UCT-KIRCHBERG ALGEBRAS HAVE NUCLEAR DIMENSION ONE

EFREN RUIZ, AIDAN SIMS, AND ADAM P. W. SØRENSEN

Abstract. We prove that every Kirchberg algebra in the UCT class has nuclear dimension 1. We first show that Kirchberg 2-graph algebras with trivial K_0 and finite K_1 have nuclear dimension 1 by adapting a technique developed by Winter and Zacharias for Cuntz algebras. We then prove that every Kirchberg algebra in the UCT class is a direct limit of 2-graph algebras to obtain our main theorem.

1. Introduction

Nuclear dimension for C^*-algebras, introduced by Winter and Zacharias in [26], is a noncommutative notion of rank based on covering dimension for topological spaces. It has been shown [20, 24, 25] to be closely related to $\mathcal{Z}$-stability and hence to the classification program for simple nuclear C^*-algebras. Winter and Zacharias showed that all UCT-Kirchberg algebras (i.e., separable, nuclear, simple, purely infinite C^*-algebras in the UCT class) have nuclear dimension at most 5 and asked whether the precise value of their dimension is determined by algebraic properties of their K-groups, such as torsion [26, Problem 9.2]. Matui and Sato [13] subsequently improved the estimate for simple Kirchberg algebras from 5 to 3, and their result is valid for non-UCT Kirchberg algebras, if any exist; and Barlak, Enders, Matui, Szabó and Winter have showed how to recover the Matui-Sato estimate from a general relationship between the nuclear dimension of an $\mathcal{O}_\infty$-stable C^*-algebra and its $\mathcal{O}_2$-stablization that implies, in particular, that every $\mathcal{O}_\infty$-absorbing C^*-algebra with compact metrizable primitive-ideal space has nuclear dimension at most 7 [1]. For UCT-Kirchberg algebras, a further improvement due to Enders [4] shows that every UCT-Kirchberg algebra with torsion-free K_1 has nuclear dimension 1. But the question remained open whether torsion in K_1 precludes having nuclear dimension 1. Here we prove that every UCT-Kirchberg algebra, regardless of its K-theory, has nuclear dimension 1. This settles the question posed by Winter and Zacharias, and suggests another question: does every simple separable non-AF C^*-algebra with finite nuclear dimension have nuclear dimension 1? Indeed, the appearance of the present paper in preprint form motivated Bosa–Brown–Sato–Tikuisis–White–Winter in their remarkable recent preprint [2] to pursue the optimal bound of 1 for the nuclear dimension of very broad classes of simple C^*-algebras (see [2, page 4]).

We recall the definition of nuclear dimension. A completely positive map ϕ between C^*-algebras is order zero if $ab = 0$ implies $\phi(a)\phi(b) = 0$ for positive a, b. A separable C^*-algebra A has *nuclear dimension* r, denoted by $\dim_{\rm nuc}(A) = r$, if r is the least element in $\mathbb{N} \cup \{\infty\}$ for which there is a sequence of order-r factorisations $\phi_n \circ \psi_n$ through finite dimensional C^*-algebras $\mathcal{F}_n$ that pointwise approximate the identity map on A. That is, there exist finite dimensional C^*-algebras $(\mathcal{F}_n)_{n\in\mathbb{N}}$, completely positive, contractive linear maps $(\psi_n : A \to \mathcal{F}_n)_{n\in\mathbb{N}}$, and completely positive linear maps $(\phi_n : \mathcal{F}_n \to A)_{n\in\mathbb{N}}$ such that

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(1) $\lim_{n\to\infty} \|a - \phi_n \circ \psi_n(a)\| = 0$ for all $a \in A$ and
(2) each $\mathcal{F}_n$ has a decomposition $\bigoplus_{i=0}^r \mathcal{F}_{n,i}$ such that $\phi_n|_{\mathcal{F}_{n,i}}$ is an order-zero completely positive contraction for each i.

Winter and Zacharias' calculation of nuclear dimension for Cuntz algebras in [26] is related to a construction of Kribs and Solel [8] which builds from a directed graph E a sequence of directed graphs $(E(n))_{n=1}^\infty$ comprising a kind of generalised combinatorial solenoid. The first two authors, with Tomforde, used the Kribs-Solel construction explicitly to compute nuclear dimension of many purely infinite nonsimple graph algebras in [19]. The key feature of $E(n)$ used in nuclear-dimension calculations is that there are inclusions of the Toeplitz algebras $\iota_n : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E(n))$ that can be approximated modulo compacts by order-1 factorisations through finite-dimensional C^*-algebras. These are parlayed into an approximation of the identity on $C^*(E)$ using a completely positive splitting $C^*(E) \to \mathcal{T}C^*(E)$ (which exists since every graph algebra is nuclear), a suitable sequence of homomorphisms $j_n : C^*(E(n)) \to C^*(E) \otimes \mathcal{K}$, and classification results for purely infinite C^*-algebras.

Here, we develop a version of this machinery for higher-rank graphs and their C^*-algebras as introduced in [9]. We use this, and the fact that each Kirchberg algebra with trivial K_0 and finite K_1 is isomorphic to a tensor product of purely infinite simple graph C^*-algebras, to show that such Kirchberg algebras have nuclear dimension 1. This is, on the face of it, somewhat surprising: one expects the tensor product of C^*-algebras of nuclear dimension n and m to have nuclear dimension $(n+1)(m+1)-1$, as is the case for commutative C^*-algebras. Here the underlying combinatorial model plays a key role. For graph C^*-algebras [19], following [26, Section 7], the n^{th} order-1 factorisation through finite-dimensional C^*-algebras was obtained by producing a pair of pavings of the combinatorial quadrant $\mathbb{N} \times \mathbb{N}$ by copies of a completely positive contractive $n \times n$ real matrix so that when the two pavings are superimposed, entries at any fixed distance from the diagonal, and sufficiently far from the origin, approach 1 as $n \to \infty$. For 2-graphs, we must perform an analogous decomposition in $\mathbb{N}^2 \times \mathbb{N}^2$, and the expected increase in dimension turns out to be illusory: facing no topological constraints, we can just pick convenient bijections of $\mathbb{N}^2 \times \mathbb{N}^2$ onto $\mathbb{N} \times \mathbb{N}$ that carry points close to the diagonal to points close to the diagonal. In essence, this is the idea underlying the technical argument of Theorem 5.6.

To deduce our main theorem, we apply the Kirchberg-Phillips theorem to show that every stable UCT-Kirchberg algebra is a direct limit of purely infinite 2-graph C^*-algebras that are known to have nuclear dimension 1 either by Enders' results or by the result of the preceding paragraph. We believe that this realisation of Kirchberg algebras using 2-graph C^*-algebras has independent interest: many important C^*-algebraic properties are preserved under direct limits, and as our approach here shows, the combinatorial structure of 2-graphs can provide a good line of attack in establishing structural properties of the associated C^*-algebras. Related combinatorial models have been used to great effect to study weak semiprojectivity [21] and prime-order automorphisms [22] of UCT-Kirchberg algebras. Our techniques also have potential applications to calculations of nuclear dimension for large classes of nonsimple Kirchberg algebras along the lines of [19] (see Remark 6.3).

We start with some background on higher-rank graphs in Section 2. In Section 3, we show how to generalise the Kribs-Solel construction to higher-rank graphs, and produce analogues of the homomorphisms ι_n and j_n discussed in the preceding paragraph. In Section 4, we investigate how our construction behaves with respect to the cartesian-product construction for k-graphs [9]; this allows us to relate the results of the preceding section to tensor products of graph C^*-algebras. In Section 5, we show that for 2-graphs, the maps $\tilde{\iota}_n : C^*(\Lambda) \to C^*(\Lambda(n))$ induced by the ι_n can be asymptotically approximated by sums of two order-zero maps through AF-algebras. In Section 6, we prove our main result. We first show that if E and F are 1-graphs whose C^*-algebras are Kirchberg algebras with K-theory $(T, 0)$ and $(0, \mathbb{Z})$ respectively, where

T is a finite abelian group, then for the 2-graph $\Lambda = E \times F$, the composition $j_{(n_1,n_2)} \circ \tilde{\iota}_{(n_1,n_2)}$ implements multiplication by $n_1 n_2$ in $K_*(C^*(\Lambda)) \cong (0, T)$. By choosing increasing (n_1, n_2) for which multiplication by $n_1 n_2$ is the identity on T, and applying classification machinery, we deduce that UCT-Kirchberg algebras with trivial K_0 and finite K_1 have nuclear dimension 1. We then prove our main result by combining this with Enders' results and a direct-limit argument.

2. Higher rank graphs and their C^*-algebras

We recall the standard conventions for k-graphs and their C^*-algebras introduced in [9]. We regard $\mathbb{N}^k$ as an additive semigroup with identity 0. For $m, n \in \mathbb{N}^k$, we write $m \vee n$ for their coordinatewise maximum. We write $n \leq m$ if $n_i \leq m_i$ for all i. We also write $n < m$ to mean $n_i < m_i$ for all i. Warning: this convention means that $n < m$ and $n \lneq m$ mean different things.

Definition 2.1 (See [9])**.** Let $k \in \mathbb{N} \setminus \{0\}$. A *graph of rank k*, or *k-graph*, is a pair (Λ, d) where Λ is a countable category and d is a functor from Λ to $\mathbb{N}^k$ that satisfies the *factorisation property*: for all $\lambda \in \mathrm{Mor}(\Lambda)$ and all $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique morphisms μ and ν in $\mathrm{Mor}(\Lambda)$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Since we are regarding k-graphs as generalised directed graphs, we refer to elements of $\mathrm{Mor}(\Lambda)$ as *paths*. The factorisation property implies that $\{\mathrm{id}_o \mid o \in \mathrm{Obj}(\Lambda)\} = \{\lambda \in \mathrm{Mor}(\Lambda) \mid d(\lambda) = 0\}$. So the codomain and domain maps $\mathrm{cod}, \mathrm{dom} : \mathrm{Mor}(\Lambda) \to \mathrm{Obj}(\Lambda)$ determine maps $r : \lambda \mapsto \mathrm{id}_{\mathrm{cod}(\lambda)}$ and $s : \lambda \mapsto \mathrm{id}_{\mathrm{dom}(\lambda)}$ from $\mathrm{Mor}(\Lambda)$ to $d^{-1}(0)$. We refer to the elements of $d^{-1}(0)$ as *vertices*, and call $r(\lambda)$ and $s(\lambda)$ the range and source of λ. We have $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$. We write $\lambda \in \Lambda$ to mean $\lambda \in \mathrm{Mor}(\Lambda)$.

We use the following notation from [14]: given $\lambda \in \Lambda$ and $E \subseteq \Lambda$, we define

$$\lambda E := \{\lambda\mu \mid \mu \in E, r(\mu) = s(\lambda)\} \quad \text{and} \quad E\lambda := \{\mu\lambda \mid \mu \in E, s(\mu) = r(\lambda)\}.$$

In particular if $d(v) = 0$, then $vE = \{\lambda \in E \mid r(\lambda) = v\}$, and $Ev = \{\lambda \in E \mid s(\lambda) = v\}$.

For $n \in \mathbb{N}^k$, we let $\Lambda^n = d^{-1}(n)$. For $n < m$, we set $\Lambda^{[n,m)} = \{\lambda \in \Lambda \mid n \leq d(\mu) < m\}$. We use the convention that for $m \leq n \leq d(\lambda)$, the path $\lambda(m, n)$ is the unique element of Λ^{n-m} such that $\lambda = \lambda'\lambda(m, n)\lambda''$ for some $\lambda' \in \Lambda^m$. An application of the factorisation property shows that for $m \leq d(\lambda)$ we have $\lambda = \lambda(0, m)\lambda(m, d(\lambda))$.

As in [9], we say that Λ is *row-finite* if $v\Lambda^n$ is finite for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We say that Λ has *no sources* if each $v\Lambda^n \neq \emptyset$. All k-graphs in this paper will be row-finite with no sources.

Definition 2.2 (See [17])**.** Let (Λ, d) be a k-graph. Given $\mu, \nu \in \Lambda$, we say that λ is a *minimal common extension* of μ and ν if $\lambda \in \mu\Lambda \cap \nu\Lambda$ and $d(\lambda) = d(\mu) \vee d(\nu)$. We denote the collection $\mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$ of all minimal common extensions of μ and ν by $\mathrm{MCE}(\mu, \nu)$. We define

$$\Lambda^{\min}(\mu, \nu) := \{(\alpha, \beta) \in \Lambda \times \Lambda \mid \mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)\}.$$

For a row-finite k-graph Λ, the set $\mathrm{MCE}(\mu, \nu)$ is finite for all $\mu, \nu \in \Lambda$, since each $\mathrm{MCE}(\mu, \nu) \subseteq r(\mu)\Lambda^{d(\mu)\vee d(\nu)}$. The factorisation property ensures that $(\alpha, \beta) \mapsto \mu\alpha$ is a bijection from $\Lambda^{\min}(\mu, \nu)$ to $\mathrm{MCE}(\mu, \nu)$.

The following definition of a Toeplitz-Cuntz-Krieger family for a higher-rank graph is essentially [16, Definition 7.1], with the appropriate changes of conventions to translate from product-systems of graphs to k-graphs (see also [6, Section 2.2]).

Definition 2.3. Let Λ be a row-finite k-graph with no sources. A *Toeplitz-Cuntz-Krieger* Λ*-family* is a collection $\{t_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in a C^*-algebra satisfying

(TCK1) $\{t_v\}_{v\in\Lambda^0}$ is a collection of mutually orthogonal projections;
(TCK2) $t_\lambda t_\mu = \delta_{s(\lambda),r(\mu)} t_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$;
(TCK3) $t^*_\lambda t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and

(TCK4) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$.

