil$, let $P_{m,l} = \sum_{i=l}^{m-l+1} \theta_{i,i}$ denote the projection onto $\text{span}\{e_l, \dots, e_{m-l+1}\}$. Then Schur multiplication by κ_m is given by $\kappa_m * A = \sum_{l=1}^{\lceil \frac{m}{2} \rceil} \frac{1}{\lceil \frac{m}{2} \rceil + 1} P_{m,l} A P_{m,l}$. Hence Schur multiplication by κ_m is completely positive with completely bounded norm at most $\lceil \frac{m}{2} \rceil / (\lceil \frac{m}{2} \rceil + 1) < 1$.

Lemma 2.3. *For each $m \in \mathbb{N}$, there are completely positive contractions $P_m, Q_m : \mathcal{TC}^*(E) \rightarrow \overline{\text{span}}\{t_\mu \Delta_{r(\mu)} t_\nu^* : \mu, \nu \in E^*\}$ such that, putting $l = \lceil \frac{m}{2} \rceil$, we have*

$$P_m(t_\mu t_\nu^*) = \sum_{\substack{\tau \in r(\mu)E^* \\ m \leq |\mu\tau|, |\nu\tau| < 2m}} \kappa_m(|\mu\tau| - m, |\nu\tau| - m) t_{\mu\tau} \Delta_{r(\tau)} t_{\nu\tau}^* \quad \text{and}$$

$$Q_m(t_\mu t_\nu^*) = \sum_{\substack{\tau \in r(\mu)E^* \\ m+l \leq |\mu\tau|, |\nu\tau| < 2m+l}} \kappa_m(|\mu\tau| - (m+l), |\nu\tau| - (m+l)) t_{\mu\tau} \Delta_{r(\tau)} t_{\nu\tau}^*.$$

Proof. We argue the case for P_m ; the situation for Q_m is similar. The multiplier $\Phi_{2m} - \Phi_m$ obtained from Lemma 2.1 is a projection, and so compression by this element determines a completely positive contraction. Let M be the block operator matrix $M = \sum_{p,q} \kappa_m(p - m, q - m) 1_{E^p \times E^q}$, where $1_{X,Y}$ denotes the matrix in $M_{X,Y}(\mathbb{C})$ whose entries are all 1. Since Schur multiplication by κ_m is a completely positive complete contraction, Schur multiplication $a \mapsto M * a$ by M is a completely positive contraction. So $P_m : a \mapsto M * (\Phi_{2m} - \Phi_m) a (\Phi_{2m} - \Phi_m)$ is a completely positive contraction satisfying the desired formula. \square

For what follows, we need to recall a construction from Section 4 of [19]. If E is a directed graph and m is a positive integer, we define $E(m)$ to be the directed graph with

$$E(m)^0 = E^{<m}, \quad E(m)^1 = \{(e, \mu) : e \in E^1, \mu \in E^{<m}, r(e) = s(\mu)\},$$

$$r(e, \mu) = \mu, \quad s(e, \mu) = \begin{cases} e\mu & \text{if } |\mu| < m - 1 \\ s(e) & \text{if } |\mu| = m - 1. \end{cases}$$

For $\mu \in E^*$, we write $[\mu]_m$ for the unique element of $E^{<m}$ such that $\mu = [\mu]_m \mu'$ with $|\mu'| \in m\mathbb{N}$. There is an injection $i_m : E^* \rightarrow E(m)^*$ given by

$$i_m(v) = v \quad \text{for } v \in E^0,$$

and

$$i_m(\mu) = (\mu_1, \mu_2 \mu_3 \dots \mu_{|\mu|}) (\mu_2, \mu_3 \dots \mu_{|\mu|}) \dots (\mu_{|\mu|}, r(\mu)) \quad \text{if } 1 \leq |\mu| \leq m,$$

and recursively by $i_m(\mu) = i_m([\mu]_m) i_m(\mu') i_m(\mu'')$ when μ is factorized as $\mu = [\mu]_m \mu' \mu''$ with $|\mu'| = m$. We have $s_{E(m)}(i_m(\mu)) = [\mu]_m$ and $r_{E(m)}(i_m(\mu)) = r_E(\mu)$.

In the following, given $p < q \in \mathbb{N}$, we shall write $E^{[p,q]}$ for the set $\{\mu \in E^* : p \leq |\mu| < q\}$.

Lemma 2.4. *Let E be a row-finite directed graph. For $p, m \in \mathbb{N}$ there is a homomorphism $\Lambda_p^{p+m} : \bigoplus_{v \in E^0} \mathcal{K}_{E^{[p,p+m]}_v} \rightarrow \mathcal{TC}^*(E(m))$ such that*

$$\Lambda_p^{p+m}(\theta_{\mu,\nu}) = t_{i_m(\mu)} t_{i_m(\nu)}^*$$

for all $\mu, \nu \in E^{[p,p+m]}$ with $r(\mu) = r(\nu)$.

Proof. We clearly have $(t_{i_m(\mu)} t_{i_m(\nu)}^*)^* = t_{i_m(\nu)} t_{i_m(\mu)}^*$. We show that

$$t_{i_m(\mu)} t_{i_m(\nu)}^* t_{i_m(\alpha)} t_{i_m(\beta)}^* = \delta_{\nu,\alpha} t_{i_m(\mu)} t_{i_m(\beta)}^*.$$

First suppose $\nu = \alpha$. Then $i_m(\nu) = i_m(\alpha)$. Hence (TCK1) gives $t_{i_m(\mu)} t_{i_m(\nu)}^* t_{i_m(\alpha)} t_{i_m(\beta)}^* = t_{i_m(\mu)} q_{r(\nu)} t_{i_m(\beta)}^*$. Since $r(\mu) = r(\nu)$, we have $r_{E(m)}(i(\mu)) = r(\nu)$, and so applying (TCK1) again gives $t_{i_m(\mu)} q_{r(\nu)} t_{i_m(\beta)}^* = t_{i_m(\mu)} t_{i_m(\beta)}^*$ as required. Now suppose that $\nu \neq \alpha$. Assume without loss of generality that $|\nu| \geq |\alpha|$. We consider two cases. First suppose that $\nu \neq \alpha \nu'$ for any ν' . Then there exists $i < |\alpha|$ such that $\nu_i \neq \alpha_i$. Since $i_m(\nu)_i = (\nu_i, \tau)$

and $i_m(\alpha)_i = (\alpha_i, \rho)$ for some $\tau, \rho \in E^{<m}$, we have $i_m(\nu)_i \neq i_m(\alpha)_i$. In particular $i_m(\nu)$ does not have the form $i_m(\alpha)\beta$ for $\beta \in E(m)^*$, and so (TCK1) in $\mathcal{TC}^*(E(m))$ gives $t_{i_m(\nu)}^* t_{i_m(\alpha)} = 0$. Now suppose that $\nu = \alpha\nu'$. Then $|\nu'| > 0$, and since $\alpha, \nu \in E^{[p, p+m)}$, we have $|\nu'| < m$. Thus $|\nu| \not\equiv |\alpha| \pmod{m}$, and in particular $s(i_m(\nu)) = [\nu]_m \neq [\alpha]_m = s(i_m(\alpha))$. So (TCK1) implies that $t_{i_m(\nu)}^* t_{i_m(\alpha)} = 0$.

Hence $\{t_{i_m(\mu)}^* t_{i_m(\nu)}^* : \mu, \nu \in E^{[p, m+p)}\}$ is a family of matrix units for each v , and it follows from the universal properties of the $\mathcal{K}_{E^{<m}_v}$ that there is a homomorphism Λ_p^{p+m} as claimed. \square

The next lemma and its proof are due to James Rout, and will appear in his PhD thesis. We thank James for providing us with the details.

Lemma 2.5 (Rout). *If E is a row-finite directed graph and $m \geq 0$, then there is an injective homomorphism $\iota_m : \mathcal{TC}^*(E) \rightarrow \mathcal{TC}^*(E(m))$ such that*

$$(2.1) \quad \iota_m(q_v) = \sum_{\mu \in vE^{<m}} q_\mu^m \quad \text{and} \quad \iota_m(t_e) = \sum_{(e, \mu) \in E(m)^1} t_{(e, \mu)}^m.$$

The map ι_m descends to an injective homomorphism $\tilde{\iota}_m : C^*(E) \rightarrow C^*(E(m))$.

Proof. Routine calculations show that the $\iota_m(q_v)$ and $\iota_m(t_e)$ form a Toeplitz-Cuntz-Krieger E -family, and so induce a homomorphism $\iota_m : \mathcal{TC}^*(E) \rightarrow \mathcal{TC}^*(E(m))$ satisfying the desired formula. For $v \in E^0$ we have

$$(2.2) \quad \iota_m\left(q_v - \sum_{e \in vE^1} t_e t_e^*\right) = \sum_{\mu \in vE^{<m}} \left(q_\mu^m - \sum_{s_{E(m)}(e, \nu) = \mu} t_{(e, \nu)}^m (t_{(e, \nu)}^m)^*\right).$$

This is nonzero because each $q_\mu^m - \sum_{\lambda \in \mu E(m)^1} t_\lambda^m (t_\lambda^m)^*$ is nonzero in $\mathcal{TC}^*(E(m))$. Hence [12, Theorem 2.1] implies that ι_m is injective.

The quotient map $\pi_m : \mathcal{TC}^*(E(m)) \rightarrow C^*(E(m))$ carries each term in the right-hand side of (2.2) to zero, and so $\pi_m \circ \iota_m$ descends to a homomorphism $\tilde{\iota}_m : C^*(E) \rightarrow C^*(E(m))$. The $\tilde{\iota}_m(p_v)$ are all nonzero because the vertex projections $\pi_m(q_\mu^m)$ in $C^*(E(m))$ are nonzero. The homomorphism $\tilde{\iota}_m$ is clearly equivariant for the gauge actions on $C^*(E)$ and $C^*(E(m))$, and so the gauge-invariant uniqueness theorem [3, Theorem 3.1] implies that $\tilde{\iota}_m$ is injective. \square

Lemma 2.6. *Let E be a row-finite directed graph and let $m \geq 1$. Let q, t be the universal Toeplitz-Cuntz-Krieger family in $\mathcal{TC}^*(E)$, and let Q, T be the universal Toeplitz-Cuntz-Krieger family in $\mathcal{TC}^*(E(m))$. For $\mu \in E^*$, we have*

$$T_{i_m(\mu)} = \iota_m(t_\mu) Q_{r(\mu)} = Q_{[\mu]_m} \iota_m(t_\mu).$$

Proof. First suppose that $l := |\mu| < m$. Then

$$\begin{aligned}
\iota_m(t_\mu)Q_{r(\mu)} &= \iota_m(t_{\mu_1}) \cdots \iota_m(t_{\mu_{l-1}})\iota_m(t_{\mu_l})Q_{r(\mu)} \\
&= \left(\sum_{(\mu_1, \nu) \in E(m)^1} T_{(\mu_1, \nu)} \right) \cdots \left(\sum_{(\mu_{l-1}, \nu) \in E(m)^1} T_{(\mu_{l-1}, \nu)} \right) \left(\sum_{(\mu_l, \nu) \in E(m)^1} T_{(\mu_l, \nu)} \right) Q_{r(\mu)} \\
&= \left(\sum_{(\mu_1, \nu) \in E(m)^1} T_{(\mu_1, \nu)} \right) \cdots \left(\sum_{(\mu_{l-1}, \nu) \in E(m)^1} T_{(\mu_{l-1}, \nu)} \right) T_{(\mu_l, r(\mu))} \\
&= \left(\sum_{(\mu_1, \nu) \in E(m)^1} T_{(\mu_1, \nu)} \right) \cdots \left(\sum_{(\mu_{l-1}, \nu) \in E(m)^1} T_{(\mu_{l-1}, \nu)} \right) Q_{\mu_l} T_{(\mu_l, r(\mu))} \\
&\quad \vdots \\
&= T_{(\mu_1, \mu_2 \dots \mu_l)} \cdots T_{(\mu_l, r(\mu))} \\
&= T_{i_m(\mu)},
\end{aligned}$$

and

$$\begin{aligned}
Q_{[\mu]_m} \iota_m(t_\mu) &= Q_\mu \iota_m(t_\mu) \\
&= Q_\mu \left(\sum_{(\mu_1, \nu) \in E(m)^1} T_{(\mu_1, \nu)} \right) \cdots \left(\sum_{(\mu_l, \nu) \in E(m)^1} T_{(\mu_l, \nu)} \right) \\
&= T_{(\mu_1, \mu_2 \dots \mu_l)} \left(\sum_{(\mu_2, \nu) \in E(m)^1} T_{(\mu_2, \nu)} \right) \cdots \left(\sum_{(\mu_l, \nu) \in E(m)^1} T_{(\mu_l, \nu)} \right) \\
&\quad \vdots \\
&= T_{(\mu_1, \mu_2 \dots \mu_l)} \cdots T_{(\mu_l, r(\mu))} \\
&= T_{i_m(\mu)}.
\end{aligned}$$

Now suppose that $|\mu| = m$. Then, using the above calculations, we have

$$\begin{aligned}
\iota_m(t_\mu)Q_{r(\mu)} &= \left(\sum_{(\mu_1, \nu) \in E(m)^1} T_{(\mu_1, \nu)} \right) Q_{\mu_2 \dots \mu_m} T_{(\mu_2, \mu_3 \dots \mu_l)} \cdots T_{(\mu_m, r(\mu))} \\
&= Q_{s(\mu_1, \mu_2 \dots \mu_m)} T_{(\mu_1, \mu_2 \dots \mu_m)} T_{(\mu_2, \mu_3 \dots \mu_l)} \cdots T_{(\mu_m, r(\mu))} = T_{i_m(\mu)},
\end{aligned}$$

and

$$\begin{aligned}
Q_{[\mu]_m} \iota_m(t_\mu) &= Q_{s(\mu)} \left(\sum_{(\mu_1, \nu) \in E(m)^1} T_{(\mu_1, \nu)} \right) \cdots \left(\sum_{(\mu_m, \nu) \in E(m)^1} T_{(\mu_m, \nu)} \right) \\
&= \sum_{\nu \in r(\mu_1)E^{m-1}} \left(T_{(\mu_1, \nu)} \left(\sum_{(\mu_2, \eta) \in E(m)^1} Q_\nu T_{(\mu_2, \eta)} \right) \cdots \left(\sum_{(\mu_m, \eta) \in E(m)^1} T_{(\mu_m, \eta)} \right) \right) \\
&= \sum_{\nu \in r(\mu_2)E^{m-2}} \left(T_{(\mu_1, \mu_2 \nu)} T_{(\mu_2, \nu)} \left(\sum_{(\mu_3, \eta) \in E(m)^1} T_{(\mu_3, \eta)} \right) \cdots \left(\sum_{(\mu_m, \nu) \in E(m)^1} T_{(\mu_m, \nu)} \right) \right) \\
&\quad \vdots \\
&= T_{(\mu_1, \mu_2 \dots \mu_m)} T_{(\mu_2, \mu_3 \dots \mu_l)} \cdots T_{(\mu_m, r(\mu))} = T_{i_m(\mu)}.
\end{aligned}$$

Now a straightforward induction on $|\mu|$ using the case $|\mu| \leq m$ as a base case establishes the result. \square

Proposition 2.7. *Let E be a row-finite directed graph. For each integer $m \geq 1$, let $q_m : \mathcal{TC}^*(E(m)) \rightarrow C^*(E(m))$ be the quotient map. Then for $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$, we have*

$$\left\| q_m \left(\Lambda_m^{2m} (P_m(t_\mu t_\nu^*)) + \Lambda_{\lceil \frac{3m}{2} \rceil} (Q_m(t_\mu t_\nu^*)) \right) - q_m(t_m(t_\mu t_\nu^*)) \right\| \rightarrow 0$$

as $m \rightarrow \infty$.

Proof. Identify each $\mathcal{K}_{E^{<p}}$ with $\overline{\text{span}}\{s_\mu \Delta_{r(\mu)} s_\nu^* : \mu, \nu \in E^{<p}\}$ as in Lemma 2.1. Fix μ, ν with $r(\mu) = r(\nu) = v$. Combining the formula for $P_m(t_\mu t_\nu^*)$ from Lemma 2.3 with our convention that $\kappa_m(i, j) = 0$ if $(i, j) \notin \{1, \dots, m\} \times \{1, \dots, m\}$, we have

$$P_m(t_\mu t_\nu^*) = \sum_{\tau \in r(\mu)E^*} \kappa_m(|\mu\tau| - m, |\nu\tau| - m) t_{\mu\tau} \Delta_{r(\tau)} t_{\nu\tau}^*.$$

Applying the definition of Λ_m^{2m} and then Lemma 2.6, we obtain

$$\begin{aligned} \Lambda_m^{2m}(P_m(t_\mu t_\nu^*)) &= \sum_{\tau \in r(\mu)E^*} \kappa_m(|\mu\tau| - m, |\nu\tau| - m) t_{i_m(\mu\tau)} t_{i_m(\nu\tau)}^* \\ &= \sum_{i=0}^{m-1} \sum_{j=1}^{\infty} \kappa_m(|\mu| + i + m(j-1), |\nu| + i + m(j-1)) \\ &\quad \left(\sum_{\alpha \in r(\mu)E^i} t_m(t_{\mu\alpha}) \left(\sum_{\rho \in r(\alpha)E^{mj}} t_m(t_{\rho} t_\rho^*) \right) Q_\alpha t_m(t_{\nu\alpha}^*) \right). \end{aligned}$$

Suppose that $m > |\mu|, |\nu|$. Then for each $i < m$ there exists a unique $j(i) \in \mathbb{Z}$ such that $0 \leq |\mu| + i + m(j(i) - 1) < m$. So applying q_m to both sides of the preceding calculation yields

$$\begin{aligned} q_m(\Lambda_m^{2m}(P_m(t_\mu t_\nu^*))) &= \sum_{i=0}^{m-1} \kappa_m(|\mu| + i + m(j(i) - 1), |\nu| + i + m(j(i) - 1)) \\ &\quad \left(\sum_{\alpha \in r(\mu)E^i} \tilde{t}_m(s_{\mu\alpha}) \tilde{t}_m \left(\sum_{\tau \in r(\alpha)E^{mj(i)}} s_\tau s_\tau^* \right) \tilde{t}_m(s_\alpha^*) P_\alpha \tilde{t}_m(s_\nu^*) \right) \\ &= \sum_{i=0}^{m-1} (\kappa_m(|\mu| + i + m(j(i) - 1), |\nu| + i + m(j(i) - 1))) \\ &\quad \left(\sum_{\alpha \in r(\mu)E^i} \tilde{t}_m(s_{\mu\alpha}) P_{r(\alpha)} \tilde{t}_m(s_{\nu\alpha}^*) \right), \end{aligned}$$

where the last equality comes from the Cuntz-Krieger relation and Lemma 2.6.

Similar reasoning gives

$$\begin{aligned} q_m(\Lambda_{\lceil \frac{3m}{2} \rceil} (Q_m(t_\mu t_\nu^*))) &= \sum_{i=0}^{m-1} \kappa_m(|\mu| + i + \lceil m(k(i) - 3/2) \rceil, |\nu| + i + \lceil m(k(i) - 3/2) \rceil) \\ &\quad \left(\sum_{\alpha \in r(\mu)E^i} \tilde{t}_m(s_{\mu\alpha}) P_{r(\alpha)} \tilde{t}_m(s_{\nu\alpha}^*) \right), \end{aligned}$$

where each $k(i)$ is the unique integer such that $0 \leq |\mu| + i + \lceil m(k(i) - 3/2) \rceil < m$. Since $|\mu| + i + m(j(i) - 1)$ and $|\mu| + i + \lceil m(k(i) - 3/2) \rceil$ differ by an odd multiple of $\lceil \frac{m}{2} \rceil$, the sum

$$K_{m,i} := \kappa_m \left(|\mu| + i + m(j(i) - 1), |\nu| + i + m(j(i) - 1) \right) \\ + \kappa_m \left(|\mu| + i + \lceil m(k(i) - 3/2) \rceil, |\nu| + i + \lceil m(k(i) - 3/2) \rceil \right)$$

is either $\frac{m - \lceil |\mu| - |\nu| \rceil}{m+1}$ or $\frac{m+1 - \lceil |\mu| - |\nu| \rceil}{m+1}$ for each $i \leq m$. In particular

$$(2.3) \quad 0 < 1 - K_{m,i} < \frac{\lceil |\mu| - |\nu| \rceil}{m+1}$$

for all m and all $i \leq m$.