Remark 2.4. In previous treatments (see, for example, [6]), the definition of a Toeplitz-Cuntz-Krieger Λ-family for a row-finite k-graph Λ with no sources has included the additional relation

$$\sum_{\lambda\in v\Lambda^n} t_\lambda t_\lambda^* \le t_v \qquad \text{for all } v \in \Lambda^0 \text{ and } n \in \mathbb{N}^k. \tag{2.1}$$

But in fact this relation follows from the other four. To see this, observe that if $\mu, \nu \in \Lambda^n$, then $\Lambda^{\min}(\mu,\nu) = \{s(\mu), s(\mu)\}$ if $\mu = \nu$ and is empty otherwise. So (TCK4) shows that the $t_\lambda t_\lambda^*$ for $\lambda \in \Lambda^n$ are mutually orthogonal, and so the sum on the left-hand side of (2.1) is a projection. For each Λ, we have $\Lambda^{\min}(\lambda, r(\lambda)) = \{(s(\lambda), \lambda)\}$. Hence

$$\Big(\sum_{\lambda\in v\Lambda^n} t_\lambda t_\lambda^*\Big) t_v = \sum_{\lambda\in v\Lambda^n} \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,r(\lambda))} t_\lambda t_\alpha t_\beta^* = \sum_{\lambda\in v\Lambda^n} t_\lambda t_{s(\lambda)} t_\lambda^* = \sum_{\lambda\in v\Lambda^n} t_\lambda t_\lambda^*.$$

As in [9], a *Cuntz-Krieger Λ-family* is a collection $\{s_\lambda\}_{\lambda\in\Lambda}$ of partial isometries in a C^*-algebra satisfying (TCK1), (TCK2), (TCK3), and

(CK) $s_v = \sum_{\lambda\in v\Lambda^n} s_\lambda s_\lambda^*$ for each $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Let Λ be a row-finite k-graph with no sources. There is a universal C^*-algebra $\mathcal{T}C^*(\Lambda)$ generated by a universal Toeplitz-Cuntz-Krieger Λ-family $\{t_\lambda\}_{\lambda\in\Lambda}$. We call this C^*-algebra the *Toeplitz algebra of* Λ. There is also a universal C^*-algebra $C^*(\Lambda)$ generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda\}_{\lambda\in\Lambda}$. We call this C^*-algebra the *Cuntz-Krieger algebra of* Λ, or just the C^*-algebra of Λ.

3. The Kribs-Solel construction for k-graphs

For the duration of this section, we fix a row-finite k-graph Λ with no sources. The key tool for understanding nuclear dimension of graph algebras in [19] was a construction due to Kribs and Solel [8]. The first step in our analysis here is to adapt this construction to k-graphs.

Choose $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ with each $n_i \ge 1$. Let

$$H_n := \{an \mid a \in \mathbb{Z}^k\} = \{m \in \mathbb{Z}^k \mid m_i/n_i \in \mathbb{Z} \text{ for all } i\}.$$

We will often just write H for H_n. For $m \in \mathbb{N}^k$, we write $[m]$ for $m + H \in \mathbb{Z}^k/H$. We often identify $\mathbb{Z}^k/H$ as a set with $\{m \in \mathbb{N}^k \mid m < n\}$.

For $\lambda \in \Lambda$, we define

$$[\lambda]_H := \lambda(0, [d(\lambda)]),$$

and we usually write $[\lambda]$ for $[\lambda]_H$. So $[\lambda]$ is the unique element of Λ such that $d([\lambda]) < n$ and $\lambda = [\lambda]\lambda'$ with $d(\lambda') \in H$. The factorisation property implies that if $d(\mu) \in H$, then $[\lambda\mu] = [\lambda]$.

Following [19], we write $\Lambda^{<n} := \{\lambda \in \Lambda \mid d(\lambda) < n\}$. Let

$$\Lambda(n) := \{(\lambda, \lambda') \in \Lambda \times \Lambda^{<n} \mid s(\lambda) = r(\lambda')\}.$$

We aim to make this set into a k-graph. For $(\lambda, \lambda') \in \Lambda(n)$, define

$$d((\lambda, \lambda')) := d(\lambda).$$

So $\Lambda(n)^0 = \{(r(\lambda), \lambda) \mid \lambda \in \Lambda^{<n}\}$. Define $r, s : \Lambda(n) \to \Lambda(n)^0$ by

$$s((\lambda,\lambda')) := (s(\lambda), \lambda'), \quad \text{and}$$
$$r((\lambda,\lambda')) := (r(\lambda), [\lambda\lambda']).$$

Identify $\Lambda(n)^0$ with $\Lambda^{<n}$ via $(r(\lambda), \lambda) \mapsto \lambda$. Then $s((\lambda, \lambda')) = \lambda'$ and $r((\lambda, \lambda')) = [\lambda\lambda']$. Suppose that $s((\lambda, \lambda')) = r((\mu, \mu'))$; that is, $\lambda' = [\mu\mu']$. Then we define

$$(\lambda, \lambda')(\mu, \mu') := \big(\lambda\mu, \mu'\big).$$

Lemma 3.1. *Under the operations just described, $\Lambda(n)$ is a row-finite k-graph with no sources.*

Proof. We show that $\Lambda(n)$ is a category. We first check that s and r are compatible with composition. Suppose that $s((\lambda,\lambda')) = r((\mu,\mu'))$. Then

$$s((\lambda,\lambda')(\mu,\mu')) = s((\lambda\mu,\mu')) = \mu' = s((\mu,\mu')).$$

Writing $\mu\mu' = [\mu\mu']\tau = \lambda'\tau$, we have

$$r\big((\lambda,\lambda')(\mu,\mu')\big) = r((\lambda\mu,\mu')) = \big[\lambda\mu\mu'\big] = \big[\lambda[\mu\mu']\tau\big] = [\lambda\lambda'\tau];$$

and since $d(\tau) \in H$, we have $r((\lambda,\lambda')(\mu,\mu')) = [\lambda\lambda'\tau] = [\lambda\lambda'] = r((\lambda,\lambda'))$.

We now check that $r((\lambda,\lambda'))$ and $s((\lambda,\lambda'))$ act as left- and right identities for (λ,λ'):

$$r((\lambda,\lambda'))(\lambda,\lambda') = (r(\lambda),[\lambda\lambda'])(\lambda,\lambda') = (r(\lambda)\lambda,\lambda') = (\lambda,\lambda'),$$

and

$$(\lambda,\lambda')s((\lambda,\lambda')) = (\lambda,\lambda')\big(s(\lambda),\lambda'\big) = (\lambda s(\lambda),\lambda') = (\lambda,\lambda').$$

To check associativity, suppose that $s((\lambda,\lambda')) = r((\mu,\mu'))$ and $s((\mu,\mu')) = r((\nu,\nu'))$. Then

$$\begin{aligned}((\lambda,\lambda')(\mu,\mu'))(\nu,\nu') &= (\lambda\mu,\mu')(\nu,\nu') = (\lambda\mu\nu,\nu'),\\ &= (\lambda,\lambda')(\mu\nu,\nu') = (\lambda,\lambda')((\mu,\mu')(\nu,\nu')).\end{aligned}$$

So $\Lambda(n)$ is a category.

We check that d is a functor:

$$d((\lambda,\lambda')(\mu,\mu')) = d((\lambda\mu,\mu')) = d(\lambda\mu) = d(\lambda) + d(\mu) = d((\lambda,\lambda')) + d((\mu,\mu')).$$

Now we check the factorisation property. Suppose that $d((\lambda,\lambda')) = p+q$. Then $d(\lambda) = p+q$, and the factorisation property in Λ gives $\mu \in \Lambda^p$ and $\nu \in \Lambda^q$ such that $\lambda = \mu\nu$. Now $(\nu,\lambda') \in \Lambda(n)^q$ and has range $r((\nu,\lambda')) = [\nu\lambda']$. Hence $(\mu,[\nu\lambda']) \in \Lambda(n)^p r((\nu,\lambda'))$, and $(\mu,[\nu\lambda'])(\nu,\lambda') = (\mu\nu,\lambda') = (\lambda,\lambda')$. For uniqueness, suppose that $(\alpha,\alpha') \in \Lambda(n)^p$ and $(\beta,\beta') \in \Lambda(n)^q$ satisfy $(\alpha,\alpha')(\beta,\beta') = (\lambda,\lambda')$. By definition of composition, we have $(\alpha\beta,\beta') = (\lambda,\lambda')$. This forces $\beta' = \lambda'$ and $\alpha\beta = \lambda$. Since $d(\alpha) = d((\alpha,\alpha')) = p$ and $d(\beta) = d((\beta,\beta')) = q$, the factorisation property in Λ forces $\alpha = \mu$ and $\beta = \nu$. Since (α,α') and (β,β') are composable, we have $\alpha' = s((\mu,\alpha')) = r((\nu,\lambda')) = [\nu\lambda']$. Hence $\Lambda(n)$ is a k-graph.

To see that $\Lambda(n)$ is row-finite with no sources, take $(r(\lambda),\lambda) \in \Lambda(n)^0$ and $m \in \mathbb{N}^k$. Then

$$\begin{aligned}(r(\lambda),\lambda)\Lambda(n)^m &= \{(\mu,\mu') \mid \mu \in \Lambda^m, \mu' \in s(\mu)\Lambda^{<n}, [\mu\mu'] = \lambda\}\\ &= \big\{(\mu,\mu') \mid \mu \in \Lambda^m, \mu' \in s(\mu)\Lambda^{[d(\lambda)-m]}, [\mu\mu'] = \lambda\big\}.\end{aligned}$$

Since $m + [d(\lambda) - m]$ is positive and congruent to $d(\lambda)$ mod H, we have $m + [d(\lambda) - m] \geq d(\lambda)$. Let $p := m + [d(\lambda) - m] - d(\lambda) \in H$. Then

$$(r(\lambda),\lambda)\Lambda(n)^m = \big\{\big((\lambda\nu)(0,m),(\lambda\nu)(m,p+d(\lambda))\big) \bigm| \nu \in s(\lambda)\Lambda^p\big\},$$

which is finite and nonempty because $s(\lambda)\Lambda^p$ is finite and nonempty. □

To work with Toeplitz-Cuntz-Krieger $\Lambda(n)$-families we first compute $\Lambda(n)^{\min}((\lambda,\lambda'),(\mu,\mu'))$.

Lemma 3.2. *For $(\lambda,\lambda'),(\mu,\mu') \in \Lambda(n)$, if $[\lambda\lambda'] \neq [\mu\mu']$, then $\Lambda(n)^{\min}((\lambda,\lambda'),(\mu,\mu')) = \emptyset$; otherwise,*

$$\begin{aligned}\Lambda(n)^{\min}((\lambda,\lambda'),(\mu,\mu')) = \big\{((\alpha,\tau),(\beta,\tau)) \bigm| &(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu),\\ &\tau \in s(\alpha)\Lambda^{<n}, [\alpha\tau] = \lambda' \text{ and } [\beta\tau] = \mu'\big\}.\end{aligned} \tag{3.1}$$

Proof. If $[\lambda\lambda'] \neq [\mu\mu']$, then $r((\lambda,\lambda')) \neq r((\mu,\mu'))$, and so $\Lambda(n)^{\min}((\lambda,\lambda'),(\mu,\mu')) = \emptyset$.

Suppose that $[\lambda\lambda'] = [\mu\mu']$. Suppose further that $(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)$, that $\tau \in s(\alpha)\Lambda^{<n}$, and that $[\alpha\tau] = \lambda'$ and $[\beta\tau] = \mu'$. Then $(\alpha,\tau),(\beta,\tau) \in \Lambda(n)$, and $r((\alpha,\tau)) = \lambda' = s((\lambda,\lambda'))$ and $r((\beta,\tau)) = \mu' = s((\mu,\mu'))$. We have

$$(\lambda,\lambda')(\alpha,\tau) = (\lambda\alpha,\tau) = (\mu\beta,\tau) = (\mu,\mu')(\beta,\tau).$$

Since $d((\lambda\alpha, \tau)) = d(\lambda\alpha) = d(\lambda) \vee d(\mu) = d((\lambda, \lambda')) \vee d((\mu, \mu'))$, we have $((\alpha, \tau), (\beta, \tau)) \in \Lambda(n)^{\min}((\lambda, \lambda'), (\mu, \mu'))$.

Conversely, suppose that $(\alpha, \tau), (\beta, \rho) \in \Lambda(n)^{\min}((\lambda, \lambda'), (\mu, \mu'))$. Then

$$(\lambda\alpha, \tau) = (\lambda, \lambda')(\alpha, \tau) = (\mu, \mu')(\beta, \rho) = (\mu\beta, \rho). \tag{3.2}$$

So $\lambda\alpha = \mu\beta$, and

$$d(\lambda\alpha) = d((\lambda\alpha, \tau)) = d((\lambda, \lambda')(\alpha, \tau)) = d((\lambda, \lambda')) \vee d((\mu, \mu')) = d(\lambda) \vee d(\mu),$$

so $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$. By (3.2), $\tau = s((\lambda\alpha, \tau)) = s((\mu\beta, \rho)) = \rho$. Since $[\alpha\tau] = r((\alpha, \tau)) = s((\lambda, \lambda')) = \lambda'$ and $[\beta\tau] = r((\beta, \tau)) = s((\mu, \mu')) = \mu'$, we deduce that $((\alpha, \tau), (\beta, \rho)) = ((\alpha, \tau), (\beta, \tau))$ belongs to the right-hand side of (3.1). □

For each n we now construct a homomorphism from $C^*(\Lambda)$ to $C^*(\Lambda(n))$ analogous to those for directed graphs described in [19, Lemma 2.5].

Lemma 3.3. *Let $\{t_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{T}C^*(\Lambda)$ and $\{t_{(\lambda,\lambda')}\}_{(\lambda,\lambda')\in\Lambda(n)} \subseteq \mathcal{T}C^*(\Lambda(n))$ be the generating Toeplitz-Cuntz-Krieger families and let $\{s_\lambda\}_{\lambda\in\Lambda} \subseteq C^*(\Lambda)$ and $\{s_{(\lambda,\lambda')}\}_{(\lambda,\lambda')\in\Lambda(n)} \subseteq C^*(\Lambda(n))$ be the generating Cuntz-Krieger families. For $n \in \mathbb{N}^k$, there are homomorphisms $\iota_n : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda(n))$ and $\tilde{\iota}_n : C^*(\Lambda) \to C^*(\Lambda(n))$ such that*

$$\iota_n(t_\lambda) = \sum_{\lambda'\in s(\lambda)\Lambda^{<n}} t_{(\lambda,\lambda')} \quad \text{and} \quad \tilde{\iota}_n(s_\lambda) = \sum_{\lambda'\in s(\lambda)\Lambda^{<n}} s_{(\lambda,\lambda')}. \tag{3.3}$$

The homomorphism ι_n descends to the homomorphism $\tilde{\iota}_n$ under the canonical quotient maps from Toeplitz algebras to Cuntz-Krieger algebras.

Proof. For $\lambda \in \Lambda$, define $T_\lambda := \sum_{\lambda'\in s(\lambda)\Lambda^{<n}} t_{(\lambda,\lambda')} \in \mathcal{T}C^*(\Lambda(n))$. We check that $\{T_\lambda\}_{\lambda\in\Lambda}$ is a Toeplitz-Cuntz-Krieger Λ-family. Take $v, w \in \Lambda^0$. Since $\{t_{(r(\nu),\nu)}\}_{\nu\in\Lambda^{<n}}$ is a set of mutually orthogonal projections,

$$T_v^* T_w = \sum_{\lambda\in v\Lambda^{<n}} t_{(v,\lambda)} \sum_{\mu\in w\Lambda^{<n}} t_{(w,\mu)} = \sum_{\lambda\in v\Lambda^{<n}, \mu\in w\Lambda^{<n}} \delta_{(v,\lambda),(w,\mu)} t_{(v,\lambda)} = \delta_{v,w} T_v,$$

and so $\{T_v\}_{v\in\Lambda^0}$ are mutually orthogonal projections, giving (TCK1).

For $(\lambda, \lambda'), (\mu, \mu') \in \Lambda(n)$, we have

$$t_{(\lambda,\lambda')} t_{(\mu,\mu')} = \delta_{s((\lambda,\lambda')),r((\mu,\mu'))} t_{(\lambda,\lambda')(\mu,\mu')} = \delta_{s(\lambda),r(\mu)} \delta_{\lambda',[\mu\mu']} t_{(\lambda\mu,\mu')}.$$

Hence, for $\lambda, \mu \in \Lambda$,

$$\begin{aligned} T_\lambda T_\mu &= \sum_{\lambda'\in s(\lambda)\Lambda^{<n}, \mu'\in s(\mu)\Lambda^{<n}} t_{(\lambda,\lambda')} t_{(\mu,\mu')} \\ &= \sum_{\lambda'\in s(\lambda)\Lambda^{<n}, \mu'\in s(\mu)\Lambda^{<n}} \delta_{s(\lambda),r(\mu)} \delta_{\lambda',[\mu\mu']} t_{(\lambda\mu,\mu')} \\ &= \sum_{\mu'\in s(\mu)\Lambda^{<n}} \delta_{s(\lambda),r(\mu)} t_{(\lambda\mu,\mu')} = \delta_{s(\lambda),r(\mu)} T_{\lambda\mu}. \end{aligned}$$

So $\{T_\lambda\}_{\lambda\in\Lambda}$ satisfies (TCK2).

For (TCK3) and (TCK4), fix $\lambda, \mu \in \Lambda$. We calculate:

$$\begin{aligned}
T_\lambda^* T_\mu &= \sum_{\lambda' \in s(\lambda)\Lambda^{<n}, \mu' \in s(\mu)\Lambda^{<n}} t^*_{(\lambda,\lambda')} t_{(\mu,\mu')} \\
&= \sum_{\substack{\lambda' \in s(\lambda)\Lambda^{<n} \\ \mu' \in s(\mu)\Lambda^{<n}}} \Big(\sum_{((\alpha,\alpha'),(\beta,\beta')) \in \Lambda(n)^{\min}((\lambda,\lambda'),(\mu,\mu'))} t_{(\alpha,\alpha')} t^*_{(\beta,\beta')} \Big) \\
&= \sum_{\substack{\lambda' \in s(\lambda)\Lambda^{<n} \\ \mu' \in s(\mu)\Lambda^{[d(\lambda\lambda') - d(\mu)]}}} \sum_{\substack{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu), \tau \in s(\alpha)\Lambda^{<n}, \\ [\alpha\tau] = \lambda', [\beta\tau] = \mu'}} t_{(\alpha,\tau)} t^*_{(\beta,\tau)} \quad \text{(by Lemma 3.2)} \\
&= \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} \Big(\sum_{\tau \in s(\alpha)\Lambda^{<n}} t_{(\alpha,\tau)} t^*_{(\beta,\tau)} \Big).
\end{aligned}$$

If $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ and $\tau, \rho \in \Lambda^{<n}$, then $t_{(\alpha,\tau)} t^*_{(\beta,\rho)} \neq 0$ forces $\tau = s((\alpha, \tau)) = s((\beta, \rho)) = \rho$. So summing over two variables $\tau \in s(\alpha)\Lambda^{<n}$ and $\rho \in s(\alpha)\Lambda^{<n} = s(\beta)\Lambda^{<n}$ adds no new nonzero terms to the final line of the preceding calculation. Hence

$$T_\lambda^* T_\mu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} \Big(\sum_{\tau \in s(\alpha)\Lambda^{<n}, \rho \in s(\beta)\Lambda^{<n}} t_{(\alpha,\tau)} t^*_{(\beta,\rho)} \Big) = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\mu)} T_\alpha T_\beta^*.$$

This gives (TCK4); and (TCK3) then follows from (TCK1) because $\Lambda^{\min}(\lambda, \lambda) = \{(s(\lambda), s(\lambda))\}$. Hence $\{T_\lambda\}_{\lambda \in \Lambda}$ is a Toeplitz-Cuntz-Krieger Λ-family.

The universal property of $\mathcal{T}C^*(\Lambda)$ gives a homomorphism $\iota_n : \mathcal{T}C^*(\Lambda) \to \mathcal{T}C^*(\Lambda(n))$ such that

$$\iota_n(t_\lambda) = T_\lambda = \sum_{\lambda' \in s(\lambda)\Lambda^{<n}} t_{(\lambda,\lambda')}$$

for all λ. To see that ι_n descends to the desired homomorphism $\tilde{\iota}_n : C^*(\Lambda) \to C^*(\Lambda(n))$, let $q_n : \mathcal{T}C^*(\Lambda(n)) \to C^*(\Lambda(n))$ denote the quotient map. We check that the family $S_\lambda := q_n(T_\lambda)$ satisfies (CK). For $v \in \Lambda^0$ and $m \in \mathbb{N}^k$,

$$\sum_{\lambda \in v\Lambda^m} S_\lambda S_\lambda^* = \sum_{\lambda \in v\Lambda^m} \sum_{\mu,\nu \in s(\lambda)\Lambda^{<n}} s_{(\lambda,\mu)} s^*_{(\lambda,\nu)}.$$

As above, $s_{(\lambda,\mu)} s^*_{(\lambda,\nu)} \neq 0$ forces $s((\lambda, \mu)) = s((\lambda, \nu))$, and so $\mu = \nu$. Using this at the first equality and relation (CK) in $C^*(\Lambda(n))$ at the second-last equality, we calculate:

$$\begin{aligned}
\sum_{\lambda \in v\Lambda^m} S_\lambda S_\lambda^* &= \sum_{\lambda \in v\Lambda^m} \sum_{\lambda' \in s(\lambda)\Lambda^{<n}} s_{(\lambda,\lambda')} s^*_{(\lambda,\lambda')} = \sum_{(\lambda,\lambda') \in \Lambda(n)^m, r(\lambda) = v} s_{(\lambda,\lambda')} s^*_{(\lambda,\lambda')} \\
&= \sum_{\alpha \in v\Lambda^{<n}} \sum_{(\lambda,\lambda') \in \Lambda(n)^m, [\lambda\lambda'] = \alpha} s_{(\lambda,\lambda')} s^*_{(\lambda,\lambda')} = \sum_{\alpha \in v\Lambda^{<n}} \sum_{(\lambda,\lambda') \in (v,\alpha)\Lambda(n)^m} s_{(\lambda,\lambda')} s^*_{(\lambda,\lambda')} \\
&= \sum_{\alpha \in v\Lambda^{<n}} s_{(v,\alpha)} = S_v.
\end{aligned}$$

So $\{S_\lambda\}_{\lambda \in \Lambda}$ is a Cuntz-Krieger Λ-family. The universal property of $C^*(\Lambda)$ now gives a homomorphism $\tilde{\iota}_n : C^*(\Lambda) \to C^*(\Lambda(n))$ such that $\tilde{\iota}_n(s_\lambda) = S_\lambda = q_n(\iota_n(t_\lambda))$. The quotient maps $q : \mathcal{T}C^*(\Lambda) \to C^*(\Lambda)$ and $q_n : \mathcal{T}C^*(\Lambda(n)) \to C^*(\Lambda(n))$ satisfy $\tilde{\iota}_n \circ q = q_n \circ \iota_n$, and the formula for $\tilde{\iota}_n$ in (3.3) follows. □

Now we construct an analogue of the map of [19, Proposition 3.1]. For $\lambda \in \Lambda$, we write $T(\lambda)$ for the unique path such that $\lambda = [\lambda]T(\lambda)$. Note that $d(T(\lambda)) = d(\lambda) - [d(\lambda)] \in H_n$.

For a set X, we write $\mathcal{K}_X$ for the C^*-algebra of compact operators on $\ell^2(X)$, with canonical matrix units $\{\theta_{x,y} \mid x, y \in X\}$.

Lemma 3.4. *Let $\{s_\lambda\}_{\lambda\in\Lambda} \subseteq C^*(\Lambda)$ and $\{s_{(\lambda,\lambda')}\}_{(\lambda,\lambda')\in\Lambda(n)} \subseteq C^*(\Lambda(n))$ be the generating Cuntz-Krieger families. There is a homomorphism $j_n : C^*(\Lambda(n)) \to C^*(\Lambda) \otimes \mathcal{K}_{\Lambda^{<n}}$ such that*

$$j_n(s_{(\lambda,\lambda')}) = s_{T(\lambda\lambda')} \otimes \theta_{[\lambda\lambda'],\lambda'} \quad \textit{for all } (\lambda, \lambda') \in \Lambda(n).$$

Proof. We just have to check the Cuntz-Krieger relations for the elements $S_{(\lambda,\lambda')} := s_{T(\lambda\lambda')} \otimes \theta_{[\lambda\lambda'],\lambda'}$. For $\lambda \in \Lambda^{<n}$, we have $T(\lambda) = s(\lambda)$ and $[\lambda] = \lambda$. Thus $\{S_{(r(\lambda),\lambda)} = s_{s(\lambda)} \otimes \theta_{\lambda,\lambda}\}_{(r(\lambda),\lambda)\in\Lambda(n)^0}$ is a collection of mutually orthogonal projections.

Let (λ, λ') and (μ, μ') be elements in $\Lambda(n)$. Then

$$\begin{aligned} S_{(\lambda,\lambda')}S_{(\mu,\mu')} &= (s_{T(\lambda\lambda')} \otimes \theta_{[\lambda\lambda'],\lambda'})(s_{T(\mu\mu')} \otimes \theta_{[\mu\mu'],\mu'}) \\ &= \delta_{\lambda',[\mu\mu']} s_{T(\lambda\lambda')} s_{T(\mu\mu')} \otimes \theta_{[\lambda\lambda'],\mu'} \\ &= \delta_{s((\lambda,\lambda')),r((\mu,\mu'))} s_{T(\lambda\lambda')} s_{T(\mu\mu')} \otimes \theta_{[\lambda\lambda'],\mu'}. \end{aligned}$$

Suppose that $\lambda' = s((\lambda, \lambda')) = r((\mu, \mu')) = [\mu\mu']$. Then $r(T(\mu\mu')) = s([\mu\mu']) = s(\lambda') = s(T(\lambda\lambda'))$. Moreover, $\lambda\lambda' T(\mu\mu') = \lambda\mu\mu'$ because $[\mu\mu'] = \lambda'$. So $T(\lambda\lambda' T(\mu\mu')) = T(\lambda\mu\mu') = T(\lambda\lambda')T(\mu\mu')$. Since $d(T(\mu\mu')) \in H$, we also have $[\lambda\lambda'] = [\lambda\lambda' T(\mu\mu')]$, and hence $[\lambda\mu\mu'] = [\lambda\lambda' T(\mu\mu')] = [\lambda\lambda']$. Putting these two observations together, we deduce that

$$\begin{aligned} S_{(\lambda,\lambda')}S_{(\mu,\mu')} &= \delta_{s((\lambda,\lambda')),r((\mu,\mu'))} s_{T(\lambda\mu\mu')} \otimes \theta_{[\lambda\mu\mu'],\mu'} \\ &= \delta_{s((\lambda,\lambda')),r((\mu,\mu'))} S_{(\lambda\mu,\mu')} \\ &= \delta_{s((\lambda,\lambda')),r((\mu,\mu'))} S_{(\lambda,\lambda')(\mu,\mu')}, \end{aligned}$$

establishing (TCK2).

For (TCK3), fix $(\lambda, \lambda') \in \Lambda(n)$. We have

$$\begin{aligned} S^*_{(\lambda,\lambda')}S_{(\lambda,\lambda')} &= \big(s^*_{T(\lambda\lambda')} \otimes \theta_{\lambda',[\lambda\lambda']}\big)\big(s_{T(\lambda\lambda')} \otimes \theta_{[\lambda\lambda'],\lambda'}\big) \\ &= s_{s(T(\lambda\lambda'))} \otimes \theta_{\lambda',\lambda'} = s_{s(\lambda')} \otimes \theta_{\lambda',\lambda'} = S_{(r(\lambda'),\lambda')} = S_{s((\lambda,\lambda'))}. \end{aligned}$$

Finally for (CK), fix $(v, \lambda) \in \Lambda(n)^0$ and $m \in \mathbb{N}^k$. Then

$$\begin{aligned} \sum_{(\mu,\mu')\in(v,\lambda)\Lambda(n)^m} S_{(\mu,\mu')}S^*_{(\mu,\mu')} &= \sum_{\mu\in v\Lambda^m,\mu'\in s(\mu)\Lambda^{<n},[\mu\mu']=\lambda} S_{(\mu,\mu')}S^*_{(\mu,\mu')} \\ &= \sum_{\mu\in v\Lambda^m,\mu'\in s(\mu)\Lambda^{<n},[\mu\mu']=\lambda} s_{T(\mu\mu')}s^*_{T(\mu\mu')} \otimes \theta_{[\mu\mu'],[\mu\mu']} \\ &= \sum_{\mu\in v\Lambda^m,\mu'\in s(\mu)\Lambda^{<n},[\mu\mu']=\lambda} s_{T(\mu\mu')}s^*_{T(\mu\mu')} \otimes \theta_{\lambda,\lambda}. \end{aligned}$$

Let $p := m + [d(\lambda) - m]$. Then $p \geq 0$ and $[p] = d(\lambda)$, so $p \geq d(\lambda)$. The factorisation property implies that $\{\mu\mu' \mid \mu \in v\Lambda^m, \mu' \in s(\mu)\Lambda^{<n}, [\mu\mu'] = \lambda\} = \big\{\lambda\nu \mid \nu \in s(\lambda)\Lambda^{p-d(\lambda)}\big\}$. For $\nu \in s(\lambda)\Lambda^{p-d(\lambda)}$, we have $T(\lambda\nu) = \nu$. We deduce that

$$\sum_{(\mu,\mu')\in(v,\lambda)\Lambda(n)^m} S_{(\mu,\mu')}S^*_{(\mu,\mu')} = \sum_{\nu\in s(\lambda)\Lambda^{p-d(\lambda)}} s_\nu s^*_\nu \otimes \theta_{\lambda,\lambda} = s_{s(\lambda)} \otimes \theta_{\lambda,\lambda} = S_{(v,\lambda)}$$

as required. Now the universal property of $C^*(\Lambda(n))$ gives the desired homomorphism j_n. □

4. Cartesian products, 1-graphs, and the Kribs-Solel construction

Kumjian and Pask show that a cartesian product $\Lambda \times \Gamma$ of higher-rank graphs is itself a higher-rank graph with $C^*(\Lambda \times \Gamma) \cong C^*(\Lambda) \otimes C^*(\Gamma)$ ([9, Corollary 3.5(iv)]). In this section we show that the construction of the preceding section is compatible with the cartesian-product operation, and also that the construction of [8] and that of the preceding section are compatible via the passage from directed graphs to 1-graphs. We will use these results to compute the map

$K_*(j_n \circ \tilde{\iota}_n) : K_*(C^*(\Lambda)) \to K_*(C^*(\Lambda) \otimes \mathcal{K}_{\Lambda^{<n}}) \cong K_*(C^*(\Lambda))$ for a particular class of 2-graphs Λ.