So for large m , we have

$$q_m \left(\Lambda_m^{2m} (P_m(t_\mu t_\nu^*)) \right) + \Lambda_{\lceil \frac{3m}{2} \rceil}^{\lceil \frac{5m}{2} \rceil} (Q_m(t_\mu t_\nu^*)) \\ = \tilde{t}_m(s_\mu) \left(\sum_{\alpha \in r(\mu)E < m} K_{m,|\alpha|} \tilde{t}_m(s_\alpha) P_{r(\alpha)} \tilde{t}_m(s_\alpha^*) \right) \tilde{t}_m(s_\nu^*) \\ = \tilde{t}_m(s_\mu) \left(\sum_{\alpha \in r(\mu)E < m} K_{m,|\alpha|} S_{i_m(\alpha)} S_{i_m(\alpha)}^* \right) \tilde{t}_m(s_\nu^*),$$

where we have once again used Lemma 2.6 in the last line. For $0 < |\alpha| < m$, we have $s_{E(m)}^{-1}(\alpha) = \{i_m(\alpha)\}$, and so $S_{i_m(\alpha)} S_{i_m(\alpha)}^* = P_\alpha$. So the final line of the preceding display becomes

$$\tilde{t}_m(s_\mu) \left(K_{m,0} S_{r(\mu)} S_{r(\mu)}^* + \sum_{\alpha \in r(\mu)E < m \setminus \{r(\mu)\}} K_{m,|\alpha|} P_\alpha \right) \tilde{t}_m(s_\nu^*).$$

Since $\tilde{t}(P_{r(\mu)}) = \sum_{\alpha \in r(\mu)E < m} P_\alpha$, the estimate (2.3) shows that

$$\left\| \left(K_{m,0} P_{r(\mu)} + \sum_{\alpha \in r(\mu)E < m \setminus \{r(\mu)\}} K_{m,|\alpha|} \right) - \tilde{t}(P_{r(\mu)}) \right\| \leq \frac{\lceil |\mu| - |\nu| \rceil}{m+1},$$

and so

$$\left\| q_m \left(\Lambda_m^{2m} (P_m(t_\mu t_\nu^*)) \right) + \Lambda_{\lceil \frac{3m}{2} \rceil}^{\lceil \frac{5m}{2} \rceil} (Q_m(t_\mu t_\nu^*)) - \tilde{t}_m(s_\mu s_\nu^*) \right\| \leq \frac{\lceil |\mu| - |\nu| \rceil}{m+1}$$

as well. \square

Definition 2.8. Suppose that $(\beta_m)_{m=1}^\infty$ is a sequence of C^* -homomorphisms $\beta_m : A \rightarrow B_m$, and let \mathcal{C} be a class of C^* -algebras. We say that a sequence (F_m, ϕ_m, ψ_m) is an *asymptotic order- n factorization of the sequence (β_m) through elements of \mathcal{C}* if each $F_m \in \mathcal{C}$, each ψ_m is a completely positive contraction from A to F_m , each F_m decomposes as $\bigoplus_{i=0}^n F_m^{(i)}$, each ϕ_m is a map from F_m to B_m such that $\phi_m|_{F_m^{(i)}}$ is an order-zero completely positive contraction for each $i \leq n$, and $\|\phi_m \circ \psi_m(a) - \beta_m(a)\| \rightarrow 0$ for each $a \in A$. We say that (F_m, ϕ_m, ψ_m) is an asymptotic order- n factorization of a fixed C^* -homomorphism $\beta : A \rightarrow B$ if it is an asymptotic order- n factorization of the constant sequence $(\beta, \beta, \beta, \dots)$.

Lemma 2.9. *For each $m \in \mathbb{N}$, let $\beta_m : A \rightarrow B_m$ be a homomorphism of separable C^* -algebras. If the sequence (β_m) has an asymptotic order- n factorization through AF*

algebras, then it has an asymptotic order- n factorization through finite-dimensional C^* -algebras.

Proof. Let (F_m, ψ_m, φ_m) be an asymptotic order- n factorization of (β_m) through AF algebras. Let $\epsilon > 0$ and let X be a finite subset of A . Choose m such that $\|\varphi_m \circ \psi_m(a) - \beta_m(a)\| < \frac{\epsilon}{2}$ for all $a \in X$. Choose finite dimensional sub-algebras $\tilde{F}_{m,X}^{(i)}$ of the $F_m^{(i)}$ such that for each $a \in X$, there exists $y \in \bigoplus_{i=0}^n \tilde{F}_{m,X}^{(i)} =: \tilde{F}_{m,X}$ with $\|\psi_m(a) - y\| < \frac{\epsilon}{4}$. Since $\tilde{F}_{m,X}$ is a finite dimensional sub-algebra of F_m , there exists a completely positive contraction $\gamma : F_m \rightarrow \tilde{F}_{m,X}$ extending the identity map on $\tilde{F}_{m,X}$.

Let $\iota : \tilde{F}_{m,X} \rightarrow F_m$ be the inclusion map. Let $a \in X$ and let $y \in \tilde{F}_{m,X}$ such that $\|\psi_m(a) - y\| < \frac{\epsilon}{4}$. Note that $\varphi_m \circ \iota \circ \gamma(y) = \varphi_m(y)$ since γ extends the identity map on $\tilde{F}_{m,X}$. Then,

$$\begin{aligned} \|\varphi_m \circ \iota \circ \gamma \circ \psi_m(a) - \beta_m(a)\| &\leq \|\varphi_m \circ \iota \circ \gamma \circ \psi_m(a) - \varphi_m \circ \iota \circ \gamma(y)\| \\ &\quad + \|\varphi_m(y) - \varphi_m \circ \psi_m(a)\| + \|\varphi_m \circ \psi_m(a) - \beta_m(a)\| \\ &< \epsilon. \end{aligned}$$

Since ι is a C^* -homomorphism, it is a completely positive contraction of order zero. Therefore, $\tilde{\varphi}_m := \varphi_m \circ \iota$ restricts to a completely positive contraction of order zero on each $\tilde{F}_{m,X}^{(i)}$; and clearly $\tilde{\psi}_m := \gamma \circ \psi_m$ is a completely positive contraction.

To complete the proof, choose a dense sequence (a_m) in A , and for each m apply the argument of the preceding two paragraphs to $X_m = \{a_1, \dots, a_m\}$. This yields a sequence $(\tilde{F}_{m,X_m}, \tilde{\psi}_m, \tilde{\varphi}_m)$ such that $\|\tilde{\varphi}_m \circ \tilde{\psi}_m(a_i) - \beta_m(a_i)\| \rightarrow 0$ for all i ; since the a_i are dense, continuity then ensures that $\|\tilde{\varphi}_m \circ \tilde{\psi}_m(a) - \beta_m(a)\| \rightarrow 0$ for all a . \square

Corollary 2.10. *Let E be a row-finite directed graph. For each m , let $\tilde{t}_m : C^*(E) \rightarrow C^*(E(m))$ be the homomorphism induced by the homomorphism (2.1). Then the sequence (\tilde{t}_m) has an asymptotic order-1 factorization through finite-dimensional C^* -algebras.*

Proof. Since $C^*(E)$ is nuclear, there is a norm-1 completely positive splitting $\sigma : C^*(E) \rightarrow \mathcal{TC}^*(E)$ for the quotient map [6, Theorem 3.10]. Identify each $\mathcal{K}_{E^{<p}}$ with $\overline{\text{span}}\{s_\mu \Delta_r(\mu) s_\nu^* : \mu, \nu \in E^{<p}\}$ as in Lemma 2.1. For each m , define $\psi_m : C^*(E) \rightarrow \mathcal{K}_{E^{[m,2m]}} \oplus \mathcal{K}_{E^{[\lceil \frac{3m}{2} \rceil, \lceil \frac{5m}{2} \rceil]}}$ by $\psi_m^0(a) = P_m(\sigma(a)) \oplus Q_m(\sigma(a))$, and define $\varphi_m : \mathcal{K}_{E^{[m,2m]}} \oplus \mathcal{K}_{E^{[\lceil \frac{3m}{2} \rceil, \lceil \frac{5m}{2} \rceil]}} \rightarrow C^*(E(m))$ by $\varphi_m((a, b)) = q_m(\Lambda_m^{2m}(a) + \Lambda_{\lceil \frac{3m}{2} \rceil}^{\lceil \frac{5m}{2} \rceil}(b))$. Proposition 2.7 shows that $\|\varphi_m \circ \psi_m^0(s_\mu s_\nu^*) - \tilde{t}_m(s_\mu s_\nu^*)\| \rightarrow 0$ for all μ, ν , and then an $\epsilon/3$ -argument proves that $\|\varphi_m \circ \psi_m^0(a) - \tilde{t}_m(a)\| \rightarrow 0$ for each $a \in C^*(E)$. Lemma 2.4 shows that each φ_m restricts to a homomorphism, and hence a completely positive contraction of order zero, on each of $\mathcal{K}_{E^{[m,2m]}}$ and $\mathcal{K}_{E^{[\lceil \frac{3m}{2} \rceil, \lceil \frac{5m}{2} \rceil]}}$. The corollary now follows from Lemma 2.9. \square

3. REINCLUSION IN K -THEORY

In this section we describe, for each row-finite graph E with no sinks, an inclusion

$$C^*(E(m)) \hookrightarrow C^*(E) \otimes \mathcal{K}$$

which induces the multiplication-by- m map on K -theory when composed with Rout's inclusion $\tilde{t}_m : C^*(E) \rightarrow C^*(E(m))$.