For $i = 1, 2$, let (Λ_i, d_i) be a k_i-graph. The product category $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a $(k_1 + k_2)$-graph with degree map $(d_1 \times d_2)((\mu_1, \mu_2)) = (d_1(\mu_1), d(\mu_2))$. If each (Λ_i, d_i) is row-finite with no sources, then so is $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$. By [9, Corollary 3.5(iv)], there exists an isomorphism $\Theta_{\Lambda_1 \times \Lambda_2} : C^*(\Lambda_1 \times \Lambda_2) \to C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ such that $\Theta_{\Lambda_1 \times \Lambda_2}(s_{(\mu_1,\mu_2)}) = s_{\mu_1} \otimes s_{\mu_2}$.

Remark 4.1. Let (Λ_i, d_i) be a row-finite k_i-graph with no sources for $i = 1, 2$. For $n_1 \in \mathbb{N}^{k_1}$ and $n_2 \in \mathbb{N}^{k_2}$, the functor that sends $((\lambda_1, \lambda_2), (\lambda_1', \lambda_2')) \in (\Lambda_1 \times \Lambda_2)((n_1, n_2))$ to $((\lambda_1, \lambda_1'), (\lambda_2, \lambda_2')) \in \Lambda_1(n_1) \times \Lambda_2(n_2)$ is an isomorphism of $(k_1 + k_2)$-graphs. So there is an isomorphism $C^*((\Lambda_1 \times \Lambda_2)((n_1, n_2))) \cong C^*(\Lambda_1(n_1) \times \Lambda_2(n_2))$ sending $s_{((\lambda_1,\lambda_2),(\lambda_1',\lambda_2'))}$ to $s_{((\lambda_1,\lambda_1'),(\lambda_2,\lambda_2'))}$.

We show that the homomorphism in Lemma 3.3 is compatible with the isomorphism $C^*((\Lambda_1 \times \Lambda_2)((n_1, n_2))) \cong C^*(\Lambda_1(n_1) \times \Lambda_2(n_2))$ just described.

Lemma 4.2. *For $i = 1, 2$, let (Λ_i, d_i) be a row-finite k_i-graph with no sources. For $(n_1, n_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$, we have $(\tilde{\iota}_{n_1} \otimes \tilde{\iota}_{n_2}) \circ \Theta_{\Lambda_1 \times \Lambda_2} = (\Theta_{\Lambda_1(n_1) \times \Lambda_2(n_2)}) \circ \tilde{\iota}_{(n_1,n_2)}$.*

Proof. Let $(\mu_1, \mu_2) \in \Lambda_1 \times \Lambda_2$. Then

$$(\tilde{\iota}_{n_1} \otimes \tilde{\iota}_{n_2}) \circ \Theta_{\Lambda_1 \times \Lambda_2}(s_{(\mu_1,\mu_2)}) = \tilde{\iota}_{n_1} \otimes \tilde{\iota}_{n_2}(s_{\mu_1} \otimes s_{\mu_2}) = \sum_{\substack{\nu \in s(\mu_1)\Lambda_1^{<n_1} \\ \nu' \in s(\mu_2)\Lambda_2^{<n_2}}} s_{(\mu_1,\nu)} \otimes s_{(\mu_2,\nu')}.$$

Identifying $((\Lambda_1 \times \Lambda_2)((n_1, n_2)), d_1 \times d_2)$ with $(\Lambda_1(n_1), d_1) \times (\Lambda_2(n_2), d_2)$ as in Remark 4.1, we have

$$\begin{aligned}
\Theta_{\Lambda_1(n_1)\times\Lambda_2(n_2)} \circ \tilde{\iota}_{(n_1,n_2)}(s_{(\mu_1,\mu_2)}) &= \Theta_{\Lambda_1(n_1)\times\Lambda_2(n_2)}\Big(\sum_{(\alpha,\beta) \in s((\mu_1,\mu_2))(\Lambda_1\times\Lambda_2)^{<(n_1,n_2)}} s_{((\mu_1,\alpha),(\mu_2,\beta))} \Big) \\
&= \sum_{\substack{\alpha \in s(\mu_1)\Lambda^{<n_1} \\ \beta \in s(\mu_2)\Lambda^{<n_2}}} \Theta_{\Lambda_1(n_1)\times\Lambda_2(n_2)}(s_{((\mu_1,\alpha),(\mu_2,\beta))}) \\
&= \sum_{\substack{\alpha \in s(\mu_1)\Lambda^{<n_1} \\ \beta \in s(\mu_2)\Lambda^{<n_2}}} s_{(\mu_1,\alpha)} \otimes s_{(\mu_2,\beta)}.
\end{aligned}$$

Therefore, $\tilde{\iota}_{n_1} \otimes \tilde{\iota}_{n_2} \circ \Theta_{\Lambda_1 \times \Lambda_2} = \Theta_{\Lambda_1(n_1)\times\Lambda_2(n_2)} \circ \tilde{\iota}_{(n_1,n_2)}$. □

We will need to apply Lemma 4.2 where Λ_1 and Λ_2 are the 1-graphs associated to directed graphs E and F, and relate this to [19, Lemma 2.5] for $C^*(E)$ and $C^*(F)$. We therefore find ourselves in an unfortunate clash of conventions. The convention used in [19] is that of [10, 11] where, for historical reasons, the partial isometries in a Cuntz-Krieger family point in the opposite direction to the edges in the graph. This is at odds with the k-graph convention where the partial isometries go in the same direction as the morphisms in the k-graph. To deal with this, we take the approach that the range and source maps are interchanged when passing from a directed graph E to its path category E^*.

We recall the definition of the Toeplitz algebra $\mathcal{T}C^*(E)$ and the Cuntz-Krieger algebra $C^*(E)$ of a directed graph E as used in [19]. Let $E = (E^0, E^1, r_E, s_E)$ be a row-finite directed graph with no sinks (so $0 < |\{e \in E^1 \mid s_E(e) = v\}| < \infty$ for $v \in E^0$). Then $\mathcal{T}C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{q_v\}_{v \in E^0}$ and elements $\{t_e\}_{e \in E^1}$ such that

1) $t_e^* t_e = q_{r_E(e)}$ for all $e \in E^1$, and
2) $q_v \geq \sum_{e \in E^1, s_E(e) = v} t_e t_e^*$ for each $v \in E^0$.

The graph C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v\}_{v\in E^0}$ and elements $\{s_e\}_{e\in E^1}$ such that

3) $s_e^* s_e = p_{r_E(e)}$ for all $e \in E^1$, and
4) $p_v = \sum_{e\in E^1, s_E(e)=v} t_e t_e^*$ for each $v \in E^0$.

We recall the construction described in [8, Section 4]. Given $m \in \mathbb{N}$ and a directed graph $E = (E^0, E^1, r_E, s_E)$, we define $E(m)$ to be the directed graph with

$$E(m)^0 = E^{<m}, \qquad E(m)^1 = \left\{(e,\mu) \mid e \in E^1, \mu \in E^{<m}, r_E(e) = s_E(\mu)\right\},$$

$$r_{E(m)}((e,\mu)) = \mu, \qquad s_{E(m)}((e,\mu)) = \begin{cases} e\mu & \text{if } |\mu| < m-1 \\ s_E(e) & \text{if } |\mu| = m-1. \end{cases}$$

The next lemma is due to James Rout, and will appear in his PhD thesis. We thank James for providing us with the details (a proof appears in [19, Lemma 2.5]).

Lemma 4.3 (Rout)**.** *Let E be a row-finite directed graph and take $m \geq 1$. There is an injective homomorphism $\iota_{m,E} : \mathcal{T}C^*(E) \to \mathcal{T}C^*(E(m))$ such that*

$$\iota_{m,E}(q_v) = \sum_{\substack{\mu\in E^{<m} \\ s_E(\mu)=v}} q_\mu^m \quad \textit{and} \quad \iota_{m,E}(t_e) = \sum_{(e,\mu)\in E(m)^1} t_{(e,\mu)}^m,$$

where $\{q_\mu^m, t_{(e,\nu)}^m\}_{\mu\in E(m)^0, (e,\nu)\in E(m)^1}$ are the universal generators of $\mathcal{T}C^*(E)$. The map $\iota_{m,E}$ descends to an injective homomorphism $\tilde{\iota}_{m,E} : C^*(E) \to C^*(E(m))$.

We describe canonical isomorphisms $C^*(E) \cong C^*(E^*)$ and $C^*(E(m)) \cong C^*(E^*(m))$ and show that these isomorphisms intertwine the homomorphism $\tilde{\iota}_{m,E}$ of Lemma 4.3 and the homomorphism $\tilde{\iota}_m$ of Lemma 3.3.

Remark 4.4. Let E be a row-finite directed graph with no sinks, and let E^* be its path-category regarded as a row-finite 1-graph with no sources. Let $\{p_v, s_e\}_{v\in E^0, e\in E^1}$ be the universal generators of $C^*(E)$ and let $\{S_\lambda\}_{\lambda\in E^*}$ be the universal generators of $C^*(E^*)$. By [9, Examples 1.7], there is an isomorphism $\psi_E : C^*(E) \to C^*(E^*)$ such that $\psi_E(p_v) = S_v$ and $\psi_E(s_e) = S_e$ for all $v \in E^0$ and $e \in E^1$.

Lemma 4.5. *Let E be a row-finite directed graph with no sinks, and let E^* be its path-category regarded as a row-finite 1-graph with no sources. There is an isomorphism of 1-graphs $E(m)^* \cong E^*(m)$ extending the identity map on $(E(m)^*)^1 = E^*(m)^1$. There is an isomorphism $C^*(E(m)^*) \cong C^*(E^*(m))$ satisfying $s_{(e,\mu)} \mapsto s_{(e,\mu)}$ for $(e,\mu) \in E^*(m)^1 = (E(m)^*)^1$.*

Proof. Example 1.3 of [9] says that 1-graphs Λ and Γ are isomorphic if and only if there is a bijection $\Lambda^1 \to \Gamma^1$ that intertwines range maps and source maps. Since $(e,\mu) \mapsto (e,\mu)$ is such a bijection between $(E(m)^*)^1$ and $E^*(m)^1$, there is an isomorphism $E^*(m) \cong E(m)^*$ as claimed. Since isomorphic 1-graphs have canonically isomorphic C^*-algebras, the result follows. □

Lemma 4.6. *Let E be a row-finite directed graph with no sinks, and fix $m \in \mathbb{N} \setminus \{0\}$. Identify $C^*(E(m)^*)$ with $C^*(E^*(m))$ using Lemma 4.5. Then the isomorphisms $\psi_E : C^*(E) \to C^*(E^*)$ and $\psi_{E(m)} : C^*(E(m)) \to C^*(E^*(m))$ of Remark 4.4 satisfy $\tilde{\iota}_m \circ \psi_E = \psi_{E(m)} \circ \tilde{\iota}_{m,E}$.*

Proof. Let $v \in E^0$. Then

$$\tilde{\iota}_m \circ \psi_E(p_v) = \tilde{\iota}_m(S_v) = \sum_{\lambda\in v(E^*)^{<m}} S_{(v,\lambda)} = \sum_{\substack{\lambda\in E^{<m} \\ s_E(\lambda)=v}} S_{(v,\lambda)},$$

and

$$\psi_{E(m)} \circ \tilde{\iota}_{m,E}(p_v) = \psi_{E(m)}\Big(\sum_{\substack{\lambda\in E^{<m} \\ s_E(\lambda)=v}} p_\lambda\Big) = \sum_{\substack{\lambda\in E^{<m} \\ s_E(\lambda)=v}} S_{(s_E(\lambda),\lambda)} = \sum_{\substack{\lambda\in E^{<m} \\ s_E(\lambda)=v}} S_{(v,\lambda)}.$$

Thus, $\tilde{\iota}_m \circ \psi_E(p_v) = \psi_{E(m)} \circ \tilde{\iota}_{m,E}(p_v)$ for all $v \in E^0$. For $e \in E^1$,

$$\tilde{\iota}_m \circ \psi_E(s_e) = \tilde{\iota}_m(S_e) = \sum_{\lambda \in s(e)(E^*)^{<m}} S_{(e,\lambda)} = \sum_{\substack{\lambda \in E^{<m} \\ s_E(\lambda) = r_E(e)}} S_{(e,\lambda)} = \sum_{(e,\lambda) \in E(m)^1} S_{(e,\lambda)},$$

and

$$\psi_{E(m)} \circ \tilde{\iota}_{m,E}(s_e) = \psi_{E(m)}\Big(\sum_{(e,\lambda) \in E(m)^1} s_{(e,\lambda)} \Big) = \sum_{(e,\lambda) \in E(m)^1} S_{(e,\lambda)}.$$

So $\tilde{\iota}_m \circ \psi_E(s_e) = \psi_{E(m)} \circ \tilde{\iota}_{m,E}(s_e)$ for all $e \in E^1$. Since $C^*(E)$ is generated by $\{p_v, s_e\}_{v \in E^0, e \in E^1}$, we see that $\tilde{\iota}_m \circ \psi_E = \psi_{E(m)} \circ \tilde{\iota}_{m,E}$. □

5. Asymptotic order-1 approximations

In this section, we show that given a row-finite 2-graph with no sources, the family of homomorphisms $(\tilde{\iota}_n)_{n \in \mathbb{N}^k}$ has an asymptotic order-1 approximation through AF-algebras. Thus, the family $(j_n \circ \tilde{\iota}_n)_{n \in \mathbb{N}^k}$ has an asymptotic order-1 approximation through AF-algebras. We will use this family of homomorphisms in the next section to prove that the nuclear dimension of a UCT-Kirchberg algebra with trivial K_0 and finite K_1 is 1.

If $f : \mathbb{N}^k \to \mathbb{R}$ is a function, then we write $\lim_{n \to \infty} f(n) = 0$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}^k$ such that $|f(n)| < \varepsilon$ whenever $n \geq N$ in $\mathbb{N}^k$.