Proposition 3.1. *Let E be a row-finite directed graph with no sinks. For each $m \in \mathbb{N}$ there is an injective homomorphism $j_m : C^*(E(m)) \rightarrow C^*(E) \otimes \mathcal{K}_{E^{<m}}$ such that*

$$(3.1) \quad j_m(p_\mu) = p_{r(\mu)} \otimes \theta_{\mu,\mu} \quad \text{and} \quad j_m(s_{(e,\mu)}) = \begin{cases} s_{e\mu} \otimes \theta_{s(e),\mu} & \text{if } |\mu| = m - 1 \\ p_{r(\mu)} \otimes \theta_{e\mu,\mu} & \text{otherwise.} \end{cases}$$

Proof. We check the Cuntz-Krieger relations. For (CK1), we calculate:

$$j_m(s_{(e,\mu)})^* j_m(s_{(e,\mu)}) = \begin{cases} s_{e\mu}^* s_{e\mu} \otimes \theta_{\mu,\mu} & \text{if } |\mu| = m - 1 \\ p_{r(\mu)} \otimes \theta_{\mu,\mu} & \text{otherwise} \end{cases} = j_m(p_\mu) = j_m(p_{r(e,\mu)}).$$

For (CK2), we fix $\mu \in E^{<m} = E(m)^0$, and consider two cases. First suppose that $\mu = e\mu'$ for some $e \in E^1$ and $\mu' \in E^*$. Then $\mu E(m)^1 = \{(e, \mu')\}$ and $|\mu'| < m - 1$, so we have

$$\sum_{(f,\nu) \in \mu E(m)^1} j_m(s_{f,\nu}) j_m(s_{f,\nu})^* = j_m(s_{(e,\mu')}) j_m(s_{(e,\mu')})^* = p_{r(\mu')} \otimes \theta_{e\mu',e\mu'} = j_m(p_\mu).$$

Now suppose that $|\mu| = 0$ so that $\mu = v \in E^0$. We have $v E(m)^1 = \{(e, \nu) : s(e) = v \text{ and } \nu \in v E^{m-1}\}$. Since E has no sinks, each $p_v = \sum_{\lambda \in v E^m} s_\lambda s_\lambda^*$ in $C^*(E)$, and so

$$\begin{aligned} \sum_{(f,\nu) \in v E(m)^1} j_m(s_{f,\nu}) j_m(s_{f,\nu})^* &= \sum_{f\nu \in v E^m} s_{f\nu} s_{f\nu}^* \otimes \theta_{s(f),s(f)} \\ &= \left(\sum_{f\nu \in v E^m} s_{f\nu} s_{f\nu}^* \right) \otimes \theta_{v,v} = p_v \otimes \theta_{v,v} = j_m(p_v). \end{aligned}$$

So the $j_m(p_\mu)$ and the $j_m(s_e)$ form a Cuntz-Krieger $E(m)$ -family in $C^*(E) \otimes \mathcal{K}_{E^{<m}}$ and the universal property of $C^*(E(m))$ ensures that j_m extends to a homomorphism of C^* -algebras. The $j_m(p_\mu)$ are clearly all nonzero. Let γ denote the gauge action on $C^*(E)$, and let β denote the action of \mathbb{T} on $\mathcal{K}_{E^{<m}}$ determined by $\beta_z(\theta_{\mu,\nu}) = z^{|\mu| - |\nu|} \theta_{\mu,\nu}$. It is routine to check that $(\gamma \otimes \beta)_z(j_m(p_\mu)) = j_m(p_\mu)$ for all μ and that $(\gamma \otimes \beta)_z(j_m(s_{(e,\mu)})) = z j_m(s_{(e,\mu)})$ for all (e, μ) . So the gauge-invariant uniqueness theorem [3, Theorem 2.1] (see also [13, Theorem 2.3]) implies that j_m is injective. \square

Recall that there is an inclusion $\tilde{j}_m : C^*(E) \rightarrow C^*(E(m))$ satisfying the formula (2.1), and so we have an inclusion $j_m \circ \tilde{j}_m : C^*(E) \rightarrow C^*(E) \otimes \mathcal{K}_{E^{<m}}$.

Lemma 3.2. *Let E be a row-finite directed graph with no sinks. Identify $K_*(C^*(E) \otimes \mathcal{K}_{E^{<m}})$ with $K_*(C^*(E))$. Then the induced map $(j_m \circ \tilde{j}_m)_* : K_*(C^*(E)) \rightarrow K_*(C^*(E))$ is given by multiplication by m .*

Proof. We first recall from [23, page 439] (Theorem 4.2.4 and the discussion immediately preceding it) the computation of the K -theory of $C^*(E)$. We can identify $K_0(C^*(E)^\gamma)$ with $\varinjlim (\mathbb{Z}E^0, A^t)$ where A is the $E^0 \times E^0$ matrix $A(v, w) = |vE^1w|$. We then have $K_1(C^*(E)) \cong \ker \phi$ and $K_0(C^*(E)) \cong \text{coker } \phi$ where $\phi : \varinjlim (\mathbb{Z}E^0, A^t) \rightarrow \varinjlim (\mathbb{Z}E^0, A^t)$ is the homomorphism induced by $1 - A^t$. Identify $\varinjlim (\mathbb{Z}E^0, A^t)$ with tail-equivalence classes of sequences $(a_j) \in (\mathbb{Z}E^0)^\infty$ such that $a_{j+1} = A^t a_j$ for large j , and define $\omega : \mathbb{Z}E^\infty \rightarrow \varinjlim (\mathbb{Z}E^0, A^t)$ by $\omega(x) = (x, A^t x, (A^t)^2 x, \dots)$. Then Theorem 4.2.4 of [23] says that ω induces isomorphisms of $\ker(1 - A^t) \cong \ker(\phi)$ and $\text{coker}(1 - A^t) \cong \text{coker } \phi$.

It is routine to check that $j_m \circ \tilde{l}_m(p_v)$ carries $C^*(E)^\gamma$ into $C^*(E)^\gamma \otimes \mathcal{K}_{E^{<m}}$. We consider the effect of this map on $K_0(C^*(E)^\gamma)$: fix $v \in E^0$ and calculate

$$j_m \circ \tilde{l}_m(p_v) = \sum_{\mu \in vE^{<m}} j_m(p_\mu) = \sum_{\mu \in vE^{<m}} p_{r(\mu)} \otimes \theta_{\mu,\mu};$$

so composing with the isomorphism $K_0(C^*(E)^\gamma) \cong K_0(C^*(E)^\gamma \otimes \mathcal{K}_{E^{<m}})$, the induced map $(j_m \circ \tilde{l}_m)_* : K_0(C^*(E)^\gamma) \rightarrow K_0(C^*(E)^\gamma)$ is given by

$$[p_v]_0 \mapsto \sum_{\mu \in vE^{<m}} [p_{r(\mu)}]_0 = \sum_{j=0}^{m-1} \sum_{w \in E^0} A^t(w, v) [p_w]_0 = \sum_{j=0}^{m-1} (A^t)^j \cdot [p_v]_0.$$

Suppose that $x \in \ker(1 - A^t)$. Then $\omega(x) = (x, x, x, \dots)$ and $A^t x = x$, and so we have

$$(j_m \circ \tilde{l}_m)_*(\omega(x)) = \sum_{j=0}^{m-1} ((A^t)^j x, (A^t)^j x, \dots) = m \cdot (x, x, \dots) = m \cdot \omega(x),$$

and we deduce that $(j_m \circ \tilde{l}_m)_*$ is given by multiplication by m on $K_1(C^*(E))$. To calculate the induced map on $K_0(C^*(E))$, observe that for each $(a_i) \in \varinjlim (\mathbb{Z}E^0, A^t)$, we have $(a_i - A^t a_i)_{i=0}^\infty \in \text{Im } \phi$, and so $(a_i) + \text{Im } \phi = (A^t a_i) + \text{Im } \phi$. So for $v \in E^0$,

$$(j_m \circ \tilde{l}_m)_*(\omega(\delta_v)) + \text{Im } \phi = \sum_{j=0}^{m-1} ((A^t)^j \delta_v, (A^t)^{j+1} \delta_v, \dots) + \text{Im } \phi = m \cdot (\delta_v, A^t \delta_v, \dots) = m \cdot \omega(\delta_v),$$

and we deduce that $(j_m \circ \tilde{l}_m)_*$ is given by multiplication by m on $K_1(C^*(E))$. \square

It will be important for our main result that this inclusion restricts to an inclusion map of the same form on gauge-invariant ideals, and induces an inclusion map of the same form on the corresponding quotients.

Lemma 3.3. *Let E be a row-finite directed graph with no sinks. If I is a gauge-invariant ideal of $C^*(E)$, then $(j_m \circ \tilde{l}_m)_*$ restricts to the multiplication-by- m map $K_*(I) \rightarrow K_*(I)$ and induces the multiplication-by- m map $K_*(C^*(E)/I) \rightarrow K_*(C^*(E)/I)$. Moreover, if J is a gauge-invariant ideal of $C^*(E)$ with $J \subseteq I$, then the induced map $K_*(I/J) \rightarrow K_*(I/J)$ is multiplication by m .*

Proof. Let $H = \{v \in E^0 : p_v \in I\}$. By [3, Lemma 1.1], the multiplier projection $P_H := \sum_{v \in H} P_v$ is full in I , and there are canonical isomorphisms $P_H C^*(E) P_H \cong C^*(EH)$ and $C^*(E)/I \cong C^*(E \setminus EH)$. Let $H(m) := \{\mu \in E^{<m} : s(\mu) \in H\} \subseteq E(m)^0$. This $H(m)$ is a hereditary set, and we have $H(m)E(m) = (HE)(m)$ as subgraphs of $E(m)$. So the proof of [3, Theorem 4.1(c)] shows that $P_{H(m)} := \sum_{\mu \in H(m)} p_\mu \in \mathcal{M}(C^*(E(m)))$ is a full projection in $I_{H(m)}$ and there is a canonical isomorphism $P_{H(m)} C^*(E(m)) P_{H(m)} \cong C^*((HE)(m))$. The definition of the homomorphism $j_m : C^*(E(m)) \rightarrow C^*(E) \otimes \mathcal{K}_{E^{<m}}$ shows that it restricts to a homomorphism from $P_{H(m)} C^*(E(m)) P_{H(m)}$ to $P_H C^*(E) P_H \otimes \mathcal{K}_{(HE)^{<m}}$. So,

setting $Q_{H,m} := \sum_{\mu \in (HE) < m} P_H \otimes \theta_{\mu,\mu} \in \mathcal{M}(C^*(E) \otimes \mathcal{K}_{E < m})$, we see that the diagram

$$\begin{array}{ccccc}
C^*(HE) & \xrightarrow{\cong} & P_H C^*(E) P_H & \xrightarrow{\subseteq} & C^*(E) \\
\downarrow j_m^{HE} \circ \tilde{t}_m^{HE} & & \downarrow j_m^E \circ \tilde{t}_m^E & & \downarrow j_m^E \circ \tilde{t}_m^E \\
C^*(HE) \otimes \mathcal{K}_{(HE) < m} & \xrightarrow{\cong} & Q_{H,m}(C^*(E) \otimes \mathcal{K}_{E < m})Q_{H,m} & \xrightarrow{\subseteq} & C^*(E)
\end{array}$$

commutes.

Lemma 3.2 implies that the vertical map on the left induces multiplication by m in K -theory, and so the map in the middle does too. This map is the restriction of $j_m^E \circ \tilde{t}_m^E$ to $P_H C^*(E) P_H$. Since P_H and $Q_{H,m}$ are full in I_H and $I_H \otimes \mathcal{K}_{E < m}$, compression by these projections induces isomorphisms in K -theory, so we deduce that the restriction of $j_m^E \circ \tilde{t}_m^E$ to I_H also induces the multiplication-by- m map.

We have already showed that j_m carries I into the ideal $I_{H(m)} \triangleleft C^*(E(m))$ generated by $\{p_\mu : \mu \in H(m)\}$. Hence j_m induces a homomorphism $\tilde{j}_m : C^*(E)/I \rightarrow C^*(E(m))/I_{H(m)}$. By [3, Theorem 4.1] there is a canonical isomorphism $C^*(E)/I \cong C^*(E \setminus EH)$. It is routine to check that the saturation of $H(m)$ is the set $K = \{\mu \in E < m : s(\mu) \in H\}$. So [3, Theorem 4.1] again implies that $C^*(E(m))/I_{H(m)}$ is canonically isomorphic to $C^*(E(m) \setminus E(m)K)$. It is also routine to check that $(E \setminus EH)(m) = E(m) \setminus E(m)K$ as subsets of $E(m)$. By comparing formulas on generators, we see that the diagram

$$\begin{array}{ccccc}
C^*(E) & \xrightarrow{q_I} & C^*(E)/I & \xrightarrow{\cong} & C^*(E \setminus EH) \\
\downarrow j_m^E \circ \tilde{t}_m^E & & \downarrow (j_m^E \circ \tilde{t}_m^E)^\sim & & \downarrow j_m^{E \setminus EH} \circ \tilde{t}_m^{E \setminus EH} \\
C^*(E) \otimes \mathcal{K}_{E < m} & \xrightarrow{q_I \otimes \mathcal{K}_{E < m}} & (C^*(E) \otimes \mathcal{K}_{E < m}) / (I \otimes \mathcal{K}_{E < m}) & \xrightarrow{\cong} & C^*(E \setminus EH) \otimes \mathcal{K}_{E < m}
\end{array}$$

commutes. Since $C^*(E \setminus EH) \otimes \mathcal{K}_{(E \setminus EH) < m}$ embeds as a full corner in $C^*(E \setminus EH) \otimes \mathcal{K}_{E < m}$, the corresponding inclusion map in K -theory is an isomorphism. So Lemma 3.2 implies that the vertical inclusion map $j_m^{E \setminus EH} \circ \tilde{t}_m^{E \setminus EH}$ on the right of the diagram induces multiplication by m in K -theory. Thus the middle vertical map does too. Since this is precisely the map on the quotient induced by $j_m^E \circ \tilde{t}_m^E$, we deduce that $(j_m \circ \tilde{t}_m)_*$ restricts to $m \cdot \text{id}$ on $K_*(I)$ and induces $m \cdot \text{id}$ on $K_*(C^*(E)/I)$.

For the final assertion, apply the preceding assertion to the gauge-invariant ideal $P_H J P_H$ of $P_H C^*(E) P_H \cong C^*(E \setminus EH)$, using that compression by the full projection P_H induces an isomorphism of the K -theory of I . \square

4. A TECHNIQUE OF ENDERS

In this section we apply general results of Meyer-Nest and of Kirchberg to generalize a technique of Enders to see that, under suitable hypotheses, an endomorphism κ_m of a strongly purely infinite nuclear C^* -algebra A whose image $\text{FK}(\phi)$ in filtered K -theory is

the times- m map can be post-composed with an automorphism of A so that the resulting map approximates the identity. We apply this result to prove the following theorem (the proof appears at the end of the section), which is our main result.

Theorem 4.1. *Let E be a directed graph. Suppose that $C^*(E)$ is purely infinite and has finitely many ideals. Then the nuclear dimension of $C^*(E)$ is 1.*

We start by recalling some terminology and background from [21].

We write $\text{Prim}(A)$ for the primitive-ideal space of a C^* -algebra A , and equip $\text{Prim}(A)$ with the hull-kernel topology. Recall from [21, Definition 2.3] that if X is a topological space, then a C^* -algebra over X is a pair (A, ψ) consisting of a C^* -algebra A and a continuous map $\psi : \text{Prim}(A) \rightarrow X$. As in [21, Definition 5.1], we say that (A, ψ) is a tight C^* -algebra over X if ψ is a homeomorphism.

Let (A, ψ) be a C^* -algebra over X . We write $\mathbb{O}(X)$ for the lattice of open subsets of X . Given $U \in \mathbb{O}(X)$, we write $A(U)$ for the corresponding ideal $\bigcap \{I \in \text{Prim}(A) : \psi(I) \notin U\}$. We then have $A(U \cup V) = A(U) + A(V)$ and $A(U \cap V) = A(U) \cap A(V)$ for $U, V \in \mathbb{O}(X)$ (in fact, a stronger statement is true [21, Lemma 2.8]).

As in [21, Definition 2.10], if (A, ψ) and (B, ρ) are C^* -algebras over X , we say that a homomorphism $\phi : A \rightarrow B$ is X -equivariant if $\phi(A(U)) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$. If, in addition, A is a tight C^* -algebra over X , we say that ϕ is full if, whenever a generates $A(U)$ as an ideal, $\phi(a)$ generates $B(U)$ as an ideal.

Given homomorphisms $\phi, \psi : A \rightarrow B$ between C^* -algebras, we say that ϕ and ψ are asymptotically unitarily equivalent, and write $\phi \sim_{\text{au}} \psi$, if there is a continuous family of unitaries U_t ($t \in [1, \infty)$) in $\mathcal{M}(B)$ such that $\text{Ad } U_t \circ \phi(a) \rightarrow \psi(a)$ as $t \rightarrow \infty$ for all $a \in A$. We say that ϕ and ψ are approximately unitarily equivalent, and write $\phi \approx_u \psi$, if there exists a sequence U_n of unitaries in $\mathcal{M}(B)$ such that $U_n \phi(a) U_n^* \rightarrow \psi(a)$ for all $a \in A$.

Lemma 4.2. *Let X be a finite topological space, and let (A, ψ) be a tight C^* -algebra over X . Fix a unital homomorphism $j_1 : \mathcal{O}_\infty \rightarrow \mathcal{O}_2$ and a nonzero homomorphism $j_2 : \mathcal{O}_2 \rightarrow \mathcal{O}_\infty$, and let $j = (j_2 \circ j_1) \otimes \text{id}_\mathcal{K} : \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow \mathcal{O}_\infty \otimes \mathcal{K}$. Suppose that A is separable and nuclear, and define $j_X^A := \text{id}_A \otimes j : A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow A \otimes \mathcal{O}_\infty \otimes \mathcal{K}$. Then*

- (1) $\text{id}_A \oplus j_X^A \sim_{\text{au}} \text{id}_{A \otimes \mathcal{O}_\infty \otimes \mathcal{K}}$;
- (2) $j_X^A \oplus j_X^A \sim_{\text{au}} j_X^A$; and
- (3) j_X^A is a full X -equivariant homomorphism.

If (B, ρ) is a second tight C^ -algebra over X and if $\phi : A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is a homomorphism, then $(j_X^B \circ \phi) \oplus (j_X^B \circ \phi) \sim_{\text{au}} j_X^B \circ \phi$.*

Proof. Since $1 \otimes \mathcal{M}(\mathcal{K}) \cong 1 \otimes \mathcal{B}(\ell^2)$ is contained in $\mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})$, there exist isometries $s_1, s_2 \in \mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})$ such that $s_1 s_1^* + s_2 s_2^* = 1_{\mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})}$. Hence the isometries $t_i := 1_{\mathcal{M}(A)} \otimes s_i \in \mathcal{M}(A \otimes \mathcal{O}_\infty \otimes \mathcal{K})$ satisfy $t_1 t_1^* + t_2 t_2^* = 1$ as well.

We have $KK(j_1) \in KK(\mathcal{O}_\infty, \mathcal{O}_2) = \{0\}$, and hence $KK(j) = 0$. So [24, Theorem 4.13] implies that

- (j1) $\text{id}_{\mathcal{O}_\infty \otimes \mathcal{K}} \oplus j \sim_{\text{au}} \text{id}_{\mathcal{O}_\infty \otimes \mathcal{K}}$,
- (j2) $j(a \otimes x) = j_2(1) a j_2(1) \otimes x$, and
- (j3) $j \oplus j \sim_{\text{au}} j$.

We have

$$\text{id}_A \oplus j_X^A = \text{Ad}(t_1) \circ \text{id}_{A \otimes \mathcal{O}_\infty \otimes \mathcal{K}} + \text{Ad}(t_2) \circ j_X^A = \text{id}_A \otimes (\text{Ad}(s_1) \circ \text{id}_{\mathcal{O}_\infty \otimes \mathcal{K}} + \text{Ad}(s_2) \circ j).$$

Property (j1) of j therefore gives $\text{id}_A \oplus j_X^A \sim_{\text{au}} \text{id}_A \otimes \text{id}_{\mathcal{O}_\infty \otimes \mathcal{K}} = \text{id}_{A \otimes \mathcal{O}_\infty \otimes \mathcal{K}}$. Likewise, using property (j2) at the last step, we obtain

$$\begin{aligned} j_X^A \oplus j_X^A &= \text{Ad}(t_1) \circ j_X^A + \text{Ad}(t_2) \circ j_X^A \\ &= \text{id}_A \otimes (\text{Ad}(s_1) \circ j + \text{Ad}(s_2) \circ j) \sim_{\text{au}} \text{id}_A \otimes j = j_X^A. \end{aligned}$$

For $U \in \mathcal{O}(X)$, we have

$$\begin{aligned} j_X^A((A \otimes \mathcal{O}_\infty \otimes \mathcal{K})(U)) &= A(U) \otimes j(\mathcal{O}_\infty \otimes \mathcal{K}) \\ &= A(U) \otimes j_2(1) \mathcal{O}_\infty j_2(1) \otimes \mathcal{K} \subseteq A(U) \otimes \mathcal{O}_\infty \otimes \mathcal{K}, \end{aligned}$$

so that j_X^A is X -equivariant. To see that it is full, fix a full element $y \in (A \otimes \mathcal{O}_\infty \otimes \mathcal{K})(U)$. Fix a full element $a \in A(U)$ and let e_{ij} be the canonical matrix units in \mathcal{K} . Then $a \otimes 1 \otimes e_{11}$ belongs to the ideal generated by y , and so $a \otimes j_2(1) \otimes e_{11}$ belongs to the ideal generated by $j_X^A(y)$. Since $j_2(1)$ is nonzero and \mathcal{O}_∞ and \mathcal{K} are simple, the ideal generated by $a \otimes j_2(1) \otimes e_{11}$ is $\overline{AaA} \otimes \mathcal{O}_\infty \otimes \mathcal{K} = (A \otimes \mathcal{O}_\infty \otimes \mathcal{K})(U)$.