Recall that a completely positive map $\phi : A \to B$ has *order-zero* if for $a, b \in A_+$ with $ab = 0$, we have $\phi(a)\phi(b) = 0$. Suppose that $(\beta_n)_{n \in \mathbb{N}^k}$ is a family of homomorphisms $\beta_n : A \to B_n$, and let $\mathcal{C}$ be a class of C^*-algebras. Following [19, Definition 2.8][1], a family $(F_n, \phi_n, \psi_n)_{n \in \mathbb{N}^k}$ is an *asymptotic order-r factorisation of the family (β_n) through elements of $\mathcal{C}$* if each F_n is a direct sum $F_n = \bigoplus_{i=0}^r F_n^{(i)}$ of C^*-algebras $F_n^{(i)} \in \mathcal{C}$, each $\psi_n : A \to F_n$ is a completely positive contraction, each $\phi_n : F_n \to B_n$ restricts to an order-zero completely positive contraction on each $F_n^{(i)}$, and $\lim_{n \to \infty} \|\phi_n \circ \psi_n(a) - \beta_n(a)\| = 0$ for each $a \in A$. We say that $(F_n, \phi_n, \psi_n)_{n \in \mathbb{N}^k}$ is an *asymptotic order-r factorisation of $\beta : A \to B$* if it is an asymptotic order-r factorisation of $(\beta)_{n \in \mathbb{N}^k}$.

Remark 5.1. Suppose that $(\beta_n : A \to B_n)_{n \in \mathbb{N}^k}$ has an asymptotic order-r factorisation through elements of $\mathcal{C}$. Then for any strictly increasing sequence $(n^m)_{m \in \mathbb{N}}$ in $\mathbb{N}^k$ such that $n_j^m \to \infty$ as $m \to \infty$ for each $j \leq k$, the sequence $(\beta_{n^m})_{m \in \mathbb{N}}$ has an asymptotic order-r factorisation through elements of $\mathcal{C}$ in the sense of [19, Definition 2.8].

Throughout this section, we use the following notation. Let Λ be a row-finite k-graph with no sources and let $n \in \mathbb{N}^k$. Then $\{t_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{T}C^*(\Lambda)$ and $\{T_{(\lambda,\lambda')}\}_{(\lambda,\lambda') \in \Lambda(n)} \subseteq \mathcal{T}C^*(\Lambda(n))$ will be the universal generating Toeplitz-Cuntz-Krieger families, and $\{s_\lambda\}_{\lambda \in \Lambda} \subseteq C^*(\Lambda)$ and $\{S_{(\lambda,\lambda')}\}_{(\lambda,\lambda') \in \Lambda(n)} \subseteq C^*(\Lambda(n))$ will be the universal generating Cuntz-Krieger families. We will regard $\mathcal{T}C^*(\Lambda)$ as a sub-C^*-algebra of $B(\ell^2(\Lambda))$. When $s(\mu) = s(\nu)$, we have

$$t_\mu t_\nu^* = \sum_{\tau \in s(\mu)\Lambda} \theta_{\mu\tau, \nu\tau}, \tag{5.1}$$

where the series converges in the strict topology.

First we construct a homomorphism that we will use to define the maps ϕ_n in our asymptotic factorisation.

Lemma 5.2. *Let Λ be a row-finite k-graph with no sources. For $p, n \in \mathbb{N}^k$, there is a homomorphism $\Gamma_p^{p+n} : \bigoplus_{v \in \Lambda^0} \mathcal{K}_{\Lambda^{[p,p+n)}v} \to \mathcal{T}C^*(\Lambda(n))$ such that*

$$\Gamma_p^{p+n}(\theta_{\mu,\nu}) = T_{(\mu,s(\mu))} T_{(\nu,s(\nu))}^*$$

[1] In the preprint version of [19] the authors mistakenly require just that each F_n, rather than each $F_n^{(i)}$, belonged to $\mathcal{C}$; the intention was that $\mathcal{C}$ should be closed under hereditary subalgebras and direct sums.

for all $\mu,\nu \in \Lambda^{[p,p+n)}$ *with* $s(\mu) = s(\nu)$.

Proof. We just have to check that the elements $\{T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))}\}_{\mu,\nu\in\Lambda^{[p,p+n)},s(\mu)=s(\nu)}$ are nonzero and are matrix units in the sense that $(T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))})^* = T_{(\nu,s(\nu))}T^*_{(\mu,s(\mu))}$ and

$$T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))}T_{(\mu',s(\mu'))}T^*_{(\nu',s(\nu'))} = \delta_{\nu,\mu'}T_{(\mu,s(\mu))}T^*_{(\nu',s(\nu'))}.$$

(It follows from the displayed equation that $\operatorname{span}\{(T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))})^* : s(\mu) = s(\nu) = v\} \perp \operatorname{span}\{(T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))})^* : s(\mu) = s(\nu) = w\}$ for distinct v, w.)

The $T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))}$ are nonzero by (5.1). Let $\mu,\nu \in \Lambda^{[p,p+n)}$. By Lemma 3.2,

$$\begin{aligned}&\Lambda(n)^{\min}((\nu,s(\nu)),(\mu,s(\mu)))\\ &\quad= \begin{cases}\{((\alpha,\tau),(\beta,\tau)) \mid (\alpha,\beta)\in\Lambda^{\min}(\nu,\mu), \tau\in s(\alpha)\Lambda^{<n}, [\alpha\tau]=s(\nu), [\beta\tau]=s(\mu)\} & \text{if } [\mu]=[\nu]\\ \emptyset & \text{otherwise.}\end{cases}\end{aligned}$$

We claim that

$$\Lambda(n)^{\min}((\nu,s(\nu)),(\mu,s(\mu))) = \begin{cases}\{\big((s(\nu),s(\nu)),(s(\nu),s(\nu))\big)\} & \text{if } \mu=\nu\\ \emptyset & \text{otherwise.}\end{cases}$$

Indeed, if $\Lambda(n)^{\min}((\nu,s(\nu)),(\mu,s(\mu))) \neq \emptyset$, say $((\alpha,\tau),(\beta,\tau)) \in \Lambda(n)^{\min}((\nu,s(\nu)),(\mu,s(\mu)))$, then $[\mu]=[\nu]$. In particular, $[d(\mu)] = d([\mu]) = d([\nu]) = [d(\nu)]$. Since $p \le d(\nu), d(\mu) < p+n$, we have that $d(\nu) = d(\mu)$. Since $\mu\alpha = \nu\beta$, the factorisation property forces $\mu = \nu$. We then have $\Lambda^{\min}(\nu,\mu) = \Lambda^{\min}(\nu,\nu) = \{(s(\nu),s(\nu))\}$, giving

$$\Lambda(n)^{\min}((\nu,s(\nu)),(\mu,s(\mu))) = \{((s(\nu),s(\nu)),(s(\nu),s(\nu)))\}$$

as claimed.

We now show that $\{T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))}\}_{\mu,\nu\in\Lambda^{[p,p+n)},s(\mu)=s(\nu)}$ form a system of matrix units, so that the formula given for Γ_p^{p+n} indeed defines a homomorphism. For $\mu,\nu,\mu',\nu' \in \Lambda^{[p,p+n)}$ with $s(\mu)=s(\nu)$ and $s(\mu')=s(\nu')$,

$$\begin{aligned}&T_{(\mu,s(\mu))}T^*_{(\nu,s(\nu))}T_{(\mu',s(\mu'))}T^*_{(\nu',s(\nu'))}\\ &\qquad= T_{(\mu,s(\mu))}\Big(\sum_{((\alpha,\gamma),(\beta,\delta))\in\Lambda(n)^{\min}((\nu,s(\nu)),(\mu',s(\mu')))} T_{(\alpha,\gamma)}T^*_{(\beta,\delta)}\Big)T^*_{(\nu',s(\nu'))}\\ &\qquad= \delta_{\nu,\mu'}T_{(\mu,s(\mu))}T_{(s(\nu),s(\nu))}T^*_{(\nu',s(\nu'))}\\ &\qquad= \delta_{\nu,\mu'}T_{(\mu,s(\mu))}T_{(s(\mu),s(\mu))}T_{(s(\nu'),s(\nu'))}T^*_{(\nu',s(\nu'))}\\ &\qquad= \delta_{\nu,\mu'}\delta_{s(\mu),s(\nu')}T_{(\mu,s(\mu))}T^*_{(\nu',s(\nu'))}.\end{aligned}$$
□

Next we provide a technical lemma and a proposition that summarises what we require to construct an approximate order-1 factorisation of the family $(\tilde{\iota}_n)_{n\in\mathbb{N}^k}$ obtained from Lemma 3.3.

Lemma 5.3. *Let* Λ *be a row-finite* k*-graph with no sources and let* $n \in \mathbb{N}^k$. *For each* $\mu \in \Lambda$

$$\iota_n(t_\mu)T_{(s(\mu),s(\mu))} = T_{(\mu,s(\mu))} = T_{(r(\mu),[\mu])}\iota_n(t_\mu). \tag{5.2}$$

For $\mu,\nu,\tau \in \Lambda$ *with* $s(\mu)=s(\nu)=r(\tau)$,

$$T_{(\mu\tau,s(\mu\tau))}T^*_{(\nu\tau,s(\nu\tau))} = \iota_n(t_\mu)\iota_n(t_\tau t_\tau^*)T_{(r(\tau),[\tau])}\iota_n(t_\nu^*) \quad \textit{and} \quad T_{(\mu,s(\mu))}T^*_{(\mu,s(\mu))} = \iota_n(t_\mu t_\mu^*)T_{(r(\mu),[\mu])}.$$

Proof. Recall that $\iota_n(t_\mu) = \sum_{\lambda\in s(\mu)\Lambda^{<n}} T_{(\mu,\lambda)}$. So

$$\begin{aligned}\iota_n(t_\mu)T_{(s(\mu),s(\mu))} &= \Big(\sum_{\lambda\in s(\mu)\Lambda^{<n}} T_{(\mu,\lambda)}\Big)T_{(s(\mu),s(\mu))}\\ &= \Big(\sum_{\lambda\in s(\mu)\Lambda^{<n}} T_{(\mu,\lambda)}T_{(s(\mu),\lambda)}\Big)T_{(s(\mu),s(\mu))} = T_{(\mu,s(\mu))}.\end{aligned}$$

We now prove that $T_{(\mu,s(\mu))} = T_{(r(\mu),[\mu])}\iota_n(t_\mu)$. We have

$$T_{(r(\mu),[\mu])}\iota_n(t_\mu) = T_{(r(\mu),[\mu])} \sum_{\lambda\in s(\mu)\Lambda^{<n}} T_{(\mu,\lambda)} = T_{(r(\mu),[\mu])} \sum_{\lambda\in s(\mu)\Lambda^{<n}} T_{(r(\mu),[\mu\lambda])}T_{(\mu,\lambda)}.$$

Note that $T_{(r(\mu),[\mu])}T_{(r(\mu),[\mu\lambda])} \neq 0$ if and only if $[\mu] = [\mu\lambda]$. Let $\lambda \in s(\mu)\Lambda^{<n}$ with $[\mu] = [\mu\lambda]$. Since $[\mu] = \mu(0, [d(\mu)])$ and $[\mu\lambda] = (\mu\lambda)(0, [d(\mu\lambda)])$, we see that $d(\lambda) = d(\mu\lambda) - d(\mu) \in H_n$. Since $d(\lambda) < n$, we deduce that $d(\lambda) = 0$, giving $\lambda = r(\lambda) = s(\mu)$. Hence,

$$T_{(r(\mu),[\mu])}\iota_n(t_\mu) = T_{(r(\mu),[\mu])} \sum_{\lambda\in s(\mu)\Lambda^{<n}} T_{(r(\mu),[\mu\lambda])}T_{(\mu,\lambda)} = T_{(\mu,s(\mu))}.$$

This proves (5.2).

For the second assertion, take $\mu,\nu,\tau \in \Lambda$ with $s(\mu) = s(\nu) = r(\tau)$. Then (5.2) gives

$$\begin{aligned} T_{(\mu\tau,s(\mu\tau))}T^*_{(\nu\tau,s(\nu\tau))} &= \iota_n(t_{\mu\tau})T_{(s(\mu\tau),s(\mu\tau))}T_{(s(\nu\tau),s(\nu\tau))}\iota_n(t^*_{\nu\tau}) = \iota_n(t_\mu)\iota_n(t_\tau)T_{(s(\tau),s(\tau))}\iota_n(t^*_\tau)\iota_n(t^*_\nu) \\ &= \iota_n(t_\mu)\iota_n(t_\tau)\iota_n(t^*_\tau)T_{(r(\tau),[\tau])}\iota_n(t^*_\nu) = \iota_n(t_\mu)\iota_n(t_\tau t^*_\tau)T_{(r(\tau),[\tau])}\iota_n(t^*_\nu), \end{aligned}$$

and

$$\begin{aligned} T_{(\mu,s(\mu))}T^*_{(\mu,s(\mu))} &= \iota_n(t_\mu)T_{(s(\mu),s(\mu))}\big(\iota_n(t_\mu)T_{(s(\mu),s(\mu))}\big)^* \\ &= \iota_n(t_\mu)T_{(s(\mu),s(\mu))}\iota_n(t^*_\mu) = \iota_n(t_\mu t^*_\mu)T_{(r(\mu),[\mu])}. \end{aligned}$$ □

Recall that for $n \in \mathbb{N}^k$ with each $n_i \geq 1$, the group H_n is the subgroup

$$\{p \in \mathbb{Z}^k \mid n_i \text{ divides } p_i \text{ for each } i \leq k\}.$$

For $x \in \mathbb{R}^k$, let $\lceil x\rceil = \big(\lceil x_1\rceil, \dots, \lceil x_k\rceil\big) \in \mathbb{Z}^k$, and for $a \in \mathbb{R}\setminus\{0\}$, put $\frac{x}{a} = \big(\frac{x_1}{a}, \dots, \frac{x_k}{a}\big)$.