Now suppose that (B, ρ) is a second tight C^* -algebra over X and that $\phi : A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is a homomorphism. The above argument shows that $j_X^B \oplus j_X^B \sim_{\text{au}} j_X^B$, so there exists a continuous family of unitaries U_t such that $U_t(j_X^B \oplus j_X^B)(b)U_t^* \rightarrow j_X^B(b)$ for all b . Hence

$$U_t((j_X^B \circ \phi) \oplus (j_X^B \circ \phi))(a)U_t^* = U_t(j_X^B \oplus j_X^B)(\phi(a))U_t^* \rightarrow j_X^B(\phi(a)). \quad \square$$

Lemma 4.3. *Suppose that (A, ψ) and (B, ρ) are separable, nuclear, tight C^* -algebras over a finite topological space X . For $i = 1, 2$, suppose that $\phi_i : A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ is a full X -equivariant homomorphism such that $\phi_i \oplus \phi_i \sim_{\text{au}} \phi_i$. Then $\phi_1 \sim_{\text{au}} \phi_2$.*

Proof. This follows from [16, Hauptsatz 2.15]. \square

We can now deduce that, amongst full equivariant homomorphisms between strongly purely infinite C^* -algebras, asymptotically unitary equivalence is characterized by equivariant KK -equivalence.

Theorem 4.4. *Let (A, ψ) and (B, ρ) be separable, nuclear, tight C^* -algebras over a finite topological space X . For $i = 1, 2$ let $\phi_i : A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ be a full X -equivariant homomorphism. Then $KK(X; \phi_1) = KK(X; \phi_2)$ if and only if $\phi_1 \sim_{\text{au}} \phi_2$.*

Consequently, if $KK(X; \phi_1) = KK(X; \phi_2)$, then $\phi_1 \approx_u \phi_2$.

Proof. If $\phi_1 \sim_{\text{au}} \phi_2$, then they are $KK(X; \cdot)$ -equivalent by definition of the $KK(X; \cdot)$ functor.

Suppose that $KK(X; \phi_1) = KK(X; \phi_2)$. By [16, Hauptsatz 4.2], there are full X -equivariant homomorphisms $h_1, h_2 : A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \rightarrow B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that $h_i \oplus h_i \sim_{\text{au}} h_i$ for $i = 1, 2$ and such that $\phi_1 \oplus h_1 \sim_{\text{au}} \phi_2 \oplus h_2$.

The final statement of Lemma 4.2 implies that each $j_X^B \circ \phi_i$ satisfies $j_X^B \circ \phi_i \oplus j_X^B \circ \phi_i \sim_{\text{au}} j_X^B \circ \phi_i$, and so we may apply Lemma 4.3 twice to see that $j_X^B \circ \phi_i \sim_{\text{au}} h_i$ for $i = 1, 2$. Lemma 4.2(1) implies that $\text{id}_B \oplus j_X^B \sim_{\text{au}} \text{id}_B$, and so for each i we have

$$\phi_i \sim_{\text{au}} (\text{id}_B \oplus j_X^B) \circ \phi_i = \phi_i \oplus j_X^B \circ \phi_i \sim_{\text{au}} \phi_i \oplus h_i.$$

Since $\phi_1 \oplus h_1 \sim_{\text{au}} \phi_2 \oplus h_2$, the result follows. \square

We now have the material we need to extend Enders' technique to purely infinite C^* -algebras with finite primitive-ideal spaces, leading to the proof of our main result about

purely infinite graph C^* -algebras. Recall that given a finite topological space X , the filtered- K -theory functor FK sends each C^* -algebra A over X to the K -groups of the subquotients of A corresponding to locally closed subsets of X , together with all the natural transformations between these groups. Morphisms between $FK(A)$ and $FK(B)$ respect the natural transformations induced by KK elements between the subquotients of A and B corresponding to any given locally closed subset of X . For details, see [22].

Proposition 4.5. *Suppose that (A, ϕ) is a tight C^* -algebra over a finite topological space X . Suppose that A is stable, nuclear, separable and purely infinite. Suppose that there is a sequence κ_m of full X -equivariant homomorphisms $\kappa_m : A \rightarrow A$ such that*

- (1) $FK(\kappa_m) = m \cdot FK(\text{id}_A)$ and
- (2) the sequence (κ_m) has an asymptotic order- n factorization through elements of \mathcal{C} .

Then id_A has an asymptotic order- n factorization through elements of \mathcal{C} .

Proof. Using [28, Theorem 4.3] and [17, Theorem 3.15], and a simple induction argument, one can show that every separable, nuclear, purely infinite C^* -algebra with finitely many ideals is \mathcal{O}_∞ -absorbing. Thus, $A \cong A \otimes \mathcal{O}_\infty$. Let (F_m, ϕ_m, ψ_m) be an asymptotic order- n factorization through elements of \mathcal{C} . The element $x = -KK(X; \text{id}_A)$ is a KK -equivalence, so [16, Folgerung 4.3] implies that there is an automorphism λ of A such that $KK(X; \lambda) = -KK(X; \text{id}_A)$. We then have $FK(\lambda) = -FK(\text{id}_A)$ because the filtered K functor is compatible with KK . Since A is stable, $\mathcal{M}(A)$ contains isometries s_1 and s_2 such that $s_1 s_1^* + s_2 s_2^* = 1$. Define $\beta_m : A \rightarrow A$ by $\beta_m(a) = s_1 \kappa_{m+1}(a) s_1^* + s_2 \lambda(\kappa_m(a)) s_2^*$. Since each κ_l is a full X -equivariant homomorphism, the same is true of each β_m , and we have

$$FK(\beta_m) = FK(\kappa_{m+1}) + FK(\lambda \circ \kappa_m) = (m+1)FK(\text{id}_A) - mFK(\text{id}_A) = FK(\text{id}_A).$$

Hence [21, Proposition 4.15] implies that $KK(X; \beta_m)$ is a $KK(X; \cdot)$ -equivalence. Now [16, Folgerung 4.3] implies that there is an automorphism γ_m of A such that $KK(X; \gamma_m \circ \beta_m) = KK(X; \text{id}_A)$. Theorem 4.4 then implies that $\gamma_m \circ \beta_m \approx_u \text{id}_A$; fix a sequence $u_n \in \mathcal{M}(A)$ of unitaries implementing the approximate unitary equivalence, so $\|u_n \gamma_n \circ \beta_n(a) u_n^* - a\| \rightarrow 0$ for each $a \in A$.

Write $\Psi_m := \psi_m \oplus \psi_{m+1} : A \rightarrow F_m \oplus F_{m+1}$; this is a completely positive contraction because each ψ_i is. Define $\Phi_m : F_m \oplus F_{m+1} \rightarrow A$ by

$$\Phi_m(a, b) = u_m \gamma_m (s_1 \phi_{m+1}(b) s_1^* + s_2 \lambda(\phi_m(a)) s_2^*) u_m^*.$$

Each of ϕ_m and ϕ_{m+1} is completely positive by definition, and then $\lambda \circ \phi_m$ is completely positive too because λ is a homomorphism. Conjugation by an isometry is likewise completely positive, so $(a, b) \mapsto s_1 \phi_{m+1}(b) s_1^* + s_2 \lambda(\phi_m(a)) s_2^*$ is also completely positive. Since $\text{Ad}(u_m) \circ \gamma_m$ is a homomorphism, it follows that Φ_m is completely positive. Since λ , $\text{Ad}(u_m)$ and γ_m are homomorphisms, and therefore norm-decreasing, and since $s_1 s_1^* \perp s_2 s_2^*$, we have $\|\Phi_m(a, b)\| = \max\{\|\phi_{m+1}(b)\|, \|\phi_m(a)\|\}$; since $\phi_{m+1} \oplus \phi_m$ is contracting on each $F_m^{(k)} \oplus F_{m+1}^{(k)}$, the same is true of Φ_m . Fix $k \in \{0, 1, \dots, n\}$ and suppose that $a = (a_m, a_{m+1})$ and $b = (b_m, b_{m+1})$ are orthogonal elements of $F_m^{(k)} \oplus F_{m+1}^{(k)} \subseteq F_m \oplus F_{m+1}$; so $a_i b_i = b_i a_i = 0$ for $i = m, m+1$. Since $s_i^* s_j = \delta_{i,j} 1$, we have

$$\gamma_m^{-1} (u_m^* \Phi_m(a) \Phi_m(b) u_m) = s_1 \phi_{m+1}(a_{m+1}) \phi_{m+1}(b_{m+1}) s_1^* + s_2 \lambda(\phi_m(a_m) \phi_m(b_m)) s_2^* = 0.$$

Since γ_m^{-1} and $\text{Ad}(u_m^*)$ are automorphisms, we deduce that $\Phi_m(a) \Phi_m(b) = 0$, and symmetrically for $\Phi_m(b) \Phi_m(a)$. So Φ_m restricts to an order-0 contraction on $F_m^{(k)} \oplus F_{m+1}^{(k)}$.

Fix $a \in A$. It now suffices to show that $\|\Phi_m \circ \Psi_m(a) - a\| \rightarrow 0$. By definition of Φ_m and Ψ_m , we have

$$\Phi_m(\Psi_m(a)) = u_m \gamma_m (s_1 \phi_{m+1}(\psi_{m+1}(a)) s_1^* + (s_2 \lambda(\phi_m(\psi_m(a)))) s_2^*) u_m^*.$$

Hence

$$\begin{aligned} & \|\Phi_m(\Psi_m(a)) - a\| \\ & \leq \left\| \Phi_m(\Psi_m(a)) - u_m \gamma_m (s_1 \kappa_{m+1}(a) s_1^* + s_2 \lambda(\kappa_m(a))) u_m^* \right\| \\ & \quad + \left\| u_m \gamma_m (s_1 \kappa_{m+1}(a) s_1^* + s_2 \lambda(\kappa_m(a))) u_m^* - a \right\| \\ & = \left\| u_m \gamma_m \left(s_1 (\phi_{m+1} \circ \psi_{m+1}(a) - \kappa_{m+1}(a)) s_1^* + s_2 \lambda(\phi_m \circ \psi_m(a) - \kappa_m(a)) s_2^* \right) u_m^* \right\| \\ & \quad + \left\| u_m \gamma_m \circ \beta_m(a) u_m^* - a \right\| \\ & = \max \left\{ \|\phi_{m+1} \circ \psi_{m+1}(a) - \kappa_{m+1}(a)\|, \|\phi_m \circ \psi_m(a) - \kappa_m(a)\| \right\} \\ & \quad + \left\| u_m \gamma_m \circ \beta_m(a) u_m^* - a \right\| \\ & \rightarrow 0, \end{aligned}$$

completing the proof. \square

Proof of Theorem 4.1. Let E' be a Drinen-Tomforde desingularization of E as in [7], and let G be a graph as in [27] such that $C^*(G) \cong C^*(E') \otimes \mathcal{K}$. The nuclear dimension of $C^*(G)$ is equal to that of $C^*(E)$ by [31, Corollary 2.8(i)]. Since $C^*(E)$ is not an AF-algebra, the nuclear dimension of $C^*(E)$ (and hence the nuclear dimension of $C^*(G)$) is at least 1 (see [31, Remarks 2.2(iii)]). We must show that it is at most 1.

Since G is row-finite, Corollary 2.10 implies that \tilde{t}_m has an asymptotic order-1 factorization through finite dimensional C^* -algebras. Using again that G is row-finite and has no sinks, we can apply Proposition 3.1 to obtain a map $j_m : C^*(G(m)) \rightarrow C^*(G) \otimes \mathcal{K}_{G < m}$ satisfying (3.1). Since $C^*(G)$ is stable, there is an isomorphism $\rho : C^*(G) \otimes \mathcal{K}_{G < m} \rightarrow C^*(G)$ such that $FK(\rho) = \text{id}$. By the construction of \tilde{t}_m and j_m , the map $j_m \circ \tilde{t}_m$ is an X -equivariant homomorphism and $j_m \circ \tilde{t}_m(a)$ is full in $C^*(E)(U)$ whenever a is full in $C^*(E)(U)$. Thus, $j_m \circ \tilde{t}_m$ is a full X -equivariant homomorphism. Lemma 3.3 implies that the maps $\kappa_m := \rho \circ j_m \circ \tilde{t}_m : C^*(G) \rightarrow C^*(G)$ satisfy $FK(\kappa_m) = m \cdot FK(\text{id}_{C^*(G)})$. Since j_m and ρ are $*$ -homomorphisms, κ_m has an asymptotic order-1 factorization through finite dimensional C^* -algebras. So Proposition 4.5 implies that $\text{id}_{C^*(G)}$ has an asymptotic order-1 factorization through finite dimensional C^* -algebras, and thus the nuclear dimension of $C^*(G)$ is at most 1. \square

Corollary 4.6. *Let X be an accordion space as defined in [4] and let A be a separable, nuclear, purely infinite, tight C^* -algebra over X . Suppose that $K_1(A(x))$ is free abelian and that $A(x)$ is in the Rosenberg-Schochet bootstrap category \mathcal{N} for each $x \in X$. Then the nuclear dimension of A is 1.*

Proof. By [2, Corollary 7.16] (see also [1]), there exists a row-finite directed graph E such that $C^*(E) \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. Now [31, Corollary 2.8(i)] and Theorem 4.1 imply that the nuclear dimension of A is 1. \square

In [14], Jeong and Park show how to describe the C^* -algebra of a row-finite graph with no sources as a direct limit of C^* -algebras of graphs with finitely many ideals. Here we adapt their technique to see that if E satisfies Condition (K) and every vertex of E connects to a cycle, then $C^*(E)$ has nuclear dimension at most 2.

Corollary 4.7. *Let E be a graph that satisfies Condition (K), and suppose that each vertex of E connects to a cycle in E . Then $1 \leq \dim_{\text{nuc}}(C^*(E)) \leq \dim_{\text{nuc}}(\mathcal{TC}^*(E)) \leq 2$.*

Proof. Since E contains cycles, $C^*(E)$ is not AF and therefore has nuclear dimension at least 1 by [31, Remark 2.2(iii)]. Since $C^*(E)$ is a quotient of $\mathcal{TC}^*(E)$, Proposition 2.9 of [31] implies that $\dim_{\text{nuc}}(C^*(E)) \leq \dim_{\text{nuc}}(\mathcal{TC}^*(E))$. It remains to show that $\dim_{\text{nuc}}(\mathcal{TC}^*(E)) \leq 2$.

Fix a finite set $V \subseteq E^0$ and a finite set $F \subseteq E^1$. We claim that there exists a finite subgraph $G_{V,F}$ of E such that $V \subseteq G_{V,F}^0$, $F \subseteq G_{V,F}^1$, every vertex in $G_{V,F}$ connects to a cycle in $G_{V,F}$, and $G_{V,F}$ satisfies Condition (K). Since every vertex in E connects to a cycle, there exists, for each $v \in V \cup r(F)$, a path $\lambda^v = \lambda_1^v \dots \lambda_{|\lambda^v|}^v$ in vE^* such that $s(\lambda_n^v) = r(\lambda^v)$ for some $n \leq |\lambda^v|$. Let G be the subgraph of E given by $G_0^1 := F \cup \{\lambda_i^v : v \in V \cup r(F), i \leq |\lambda^v|\}$, and let $G_0^0 := r(G_0^1) \cup s(G_0^1)$. By construction, every vertex in G_0 connects to a cycle in G_0 . For each $w \in G_0^0$ that lies on exactly one cycle, say μ^w , in G_0 , condition (K) in E implies that there exists $n \leq |\mu^w|$ and a cycle ν^w in E based at $s(\mu_n^w)$ such that $\nu_1^w \neq \mu_n^w$. We let $G_{V,F}^1 = G_0^1 \cup \{s(\nu_i^w) : w \text{ lies on exactly one cycle in } G_0, 1 \leq i \leq |\nu^w|\}$, and $G_{V,F}^0 = r(G_{V,F}^1) \cup s(G_{V,F}^1)$. Then $G_{V,F}$ has all the desired properties, establishing the claim.

Proposition 5.3 of [3] implies that $C^*(G_{V,F})$ is purely infinite, and [3, Theorem 4.4] implies that it has finitely many ideals. So Theorem 4.1 implies that $C^*(G_{V,F})$ has nuclear dimension 1.

The kernel of the quotient map $q : \mathcal{TC}^*(G_{V,F}) \rightarrow C^*(G_{V,F})$ is AF. (This is well-known, but to verify it, recall that the path-space representation $\pi_{T,Q}$ of $\mathcal{TC}^*(E)$ on $\ell^2(E^*)$ is faithful by [12, Theorem 4.1], and observe that each $\pi_{T,Q}(q_v - \sum_{e \in vE^1} t_e t_e^*)$ is the rank-one projection onto the basis element δ_v , so that $\pi_{T,Q}(\ker q)$ is contained in \mathcal{K}_{E^*} .) Hence Remark 2.2 of [31] (see also Example 4.1 of [18]) implies that $\dim_{\text{nuc}}(\ker q) = 0$. So $\mathcal{TC}^*(G_{V,F})$ is an extension of an algebra of nuclear dimension 1 by an algebra of nuclear dimension 0, and [31, Proposition 2.9] then implies that $\mathcal{TC}^*(G_{V,F})$ has nuclear dimension at most 2.

The generators $\{q_v : v \in G_{V,F}^0\} \cup \{t_f : f \in G_{V,F}^1\}$ constitute a Toeplitz-Cuntz-Krieger $G_{V,F}$ -family in $\mathcal{TC}^*(E)$, and so induce a homomorphism $\iota_{G_{V,F}} : \mathcal{TC}^*(G_{V,F}) \rightarrow \mathcal{TC}^*(E)$. Since the preceding paragraph implies that $\mathcal{TC}^*(G_{V,F})$ has nuclear dimension at most 2, and since $\iota_{G_{V,F}}(\mathcal{TC}^*(G_{V,F}))$ is isomorphic to a quotient² of $\mathcal{TC}^*(G_{V,F})$, Proposition 2.9 of [31] implies that $\iota_{G,V}(\mathcal{TC}^*(G_{V,F}))$ also has nuclear dimension at most 2. Hence $\mathcal{TC}^*(E) = \overline{\bigcup_{V,F} \iota_{G_{V,F}}(\mathcal{TC}^*(G_{V,F}))}$ is a direct limit of C^* -algebras of nuclear dimension at most 2, and then [31, Proposition 2.3(iii)] implies that $\mathcal{TC}^*(E)$ has nuclear dimension at most 2. \square

Corollary 4.8. *Suppose that E is a directed graph such that $C^*(E)$ is purely infinite. Then $\dim_{\text{nuc}}(C^*(E)) \leq 2$.*

Proof. The implication (b) implies (e) of [15, Theorem 2.3] implies in particular that E satisfies Condition (K) and every vertex in E connects to a cycle. So the result follows from Corollary 4.7. \square

²In fact, [12, Theorem 4.1] can be used to show that $\iota_{G,V}$ is injective, but this is not necessary for our argument.

5. QUASIDIAGONAL EXTENSIONS AND NUCLEAR DIMENSION

In this section we show that the nuclear dimension of a quasidiagonal extension $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ is equal to the maximum of the nuclear dimension of I and the nuclear dimension of A/I . By showing that ideals in graph C^* -algebras for which the quotient is AF are quasidiagonal (we are indebted to James Gabe for this argument), we deduce the following extension of Theorem 4.1.

Theorem 5.1. *Suppose that E is a directed graph, that I is a purely infinite gauge-invariant ideal of $C^*(E)$, and that $C^*(E)/I$ is AF. Then $\dim_{\text{nuc}}(C^*(E)) \leq 2$. If I has finitely many ideals, then $\dim_{\text{nuc}}(C^*(E)) = 1$.*

We prove Theorem 5.1 at the end of the section. Prior to the proof we need to establish some preliminary results on quasidiagonal extensions.

Definition 5.2. Let A be a separable C^* -algebra and I an ideal of A . We say that $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ is a *quasidiagonal extension* if there is an approximate identity $(p_n)_{n=1}^\infty$ of projections in I such that $\|p_n a - a p_n\| \rightarrow 0$ for all $a \in A$. We refer to the sequence (p_n) as *quasicentral* in A .