Proposition 5.4. *Let Λ be a row-finite k-graph with no sources. For each $n \in \mathbb{N}^k$ such that each $n_j > 0$, each $p < n$, and each $\mu \in \Lambda$, let $h_{n,\mu}(p)$ and $g_{n,\mu}(p)$ be the unique elements in H_n such that*

$$n \leq d(\mu) + p + h_{n,\mu}(p) < 2n \quad \textit{and} \quad \left\lceil \tfrac{3n}{2}\right\rceil \leq d(\mu) + p + g_{n,\mu}(p) < \left\lceil \tfrac{5n}{2}\right\rceil.$$

For each $n \in \mathbb{N}^k$, let Δ_n be a function $\Delta_n : \mathbb{N}^k \times \mathbb{N}^k \to [0,1)$. For $i = 1, 2$, define $\Delta^{\mu,\nu}_{n,i} : \mathbb{N}^k \to [0,1)$ by $\Delta^{\mu,\nu}_{n,1}(p) := \Delta_n\big(d(\mu) + p + h_{n,\mu}(p) - n, d(\nu) + p + h_{n,\mu}(p) - n\big)$ and $\Delta^{\mu,\nu}_{n,2}(p) := \Delta_n\big(d(\mu) + p + g_{n,\mu}(p) - \lceil\frac{3n}{2}\rceil, d(\nu) + p + g_{n,\mu}(p) - \lceil\frac{3n}{2}\rceil\big)$. Suppose that for each $\mu,\nu \in \Lambda$,

$$\lim_{n\to\infty} \max\big\{|\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p) - 1| \,\big|\, p < n\big\} = 0.$$

Suppose that there exist completely positive, contractive linear maps $P_n : \mathcal{T}C^*(\Lambda) \to \mathcal{K}_{\Lambda^{[n,2n)}}$ and $Q_n : \mathcal{T}C^*(\Lambda) \to \mathcal{K}_{\Lambda^{[\lceil 3n/2\rceil, \lceil 5n/2\rceil)}}$ such that

$$P_n(t_\mu t^*_\nu) = \sum_{\substack{\tau\in s(\mu)\Lambda \\ n\leq d(\mu\tau),d(\nu\tau)<2n}} \Delta_n(d(\mu\tau) - n, d(\nu\tau) - n)\theta_{\mu\tau,\nu\tau}$$

and

$$Q_n(t_\mu t^*_\nu) = \sum_{\substack{\tau\in s(\mu)\Lambda \\ \lceil\frac{3n}{2}\rceil\leq d(\mu\tau),d(\nu\tau)<\lceil\frac{5n}{2}\rceil}} \Delta_n\left(d(\mu\tau) - \left\lceil\tfrac{3n}{2}\right\rceil, d(\nu\tau) - \left\lceil\tfrac{3n}{2}\right\rceil\right)\theta_{\mu\tau,\nu\tau}$$

for all $\mu,\nu \in \Lambda$ with $s(\mu) = s(\nu)$. Then the family $(\tilde\iota_n)_{n\in\mathbb{N}^k}$ has an order-1 approximation through AF-algebras.

Proof. For each $n \in \mathbb{N}^k$, let $\pi_n : \mathcal{T}C^*(\Lambda(n)) \to C^*(\Lambda(n))$ be the quotient homomorphism. We first show that for all $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$,

$$\lim_{n\to\infty} \Big\|\pi_n\Big(\Big(\big(\Gamma_n^{2n} \circ P_n + \Gamma_{\lceil \frac{3n}{2}\rceil}^{\lceil \frac{5n}{2}\rceil} \circ Q_n\big) - \iota_n\Big)(t_\mu t_\nu^*)\Big)\Big\| = 0, \tag{5.3}$$

where the Γ's are the homomorphisms constructed in Lemma 5.2. For this, let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, and fix $n \in \mathbb{N}^k$. Lemma 5.2 gives

$$\Gamma_n^{2n} \circ P_n(t_\mu t_\nu^*) = \sum_{\substack{\tau \in s(\mu)\Lambda \\ n \le d(\mu\tau), d(\nu\tau) < 2n}} \Delta_n(d(\mu\tau) - n, d(\nu\tau) - n) T_{(\mu\tau, s(\mu\tau))} T^*_{(\nu\tau, s(\nu\tau))}.$$

Lemma 5.3 shows that $T_{(\mu\tau,s(\mu\tau))}T^*_{(\nu\tau,s(\nu\tau))} = \iota_n(t_\mu)\iota_n(t_\tau t_\tau^*)T_{(r(\tau),[\tau])}\iota_n(t_\nu^*)$. So,

$$\begin{aligned}
\Gamma_n^{2n} \circ P_n(t_\mu t_\nu^*) &= \iota_n(t_\mu)\Big(\sum_{\substack{\tau \in s(\mu)\Lambda \\ n \le d(\mu\tau), d(\nu\tau) < 2n}} \Delta_n(d(\mu\tau) - n, d(\nu\tau) - n)\iota_n(t_\tau t_\tau^*)T_{(r(\tau),[\tau])}\Big)\iota_n(t_\nu^*) \\
&= \iota_n(t_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p}\sum_{\rho \in s(\alpha)\Lambda^{h_{n,\mu}(p)}} \Delta^{\mu,\nu}_{n,1}(p)\iota_n(t_{\alpha\rho}t^*_{\alpha\rho})T_{(r(\alpha\rho),[\alpha\rho])}\Big)\iota_n(t_\nu^*) \\
&= \iota_n(t_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p}\sum_{\rho \in s(\alpha)\Lambda^{h_{n,\mu}(p)}} \Delta^{\mu,\nu}_{n,1}(p)\iota_n(t_\alpha)\iota_n(t_\rho t_\rho^*)\iota_n(t_\alpha^*)T_{(s(\mu),\alpha)}\Big)\iota_n(t_\nu^*) \\
&= \iota_n(t_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p} \Delta^{\mu,\nu}_{n,1}(p)\iota_n(t_\alpha)\Big(\sum_{\rho \in s(\alpha)\Lambda^{h_{n,\mu}(p)}} \iota_n(t_\rho t_\rho^*)\Big)\iota_n(t_\alpha^*)T_{(s(\mu),\alpha)}\Big)\iota_n(t_\nu^*).
\end{aligned}$$

Relation (CK) for $\{s_\lambda\}_{\lambda\in\Lambda}$ gives

$$\begin{aligned}
&\pi_n \circ \Gamma_n^{2n} \circ P_n(t_\mu t_\nu^*) \\
&\quad = \widetilde{\iota}_n(s_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p} \Delta^{\mu,\nu}_{n,1}(p)\widetilde{\iota}_n(s_\alpha)\widetilde{\iota}_n(s_{s(\alpha)})\widetilde{\iota}_n(s_\alpha^*)S_{(s(\mu),\alpha)}\Big)\widetilde{\iota}_n(s_\nu^*) \\
&\quad = \widetilde{\iota}_n(s_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p} \Delta^{\mu,\nu}_{n,1}(p)\widetilde{\iota}_n(s_\alpha)\widetilde{\iota}_n(s_\alpha^*)S_{(s(\mu),\alpha)}\Big)\widetilde{\iota}_n(s_\nu^*)
\end{aligned}$$

By Lemma 5.3, $S_{(\alpha,s(\alpha))}S^*_{(\alpha,s(\alpha))} = \widetilde{\iota}_n(s_\alpha s_\alpha^*)S_{(r(\alpha),[\alpha])} = \widetilde{\iota}_n(s_\alpha)\widetilde{\iota}_n(s_\alpha^*)S_{(s(\mu),\alpha)}$ for all $\alpha \in s(\mu)\Lambda^{<n}$, and hence

$$\pi_n \circ \Gamma_n^{2n} \circ P_n(t_\mu t_\nu^*) = \widetilde{\iota}_n(s_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p} \Delta^{\mu,\nu}_{n,1}(p)S_{(\alpha,s(\alpha))}S^*_{(\alpha,s(\alpha))}\Big)\widetilde{\iota}_n(s_\nu^*). \tag{5.4}$$

Take $p < n$ and $\alpha \in s(\mu)\Lambda^p$. Then

$$\begin{aligned}
&\{(\lambda, \lambda') \in \Lambda(n)^p \mid r((\lambda, \lambda')) = (s(\mu), \alpha)\} \\
&\qquad = \{(\lambda, \lambda') \in \Lambda \times \Lambda^{<n} \mid s(\lambda) = r(\lambda'), d(\lambda) = p, (r(\lambda), [\lambda\lambda']) = (s(\mu), \alpha)\}.
\end{aligned}$$

Suppose that $(\lambda, \lambda') \in \Lambda(n)^p$ with $r((\lambda, \lambda')) = (s(\mu), \alpha)$. Then $[\lambda\lambda'] = \alpha$, so $p = d(\alpha) = d([\lambda\lambda']) = [d(\lambda\lambda')] = [p + d(\lambda')]$. Hence $[p] = [p + d(\lambda')]$, and since $d(\lambda') < n$, this forces $d(\lambda') = 0$. Therefore, $\alpha = [\lambda\lambda'] = [\lambda] = \lambda$ since $d(\lambda) = p < n$ and $\lambda' = r(\lambda') = s(\lambda) = s(\alpha)$. Hence

$$\{(\lambda, \lambda') \in \Lambda(n)^p \mid r((\lambda, \lambda')) = (s(\mu), \alpha)\} = \{(\alpha, s(\alpha))\},$$

which implies that each $S_{(\alpha,s(\alpha))}S^*_{(\alpha,s(\alpha))} = S_{(s(\mu),\alpha)}$ by (CK) in $C^*(\Lambda(n))$. Combining this with (5.4) gives

$$\pi_n \circ \Gamma_n^{2n} \circ P_n(t_\mu t_\nu^*) = \widetilde{\iota}_n(s_\mu)\Big(\sum_{p<n}\sum_{\alpha \in s(\mu)\Lambda^p} \Delta^{\mu,\nu}_{n,1}(p)S_{(s(\mu),\alpha)}\Big)\widetilde{\iota}_n(s_\nu^*).$$

A similar computation gives

$$\pi_n \circ \Gamma^{\lceil \frac{5n}{2}\rceil}_{\lceil \frac{3n}{2}\rceil} \circ Q_n(t_\mu t_\nu^*) = \widetilde{\iota}_n(s_\mu)\Big(\sum_{p<n}\sum_{\alpha\in s(\mu)\Lambda^p} \Delta^{\mu,\nu}_{n,2}(p) S_{(s(\mu),\alpha)}\Big)\widetilde{\iota}_n(s_\nu^*).$$

Since $\{S_{(s(\mu),\alpha)}\}_{\alpha\in s(\mu)\Lambda^{<n}}$ is a collection of mutually orthogonal projections,

$$\begin{aligned}
&\Big\|\big(\pi_n \circ \Gamma^{2n}_n \circ P_n(t_\mu t_\nu^*) + \pi_n \circ \Gamma^{\lceil \frac{5n}{2}\rceil}_{\lceil \frac{3n}{2}\rceil} \circ Q_n(t_\mu t_\nu^*)\big) - \pi_n \circ \iota_n(t_\mu t_\nu^*)\Big\| \\
&\qquad \le \Big\|\sum_{p<n}\sum_{\alpha\in s(\mu)\Lambda^p} (\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p))S_{(s(\mu),\alpha)} - \widetilde{\iota}_n(s_{s(\mu)})\Big\| \\
&\qquad = \Big\|\sum_{p<n}\sum_{\alpha\in s(\mu)\Lambda^p} (\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p))S_{(s(\mu),\alpha)} - \sum_{p<n}\sum_{\alpha\in s(\mu)\Lambda^p} S_{(s(\mu),\alpha)}\Big\| \\
&\qquad = \max_{p<n} |\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p) - 1|.
\end{aligned}$$

By assumption, $\lim_{n\to\infty} \max_{p<n} |\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p) - 1| = 0$. This proves (5.3).

Since k-graph algebras are nuclear [9, Theorem 5.5], we may apply [3, Theorem 3.10] to obtain a contractive completely positive splitting $\sigma : C^*(\Lambda) \to \mathcal{T}C^*(\Lambda)$ for the quotient map. For each n, define $\psi_n : C^*(\Lambda) \to \mathcal{K}_{\Lambda[n,2n)} \oplus \mathcal{K}_{\Lambda[\lceil 3n/2\rceil,\lceil 5n/2\rceil)}$ by $\psi_n(a) := \big(P_n(\sigma(a)), Q_n(\sigma(a))\big)$ and $\phi_n : \mathcal{K}_{\Lambda[n,2n)} \oplus \mathcal{K}_{\Lambda[\lceil 3n/2\rceil,\lceil 5n/2\rceil)} \to C^*(\Lambda(n))$ by $\phi_n((a,b)) = \pi_n(\Gamma^{2n}_n(a) + \Gamma^{\lceil \frac{5n}{2}\rceil}_{\lceil \frac{3n}{2}\rceil}(b))$. By Lemma 5.2, ϕ_n restricts to a homomorphism (and in particular an order-zero map) on each of $\mathcal{K}_{\Lambda[n,2n)}$ and $\mathcal{K}_{\Lambda[\lceil 3n/2\rceil,\lceil 5n/2\rceil)}$. Since $\mathcal{T}C^*(\Lambda) = \overline{\operatorname{span}}\{t_\mu t_\nu^* \mid \mu,\nu \in \Lambda, s(\mu) = s(\nu)\}$ and since

$$\lim_{n\to\infty} \Big\|\pi_n \circ \Gamma^{2n}_n \circ P_n(t_\mu t_\nu^*) + \pi_n \circ \Gamma^{\lceil \frac{5n}{2}\rceil}_{\lceil \frac{3n}{2}\rceil} \circ Q_n(t_\mu t_\nu^*) - \pi_n \circ \iota_n(t_\mu t_\nu^*)\Big\| = 0$$

for all $\mu,\nu \in \Lambda$ with $s(\mu) = s(\nu)$, the family $(\mathcal{K}_{\Lambda[n,2n)} \oplus \mathcal{K}_{\Lambda[\lceil 3n/2\rceil,\lceil 5n/2\rceil)}, \psi_n, \phi_n)$ is an asymptotic order-1 approximation of $(\tilde{\iota}_n)_{n\in\mathbb{N}^k}$ through AF-algebras. □

Notation 5.5. Following [26], for each $m \in \mathbb{N}$, define $\kappa_m \in M_{\{0,\dots,m-1\}}\big([0,1)\big)$ as follows: put $l := \lceil \frac{m}{2}\rceil$, and define

$$\kappa_m = \frac{1}{l+1}\begin{pmatrix}
1 & 1 & \dots & 1 & 1 & \dots & 1 & 1\\
1 & 2 & \dots & 2 & 2 & \dots & 2 & 1\\
\vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots\\
1 & 2 & \dots & l & l & \dots & 2 & 1\\
1 & 2 & \dots & l & l & \dots & 2 & 1\\
\vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 2 & \dots & 2 & 2 & \dots & 2 & 1\\
1 & 1 & \dots & 1 & 1 & \dots & 1 & 1
\end{pmatrix} \qquad \text{if } m \text{ is even}$$

$$\kappa_m = \frac{1}{l+2}\begin{pmatrix}
1 & 1 & \dots & 1 & \dots & 1 & 1\\
1 & 2 & \dots & 2 & \dots & 2 & 1\\
\vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots\\
1 & 2 & \dots & l+1 & \dots & 2 & 1\\
\vdots & \vdots & \iddots & \vdots & \ddots & \vdots & \vdots\\
1 & 2 & \dots & 2 & \dots & 2 & 1\\
1 & 1 & \dots & 1 & \dots & 1 & 1
\end{pmatrix} \qquad \text{if } m \text{ is odd.}$$

Define $\kappa_m(i,j) = 0$ for $(i,j) \in \mathbb{Z}^2 \setminus (\{0,\dots,m-1\} \times \{0,\dots,m-1\})$.

Theorem 5.6. *Let Λ be a row-finite 2-graph with no sources. Then $(\tilde{\iota}_n)_{n\in\mathbb{N}^k}$ has an asymptotic order-1 approximation through AF-algebras.*

Proof. For $m \in \mathbb{N}$, let A_m denote the $m \times m$ matrix with all entries equal to 1. For $n \in \mathbb{N}^2$, define

$$\Delta_n := \frac{1}{2}\left(\kappa_{n_1} \otimes \mathsf{A}_{n_2} + \mathsf{A}_{n_1} \otimes \kappa_{n_2}\right) : \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \to \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}.$$

Since $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ is a finite-dimensional Hilbert space, Δ_n can be regarded as an $n_1n_2 \times n_1n_2$ matrix. Since $\kappa_{n_1} \otimes \mathsf{A}_{n_2}$ and $\mathsf{A}_{n_1} \otimes \kappa_{n_2}$ are positive elements in the C^*-algebra $M_{n_1} \otimes M_{n_2}$, the matrix Δ_n is also positive. Write $\{e_i\}$ for the canonical orthonormal basis elements of $\mathbb{C}^{n_1}$ and of $\mathbb{C}^{n_2}$. Then

$$\begin{aligned}\Delta_n(i_1, i_2, j_1, j_2) &= \langle \Delta_n e_{i_1} \otimes e_{i_2}, e_{j_1} \otimes e_{j_2}\rangle \\ &= \frac{1}{2}\big(\langle \kappa_{n_1} e_{i_1}, e_{j_1}\rangle\langle \mathsf{A}_{n_2} e_{i_2}, e_{j_2}\rangle + \langle \mathsf{A}_{n_1} e_{i_1}, e_{j_1}\rangle\langle \kappa_{n_2} e_{i_2}, e_{j_2}\rangle\big) \\ &= \frac{1}{2}\big(\kappa_{n_1}(i_1, j_1) + \kappa_{n_2}(i_2, j_2)\big).\end{aligned}$$

Define $M^{n,1} \in M_{\Lambda^{[n,2n)}}$ by $M^{n,1}_{\mu,\nu} = \Delta_n(d(\mu) - n, d(\nu) - n)$ and define $M^{n,2} \in M_{\Lambda^{[\lceil 3n/2\rceil, \lceil 5n/2\rceil)}}$ by $M^{n,2}_{\mu,\nu} = \Delta_n(d(\mu) - \lceil \frac{3n}{2}\rceil, d(\nu) - \lceil \frac{3n}{2}\rceil)$. We claim that Schur multiplication by $M^{n,i}$ is a completely positive contraction for $i = 1, 2$. We just argue the case $i = 1$ and when n_1 and n_2 are even; the other cases are similar. For $1 \le j \le n_1/2$, let $\Phi^{1,j}$ be the strong-operator sum $\sum_{|d(\lambda)_1 - (3n_1 - 1)/2| < j} \theta_{\lambda,\lambda}$, and for $1 \le j \le n_2/2$, let $\Phi^{2,j} = \sum_{|d(\lambda)_2 - (3n_2 - 1)/2| < j} \theta_{\lambda,\lambda}$, where $d(\lambda)_i$ denotes the ith coordinate of $d(\lambda)$. Each $\Phi^{i,j}$ is a projection, and so $\Phi^i : a \mapsto \sum_{j=1}^{n_i/2} \frac{1}{n_i/2+1}\Phi^{i,j} a \Phi^{i,j}$ is a completely positive contraction. Schur multiplication by $M^{n,1}$ is equal to $\frac{1}{2}(\Phi^1 + \Phi^2)$, and so is itself a completely positive contraction.

For $p < q \in \mathbb{N}^2$, define $R^q_p \in \mathcal{B}(\ell^2(\Lambda))$ to be the strong-operator sum

$$R^q_p = \textstyle\sum_{\lambda \in \Lambda^{[p,q)}} \theta_{\lambda,\lambda}.$$

Define $P_n : \mathcal{T}C^*(\Lambda) \to \mathcal{K}_{\Lambda^{[n,2n)}} \subseteq \mathcal{K}_\Lambda$ by $P_n(a) = M^{n,1} * (R^{2n}_n a R^{2n}_n)$ and define $Q_n : \mathcal{T}C^*(\Lambda) \to \mathcal{K}_{\Lambda^{[\lceil 3n/2\rceil, \lceil 5n/2\rceil)}} \subseteq \mathcal{K}_\Lambda$ by $Q_n(a) = M^{n,2} * (R^{\lceil 5n/2\rceil}_{\lceil 3n/2\rceil} a R^{\lceil 5n/2\rceil}_{\lceil 3n/2\rceil})$. Since Schur multiplication by each $M^{n,i}$ is a completely positive, contractive linear map and since R^{2n}_n and $R^{\lceil 5n/2\rceil}_{\lceil 3n/2\rceil}$ are projections, P_n and Q_n are completely positive, contractive linear maps.

We will show that Δ_n, P_n, and Q_n satisfy the hypotheses of Proposition 5.4. Fix $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$. Recall that the series $\sum_{\tau\in\Lambda} \theta_{\mu\tau,\nu\tau}$ converges strictly to $t_\mu t^*_\nu$. Since $\theta_{\lambda,\lambda}\theta_{\mu\tau,\nu\tau}\theta_{\beta,\beta} = \delta_{\lambda,\mu\tau}\delta_{\beta,\nu\tau}\theta_{\mu\tau,\nu\tau}$,

$$R^{2n}_n t_\mu t^*_\nu R^{2n}_n = \sum_{\substack{\tau\in s(\mu)\Lambda \\ n \le d(\mu\tau), d(\nu\tau) < 2n}} \theta_{\mu\tau,\nu\tau} \quad \text{and} \quad R^{\lceil 5n/2\rceil}_{\lceil 3n/2\rceil} t_\mu t^*_\nu R^{\lceil 5n/2\rceil}_{\lceil 3n/2\rceil} = \sum_{\substack{\tau\in s(\mu)\Lambda \\ \lceil \frac{3n}{2}\rceil \le d(\mu\tau), d(\nu\tau) < \lceil \frac{5n}{2}\rceil}} \theta_{\mu\tau,\nu\tau}.$$

Hence,

$$P_n(t_\mu t^*_\nu) = \sum_{\substack{\tau\in s(\mu)\Lambda \\ n \le d(\mu\tau), d(\nu\tau) < 2n}} \Delta_n(d(\mu\tau) - n, d(\nu\tau) - n)\theta_{\mu\tau,\nu\tau}$$

and

$$Q_n(t_\mu t^*_\nu) = \sum_{\substack{\tau\in s(\mu)\Lambda \\ \lceil \frac{3n}{2}\rceil \le d(\mu\tau), d(\nu\tau) < \lceil \frac{5n}{2}\rceil}} \Delta_n\left(d(\mu\tau) - \left\lceil \tfrac{3n}{2}\right\rceil, d(\nu\tau) - \left\lceil \tfrac{3n}{2}\right\rceil\right)\theta_{\mu\tau,\nu\tau}.$$

Let $p = (p_1, p_2) < n$, let $d(\mu) = (a_1, a_2)$, and let $d(\nu) = (b_1, b_2)$. Let $h_{n,\mu}(p) = (h^\mu_{p_1,n}, h^\mu_{p_2,n})$ be the unique element in H_n such that $n \le d(\mu) + p + h_{n,\mu}(p) < 2n$ and let $g_{n,\mu}(p) = (g^\mu_{p_1,n}, g^\mu_{p_2,n})$

be the unique element in H_n such that $\lceil\frac{3n}{2}\rceil \le d(\mu)+p+g_{n,\mu}(p) < \lceil\frac{5n}{2}\rceil$. Note that $h^{\mu}_{p_j,n}$ is the unique element in $n_j\mathbb{Z}$ such that $n_j \le a_j + p_j + h^{\mu}_{p_j,n} < 2n_j$ and $g^{\mu}_{p_j,n}$ is the unique element in $n_j\mathbb{Z}$ such that $\lceil\frac{3n_j}{2}\rceil \le a_j + p_j + g^{\mu}_{p_j,n} < \lceil\frac{5n_j}{2}\rceil$.

Set

$$\begin{aligned}\zeta_{n,\mu,p_j} := \kappa_{n_j}(a_j + p_j + h^{\mu}_{p_j,n} - n_j, b_j + p_j + h^{\mu}_{p_j,n} - n_j) \\ + \kappa_{n_j}\left(a_j + p_j + g^{\mu}_{p_j,n} - \lceil\tfrac{3n_j}{2}\rceil, b_j + p_j + g^{\mu}_{p_j,n} - \lceil\tfrac{3n_j}{2}\rceil\right).\end{aligned}$$

Using the definitions of the $h^{\mu}_{p_j,n}$ and $g^{\mu}_{p_j,n}$, one checks that

$$\big(a_j + p_j + h^{\mu}_{p_j,n} - n_j\big) - \big(a_j + p_j + g^{\mu}_{p_j,n} - \lceil\tfrac{3n_j}{2}\rceil\big) \in \big\{\lceil\tfrac{n_j}{2}\rceil, -\lfloor\tfrac{n_j}{2}\rfloor\big\}.$$

For any integer k and for any x, y, we have $\kappa_k(x,y) \ge \kappa_k(x,x) - |x-y|/(\lceil\frac{k}{2}\rceil + 1)$, and $(\lceil k/2\rceil + 1)/(\lceil k/2\rceil + 2) \le \kappa_k(x,x) + \kappa_k(x + \lceil\frac{k}{2}\rceil, x + \lceil\frac{k}{2}\rceil) \le 1$. Hence

$$\frac{\lceil\frac{k}{2}\rceil + 1}{\lceil\frac{k}{2}\rceil + 2} - \frac{2|x-y|}{\lceil\frac{k}{2}\rceil + 1} \le \kappa_k(x,y) + \kappa_k\left(x + \lceil\tfrac{k}{2}\rceil, y + \lceil\tfrac{k}{2}\rceil\right) \le 1.$$

Applying this inequality with $k = n_j$, $x = \min\big\{a_j + p_j + h^{\mu}_{p_j,n} - n_j, a_j + p_j + g^{\mu}_{p_j,n} - \lceil\frac{3n_j}{2}\rceil\big\}$ and $y := \min\big\{b_j + p_j + h^{\mu}_{p_j,n} - n_j, b_j + p_j + g^{\mu}_{p_j,n} - \lceil\frac{3n_j}{2}\rceil\big\}$, we see that $|\zeta_{n,\mu,p_j} - 1| \le \big(1 + 2|a_j - b_j|\big)/\big(\lceil\frac{n_j}{2}\rceil + 1\big)$.

For $p < n$, set $\Delta^{\mu,\nu}_{n,1}(p) = \Delta_n(d(\mu) + p + h_{n,\mu}(p) - n, d(\nu) + p + h_{n,\mu}(p) - n)$ and $\Delta^{\mu,\nu}_{n,2}(p) = \Delta_n(d(\mu) + p + g_{n,\mu}(p) - \lceil\frac{3n}{2}\rceil, d(\nu) + p + g_{n,\mu}(p) - \lceil\frac{3n}{2}\rceil)$. Then

$$\big|\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p) - 1\big| = \left|\frac{1}{2}(\zeta_{n,\mu,p_1} - 1) + \frac{1}{2}(\zeta_{n,\mu,p_2} - 1)\right| \le \frac{2(1 + |a_1 - b_1| + |a_2 - b_2|)}{2(\min\{\lceil\frac{n_1}{2}\rceil, \lceil\frac{n_2}{2}\rceil\} + 1)}.$$

Hence

$$\lim_{n\to\infty} \max\big\{|\Delta^{\mu,\nu}_{n,1}(p) + \Delta^{\mu,\nu}_{n,2}(p) - 1| \,\big|\, p < n\big\} = 0.$$

So Δ_n, P_n and Q_n satisfy the hypotheses of Proposition 5.4, which then says that $(\tilde\iota_n)_{n\in\mathbb{N}^k}$ has an asymptotic order-1 approximation through AF-algebras. □

Corollary 5.7. *If E and F are row-finite directed graphs with no sinks, then $(\tilde\iota_{m,E} \otimes \tilde\iota_{m,F})_{m\in\mathbb{N}}$ has an asymptotic order-1 approximation through AF-algebras.*

Proof. Let $\tilde\iota_{m,1} : C^*(E^*) \to C^*(E^*(m))$, $\tilde\iota_{m,2} : C^*(F^*) \to C^*(F^*(m))$, $\tilde\iota_{(m,m)} : C^*(E^* \times F^*) \to C^*((E^* \times F^*)((m,m)))$ be the homomorphisms defined in Lemma 3.3 for the 1-graphs E^*, F^*, and the 2-graph $E^* \times F^*$ respectively. By Lemma 4.2, $\tilde\iota_{m,1} \otimes \tilde\iota_{m,2} = \Theta_{E^*(m)\times F^*(m)} \circ \tilde\iota_{(m,m)} \circ \Theta^{-1}_{E^*\times F^*}$, where $\Theta_{E^*\times F^*}$ and $\Theta_{E^*(m)\times F^*(m)}$ are isomorphisms. By Theorem 5.6 and Remark 5.1, the sequence $(\tilde\iota_{(m,m)})_{m\in\mathbb{N}}$ has an asymptotic order-1 approximation through AF-algebras. Hence, $(\tilde\iota_{m,1} \otimes \tilde\iota_{m,2})_{m\in\mathbb{N}}$ has an asymptotic order-1 approximation through AF-algebras. By Lemma 4.6, there exist isomorphisms $\psi_E : C^*(E) \to C^*(E^*)$, $\psi_F : C^*(F) \to C^*(F^*)$, $\psi_{E(m)} : C^*(E(m)) \to C^*(E^*(m))$, and $\psi_{F(m)} : C^*(F(m)) \to C^*(F^*(m))$ such that $\tilde\iota_{m,E} = \psi^{-1}_{E(m)} \circ \tilde\iota_{m,1} \circ \psi_E$ and $\tilde\iota_{m,F} = \psi^{-1}_{F(m)} \circ \tilde\iota_{m,2} \circ \psi_F$. Hence,

$$\tilde\iota_{m,E} \otimes \tilde\iota_{m,F} = (\psi_{E(m)} \otimes \psi_{F(m)})^{-1} \circ (\tilde\iota_{m,1} \otimes \tilde\iota_{m,2}) \circ (\psi_E \otimes \psi_F).$$

Thus $(\tilde\iota_{m,E} \otimes \tilde\iota_{m,F})_{m\in\mathbb{N}}$ has an asymptotic order-1 approximation through AF-algebras. □

6. Nuclear dimension of UCT-Kirchberg algebras

In this section, we show that all UCT-Kirchberg algebras have nuclear dimension 1. We already know from [4] that every UCT-Kirchberg algebra with torsion free K_1-group has nuclear dimension 1. So we first show that each UCT-Kirchberg algebra with trivial K_0-group and finite K_1-group has nuclear dimension 1, and then prove our main theorem.