Proposition 5.3. *Suppose that A is a separable nuclear C^* -algebra and $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ is a quasidiagonal extension. Then $\dim_{\text{nuc}}(A) = \max\{\dim_{\text{nuc}}(I), \dim_{\text{nuc}}(A/I)\}$.*

Proof. We modify the argument of [18, Proposition 6.1] using the first part of the argument of [31, Proposition 2.9].

Suppose that both $\dim_{\text{nuc}}(A/I)$ and $\dim_{\text{nuc}}(I)$ are at most d . We must show that $\dim_{\text{nuc}}(A) \leq d$. Fix normalized positive elements e_1, \dots, e_L of A , and fix $\varepsilon > 0$.

Choose a finite-dimensional $F = \bigoplus_{i=0}^d F^{(i)}$, a completely positive contraction $\eta : A/I \rightarrow F$ and a completely positive $\rho : F \rightarrow A/I$ which is an order-0 contraction on each $F^{(i)}$ so that $\|\rho(\eta(q_I(e_l))) - q_I(e_l)\| < \varepsilon/5$ for each l . Let $P = \{p_\alpha\}$ be an approximate identity of projections in I which is quasicentral in A . We follow the proof of [31, Proposition 2.9] as far as the middle of the top of page 472 first to lift ρ to a completely positive $\bar{\rho} : F \rightarrow A$ which is an order-0 contraction on each $F^{(i)}$, and then to find $p \in P$ such that, with δ as in [18, Proposition 2.6],

- (1) $\|[(1-p), \bar{\rho}(x)]\| < \delta \|x\|$ for all $x \in F$,
- (2) each $\|(pe_l p + (1-p)e_l(1-p)) - e_l\| < \varepsilon/5$, and
- (3) each $\|(1-p)(\bar{\rho}(\eta(e_l)) - e_l)(1-p)\| < 2\varepsilon/5$.

Now using Proposition 1.4 of [31] and property (1) of $\bar{\rho}$ above, we obtain completely positive order-0 contractions $\hat{\rho}^{(i)} : F^{(i)} \rightarrow (1-p)A(1-p)$ such that, putting $\hat{\rho} = \sum_i \hat{\rho}^{(i)}$, we have

$$\|\hat{\rho}(x) - (1-p)\bar{\rho}(x)(1-p)\| < \varepsilon \|x\|/5 \quad \text{for all } x \in F.$$

Now choose a finite-dimensional $G = \bigoplus_{i=0}^d G^{(i)}$, a completely positive contraction $\psi : I \rightarrow G$ and a completely positive $\phi : G \rightarrow I$ which is an order-0 contraction on each $G^{(i)}$ such that $\|\phi(\psi(pe_l p)) - pe_l p\| < \varepsilon/5$ for each l . Define $H = \bigoplus_{i=1}^d (F^{(i)} \oplus G^{(i)})$, and define $\Phi : A \rightarrow F \oplus G \cong H$ by $a \mapsto (\eta(q_I(a)), \psi(pap))$. For each i , define $\Phi^{(i)} : F^{(i)} \oplus G^{(i)} \rightarrow A$ by $\Phi^{(i)}(x, y) = \hat{\rho}(x) + \phi(y)$. Since each $\hat{\rho}(x) \in (1-p)A(1-p)$ and each $\phi(y) \in pAp$, the $\Phi^{(i)}$ are order-zero contractions. The final calculation of the proof of [31, Proposition 2.9] applies verbatim to show that $\|\Phi(\Psi(e_l)) - e_l\| < \varepsilon$ for all l . \square

To prove Theorem 5.1, we show next that if E is row-finite and I is a gauge-invariant ideal of $C^*(E)$ such that $C^*(E)/I$ is AF, then $0 \rightarrow I \rightarrow C^*(E) \rightarrow C^*(E)/I \rightarrow 0$ is quasidiagonal.

We thank James Gabe for providing us with the following result.

Theorem 5.4 (Gabe). *Let E be a row-finite graph, H a hereditary and saturated set such that the quotient graph $E/H = (E^0 \setminus H, r^{-1}(E^0 \setminus H), r, s)$ has no cycles. Then $0 \rightarrow I_H \rightarrow C^*(E) \rightarrow C^*(E)/I_H \rightarrow 0$ is a quasidiagonal extension.*

Proof. Let $\{p_v, s_e : v \in E^0, e \in E^1\}$ be the universal generating Cuntz-Krieger E -family in $C^*(E)$. Define

$$F_1(H) = H \cup \{\alpha \in E^* : \alpha = e_1 \cdots e_n \text{ with } r(e_n) \in H, s(e_n) \notin H\}.$$

Recall that if F is a directed graph, then any increasing sequence of finite subsets V_n of F^0 such that $\bigcup V_n = F^0$ yields an approximate identity $(\sum_{v \in V_n} p_v)_n$ of projections in $C^*(F)$.

We construct an approximate identity $\{e_X : X \text{ is a finite subset of } E^0\}$ for I_H that is quasicentral in $C^*(E)$ as follows. By [26, Theorem 5.1], the elements $\{Q_\alpha : \alpha \in F_1(H)\}$ given by

$$Q_\alpha = \begin{cases} p_v & \text{if } v \in H \\ s_\alpha s_\alpha^* & \text{if } \alpha \in F_1(H) \end{cases}$$

are mutually orthogonal projections in I_H and any sequence of increasing sums of these which eventually exhausts all $\alpha \in F_1(H)$ is an approximate identity of projections for I_H .

For a finite $X \subseteq E^0$, let

$$F_1(H)_X = (X \cap H) \cup \{\alpha \in F_1(H) : \alpha = e_1 \cdots e_k \text{ and } s(e_1), \dots, s(e_k) \in X \setminus H\}.$$

Since E is row-finite and E/H has no cycles, $F_1(H)_X$ is finite. We have $\bigcup_X F_1(H)_X = F_1(H)$. For $X \subseteq E^0$, set

$$e_X = \sum_{\alpha \in F_1(H)_X} Q_\alpha.$$

Since $\bigcup_X F_1(H)_X = F_1(H)$, [26, Theorem 5.1] implies that the e_X (ordered by set-inclusion on finite subsets X of E^0) constitute an approximate identity for I consisting of projections. We show that it is quasicentral.

Let $\beta, \gamma \in E^*$ with $r(\beta) = r(\gamma) = w$. If $w \in H$, then $s_\beta s_\gamma^* \in I_H$, and hence

$$\lim_X \|e_X s_\beta s_\gamma^* - s_\beta s_\gamma^* e_X\| = 0.$$

Suppose $w \notin H$. Note that $p_v s_\beta s_\gamma^* = s_\beta s_\gamma^* p_v = 0$ for $v \in H$. Let $\alpha \in F_1(H)$. Since $r(\alpha) \in H$ and $r(\beta) = r(\gamma) = w \notin H$, we have

$$s_\alpha s_\alpha^* s_\beta s_\gamma^* = \begin{cases} s_{\beta\alpha'} s_{\gamma\alpha'}^* & \text{if } \alpha = \beta\alpha' \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad s_\beta s_\gamma^* s_\alpha s_\alpha^* = \begin{cases} s_{\beta\alpha'} s_{\gamma\alpha'}^* & \text{if } \alpha = \gamma\alpha' \\ 0 & \text{otherwise} \end{cases}$$

Choose a finite $X_0 \subseteq E^0$ such that $r(\beta_i), s(\beta_i), r(\gamma_i)$ and $s(\gamma_i)$ belong to X for all i . For any X containing X_0 , we have $\beta\alpha', \gamma\alpha' \in F_1(H)_X$ for each $\alpha' \in F_1(H)_X$ with $r(\alpha') = w$.

Hence,

$$e_X s_\beta s_\gamma^* = \sum_{\alpha' \in F_1(H), r(\alpha')=w} s_{\beta\alpha'} s_{\gamma\alpha'}^* = s_\beta s_\gamma^* e_X.$$

The claim now follows since $C^*(E) = \overline{\text{span}}\{s_\beta s_\gamma^* : \beta, \gamma \in E^*, r(\beta) = r(\gamma)\}$. \square

Corollary 5.5. *Let E be a graph and let I be a gauge-invariant ideal of $C^*(E)$ such that $C^*(E)/I$ is an AF-algebra. Then $\dim_{\text{nuc}}(C^*(E)) = \dim_{\text{nuc}}(I)$.*

Proof. Let F be a Drinen-Tomforde desingularization of E [7], so that $C^*(F)$ is stably isomorphic to $C^*(E)$ and F is row-finite with no sinks. Rieffel induction over the Morita equivalence coming from a Drinen-Tomforde desingularization carries gauge-invariant ideals to gauge-invariant ideals [7]. So the ideal J of $C^*(F)$ corresponding to I is gauge invariant and is stably isomorphic to I . The quotient $C^*(F)/J$ is stably isomorphic to the AF algebra $C^*(E)/I$ and therefore itself AF. Corollary 2.8(i) of [31] implies that $\dim_{\text{nuc}}(I) = \dim_{\text{nuc}}(I \otimes \mathcal{K}) = \dim_{\text{nuc}}(J \otimes \mathcal{K}) = \dim_{\text{nuc}}(J)$. Theorem 5.4 implies that $0 \rightarrow J \rightarrow C^*(F) \rightarrow C^*(F)/J \rightarrow 0$ is quasidiagonal, and so Proposition 5.3 implies that $\dim_{\text{nuc}}(C^*(F)) = \max\{\dim_{\text{nuc}}(C^*(F)/J), \dim_{\text{nuc}}(J)\}$. Since $C^*(F)/J$ is AF, Remark 2.2 of [31] (see also Example 4.1 of [18]) implies that $\dim_{\text{nuc}}(C^*(F)/J) = 0$, and we deduce that $\dim_{\text{nuc}}(C^*(F)) = \dim_{\text{nuc}}(J)$. To finish off, we observe as above that since $C^*(F)$ and $C^*(E)$ are stably isomorphic, they have the same nuclear dimension. \square

Proof of Theorem 5.1. The result follows directly from Corollary 5.5 combined with Corollary 4.8 and Theorem 4.1. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HILO, 200 W. KAWILI ST., HILO, HAWAII, 96720-4091 USA

E-mail address: ruize@hawaii.edu

SCHOOL OF MATHEMATICS AND APPLIED STATISTICS, FACULTY OF ENGINEERING AND INFORMATION SCIENCES, ROOM 152, BUILDING 39C, UNIVERSITY OF WOLLONGONG NSW 2522

E-mail address: asims@uow.edu.au

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS, 77204- 3008, USA

E-mail address: tomforde@math.uh.edu