Definition 6.1. A *Kirchberg algebra* is a separable, nuclear, simple, purely infinite C^*-algebra. A *UCT-Kirchberg algebra* is a Kirchberg algebra in the UCT class of [18].

For each finite abelian group T, let E_T be an infinite, row-finite, strongly connected graph such that $K_*(C^*(E_T)) = (T, \{0\})$ and $C^*(E_T)$ is a UCT-Kirchberg algebra (note that strongly connected implies that E_T has no sinks and sources). Let $F_{\mathbb{Z}}$ be an infinite, row-finite, strongly connected graph such that $K_*(C^*(F_{\mathbb{Z}})) = (\{0\}, \mathbb{Z})$ and $C^*(F_{\mathbb{Z}})$ is a UCT-Kirchberg algebra. Note that E_T and $F_{\mathbb{Z}}$ exist by [23, Theorem 1.2].

Lemma 6.2. *Let T be a finite abelian group. Then the nuclear dimension of $C^*(E_T) \otimes C^*(F_{\mathbb{Z}})$ is 1. Consequently, every UCT-Kirchberg algebra with K_0 trivial and K_1 finite has nuclear dimension 1.*

Proof. Consider the directed graphs E_T and $F_{\mathbb{Z}}$. For $k \in \mathbb{N}$, let

$$\tilde{\iota}_{k,E_T} : C^*(E_T) \to C^*(E_T(k)) \quad \text{and} \quad \tilde{\iota}_{k,F_{\mathbb{Z}}} : C^*(F_{\mathbb{Z}}) \to C^*(F_{\mathbb{Z}}(k))$$

be the homomorphisms given in Lemma 4.3 for E_T and $F_{\mathbb{Z}}$ respectively. Let

$$j_{k,E_T} : C^*(E_T(k)) \to C^*(E_T) \otimes \mathcal{K} \quad \text{and} \quad j_{k,F_{\mathbb{Z}}} : C^*(F_{\mathbb{Z}}(k)) \to C^*(F_{\mathbb{Z}}) \otimes \mathcal{K}$$

be the homomorphisms given in [19, Proposition 3.1] for E_T and $F_{\mathbb{Z}}$ respectively.

By Corollary 5.7, there is an asymptotic order-1 approximation through AF-algebras for $(\tilde{\iota}_{k,E_T} \otimes \tilde{\iota}_{k,F_{\mathbb{Z}}})_{k\in\mathbb{N}}$. The composition of this sequence of homomorphisms with $j_{k,E_T} \otimes j_{k,F_{\mathbb{Z}}}$ gives an asymptotic order-1 approximation through AF-algebras for $((j_{k,E_T} \circ \tilde{\iota}_{k,E_T}) \otimes (j_{k,F_{\mathbb{Z}}} \circ \tilde{\iota}_{k,F_{\mathbb{Z}}}))_{k\in\mathbb{N}}$.

For $m \in \mathbb{N}$, let $p_m = (|T|+1)^m$. Since the order of each element of T divides $|T|$, multiplication by each p_m induces the identity map on T. Set $\gamma_m = (j_{p_m,E_T} \circ \tilde{\iota}_{p_m,E_T}) \otimes (j_{p_m,F_{\mathbb{Z}}} \circ \tilde{\iota}_{p_m,F_{\mathbb{Z}}})$. By construction, $(\gamma_m)_{m\in\mathbb{N}}$ has an asymptotic order-1 approximation through AF-algebras. The Künneth formula in [18] combined with [19, Lemma 3.2] shows that $K_1(\gamma_m)$ is multiplication by p_m^2 on $K_1(C^*(E_T) \otimes C^*(F_{\mathbb{Z}})) = T$. Thus, $K_1(\gamma_m) = \mathrm{id}_T$. Since $K_0(C^*(E_T) \otimes C^*(F_{\mathbb{Z}})) = 0$ the map $K_0(\gamma_m)$ is trivially the identity. Since E_T and $F_{\mathbb{Z}}$ are infinite directed graphs, $C^*(E_T)$ and $C^*(F_{\mathbb{Z}})$ are non-unital UCT-Kirchberg algebras, and hence stable. The Universal Coefficient Theorem in [18] and the Kirchberg-Phillips classification (cf. [7] and [15]), show that there exist an isomorphism $\beta_m : (C^*(E_T) \otimes \mathcal{K}) \otimes (C^*(F_{\mathbb{Z}}) \otimes \mathcal{K}) \to C^*(E_T) \otimes C^*(F_{\mathbb{Z}})$ and a unitary u_m in $\mathcal{M}(C^*(E_T) \otimes C^*(F_{\mathbb{Z}}))$ for each $m \in \mathbb{N}$ such that

$$\lim_{m\to\infty} \|u_m(\beta_m \circ \gamma_m)(a)u_m^* - a\| = 0 \quad \text{for all } a \in C^*(E_T) \otimes C^*(F_{\mathbb{Z}}). \tag{6.1}$$

Since $(\gamma_m)_{m\in\mathbb{N}}$ has an asymptotic order-1 approximation through AF-algebras, so does $(\mathrm{Ad}(u_m) \circ \beta_m \circ \gamma_m)_{m\in\mathbb{N}}$. So (6.1) implies that $\mathrm{id}_{C^*(E_T)\otimes C^*(F_{\mathbb{Z}})}$ has an asymptotic order-1 approximation through AF-algebras. Hence [19, Lemma 2.9] shows that $\dim_{\mathrm{nuc}}\big(C^*(E_T) \otimes C^*(F_{\mathbb{Z}})\big) = 1$.

Let A be a UCT-Kirchberg algebra with K_0 trivial and K_1 finite. Then $K_*(A) \cong K_*(C^*(E_T) \otimes C^*(F_{\mathbb{Z}}))$ where $T = K_1(A)$. By Kirchberg-Phillips classification, $A \otimes \mathcal{K} \cong C^*(E_T) \otimes C^*(F_{\mathbb{Z}})$. Hence, $\dim_{\mathrm{nuc}}(A \otimes \mathcal{K}) = 1$; so [26, Corollary 2.8] gives $\dim_{\mathrm{nuc}}(A) = 1$. □

Remark 6.3. A key step in the preceding proof was to show that the maps $j_n \circ \iota_n : C^*(\Lambda) \to C^*(\Lambda) \otimes \mathcal{K}_{\Lambda^{<n}}$ induce multiplication by $n_1 n_2$ in K-theory. We were able to do this using the Künneth formula because tensor products of 1-graph C^*-algebras provide a large enough class of models to cover all the UCT-Kirchberg algebras in question. It seems likely that our techniques could be used to compute the exact value of nuclear dimension for a large class of nonsimple Kirchberg algebras with torsion in K_1 along the lines of [19], but to do this, we need to know that $j_n \circ \iota_n$ induces multiplication by $n_1 n_2$ in the K-theory of every ideal of $C^*(\Lambda)$ for general 2-graphs Λ. This can be proved using Evans' calculation of K-theory for k-graph algebras [5] and naturality of Kasparov's spectral sequence (see [12, page 185]). But this would require introducing extraneous notation or digging into the proofs in [5], so we have not pursued this approach here.

Definition 6.4. A homomorphism $\phi : A \to B$ is called *full* if for all $a \in A$ with $a \neq 0$, the closed two-sided ideal generated by $\phi(a)$ is equal to B.

Lemma 6.5. *Let $(\phi_n : A_n \to A_{n+1})_{n=1}^{\infty}$ be a directed system of C^*-algebras, and set $A = \varinjlim(A_n, \phi_n)$. If there exists $N \in \mathbb{N}$ such that ϕ_n is full for all $n \geq N$, then A is simple. If, in addition, each A_n is a finite direct sum of UCT-Kirchberg algebras, then A is a UCT-Kirchberg algebra.*

Proof. Let I be a nonzero ideal of A. Then $I_n = \phi_{n,\infty}^{-1}(I)$ is an ideal of A_n and $\phi_n(I_n) \subseteq I_{n+1}$. Since I is nonzero, there exists M such that $I_n \neq 0$ for all $n \geq M$. So for $n \geq \max\{N, M\}$ the ideal generated by $\phi_n(I_n)$ is A_{n+1}. Since $\phi_n(I_n) \subseteq I_{n+1}$, we have $I_{n+1} = A_{n+1}$ for $n \geq \max\{N, M\}$, and so $I = A$.

Suppose each A_n is a finite direct sum of UCT-Kirchberg algebras. Then every nonzero projection of any A_n is properly infinite. Since ϕ_n is injective for $n \geq N$ (because the maps are full), ϕ_n takes properly infinite projections to properly infinite projections for all $n \geq N$. Thus, every nonzero projection of A is properly infinite. Hence, A is a purely infinite simple C^*-algebra. Since each A_n is separable, nuclear, and in the UCT class, A is too. Thus, A is a UCT-Kirchberg algebra. □

Lemma 6.6. *Let A be a stable UCT-Kirchberg algebra. Then there exist sequences $(\Lambda_n)_{n\in\mathbb{N}}$ and $(\Gamma_n)_{n\in\mathbb{N}}$ of row-finite 2-graphs with no sources, and homomorphisms $\phi_n : C^*(\Lambda_n) \oplus C^*(\Gamma_n) \to C^*(\Lambda_{n+1}) \oplus C^*(\Gamma_{n+1})$ such that: each $C^*(\Lambda_n)$ and each $C^*(\Gamma_n)$ is a stable UCT-Kirchberg algebra with finitely-generated K-theory; each $K_1(C^*(\Lambda_n))$ is free abelian; each $K_0(C^*(\Gamma_n))$ is trivial and each $K_1(C^*(\Gamma_n))$ is finite; and $A \cong \varinjlim(C^*(\Lambda_n) \oplus C^*(\Gamma_n), \phi_n)$.*

Proof. Let $(G_{n,0})_{n\in\mathbb{N}}$ and $(G_{n,1})_{n\in\mathbb{N}}$ be increasing families of finitely generated abelian groups with $\bigcup_{n=1}^{\infty} G_{n,0} = K_0(A)$ and $\bigcup_{n=1}^{\infty} G_{n,1} = K_1(A)$. Decompose each $G_{n,1}$ as $G_{n,1} = T(G_{n,1}) \oplus F(G_{n,1})$, where $T(G_{n,1})$ is finite and $F(G_{n,1})$ is free.

Let $E_{\mathcal{K}}$ be the infinite row-finite graph $\cdots \to \bullet \to \bullet \to \cdots$ (so $C^*(E_{\mathcal{K}}) \cong \mathcal{K}$). For each n apply [23, Theorem 1.2] to obtain a row-finite strongly connected graph E_n such that $C^*(E_n)$ is a UCT Kirchberg algebra with $K_*(C^*(E_n)) = (G_{n,0}, F(G_{n,1}))$. Then each $\Lambda_n := E_n^* \times E_{\mathcal{K}}^*$ is a row-finite 2-graph with no sources such that $C^*(\Lambda_n)$ is a stable UCT-Kirchberg algebra with $K_*(C^*(\Lambda_n)) = (G_{n,0}, F(G_{n,1}))$. For each n, let $E_{T(G_{n,1})}$ and $F_{\mathbb{Z}}$ be as in Lemma 6.2 and the preceding discussion. Let $\Gamma_n := E_{T(G_{n,1})}^* \times F_{\mathbb{Z}}^*$. Then Γ_n is a row-finite 2-graph with no sources, and $C^*(\Gamma_n)$ is a stable UCT-Kirchberg algebra with $K_*(C^*(\Gamma_n)) = (0, T(G_{n,1}))$.

Each $K_*(C^*(\Lambda_n) \oplus C^*(\Gamma_n)) = (G_{n,0}, G_{n,1})$. By Kirchberg-Phillips (cf. [7] and [15]), for each $n \in \mathbb{N}$, there exists a full homomorphism $\phi_n : C^*(\Lambda_n) \oplus C^*(\Gamma_n) \to C^*(\Lambda_{n+1}) \oplus C^*(\Gamma_{n+1})$ which in K-theory induces the inclusion map $G_{n,i} \hookrightarrow G_{n+1,i}$. Therefore, $K_i(\varinjlim(C^*(\Lambda_n) \oplus C^*(\Gamma_n), \phi_n)) \cong K_i(A)$. So, by Lemma 6.5 and the Kirchberg-Phillips classification, $A \cong \varinjlim(C^*(\Lambda_n) \oplus C^*(\Gamma_n), \phi_n)$. □

Theorem 6.7. *Every UCT-Kirchberg algebra has nuclear dimension 1.*

Proof. Let A be a UCT-Kirchberg algebra. Since Kirchberg algebras are not AF, [26, Remarks 2.2(iii)] shows that A has nuclear dimension at least 1. Corollary 2.8 of [26] shows that the nuclear dimension of $A \otimes \mathcal{K}$ is the same as that of A, so we may assume that A is stable.

By Lemma 6.6, $A \cong \varinjlim(B_n \oplus C_n, \phi_n)$ where B_n and C_n are UCT-Kirchberg algebras such that $K_1(B_n)$ is free, $K_0(C_n) = 0$, and $K_1(C_n)$ is a finite abelian group. By Lemma 6.2, $\dim_{\mathrm{nuc}}(C_n) = 1$. By [4, Theorem 4.1], $\dim_{\mathrm{nuc}}(B_n) = 1$. Proposition 2.3(i) of [26] implies that each $B_n \oplus C_n$ has nuclear dimension 1. It now follows from [26, Proposition 2.3(iii)] that A has nuclear dimension 1. □

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Department of Mathematics, University of Hawaii, Hilo, 200 W. Kawili St., Hilo, Hawaii, 96720-4091 USA
E-mail address: ruize@hawaii.edu

School of Mathematics and Applied Statistics, Faculty of Engineering and Information Sciences, University of Wollongong NSW 2522, Australia
E-mail address: asims@uow.edu.au

School of Mathematics and Applied Statistics, Faculty of Engineering and Information Sciences, University of Wollongong NSW 2522

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
E-mail address: apws@math.uio.no