GROUPOID FELL BUNDLES FOR PRODUCT SYSTEMS OVER QUASI-LATTICE ORDERED GROUPS

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Abstract. Consider a product system over the positive cone of a quasi-lattice ordered group. We construct a Fell bundle over an associated groupoid so that the cross-sectional algebra of the bundle is isomorphic to the Nica–Toeplitz algebra of the product system. Under the additional hypothesis that the left actions in the product system are implemented by injective homomorphisms, we show that the cross-sectional algebra of the restriction of the bundle to a natural boundary subgroupoid coincides with the Cuntz–Nica–Pimsner algebra of the product system. We apply these results to improve on existing sufficient conditions for nuclearity of the Nica–Toeplitz algebra and the Cuntz–Nica–Pimsner algebra, and for the Cuntz–Nica–Pimsner algebra to coincide with its co-universal quotient.

1. Introduction

In [20], Pimsner associated to each C^*-correspondence over a C^*-algebra A two C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$. His construction simultaneously generalised the Cuntz–Krieger algebras and their Toeplitz extensions, graph C^*-algebras and crossed products by $\mathbb{Z}$, and has been intensively studied ever since.

It is standard these days to present $\mathcal{T}_X$ as the universal C^*-algebra generated by a representation of the module X, and then $\mathcal{O}_X$ as the quotient of $\mathcal{T}_X$ determined by a natural covariance condition. However, this was not Pimsner's original definition. In [20], $\mathcal{O}_X$ is by definition the quotient of the image of the canonical representation of X as creation operators on its Fock space by the ideal of compact operators on the Fock space. Pimsner then provided two alternative presentations of $\mathcal{O}_X$, the second of which is the one in terms of its universal property. The first, which is the one germane to this paper, is an analogue of the realisation of $C(\mathbb{T})$ by dilation of the canonical representation of the classical Toeplitz algebra on ℓ^2. Pimsner constructed a direct-limit module X_∞ over the direct limit A_∞ of the algebras of compact operators on the tensor powers of X. He showed that one can make sense of $X_\infty^{\otimes n}$ for all integers n, and so form a 2-sided Fock space $\bigoplus_{n\in\mathbb{Z}} X_\infty^{\otimes n}$. This space carries a natural representation of X_∞ by translation operators, and the image generates $\mathcal{O}_{X_\infty}$ which is isomorphic to $\mathcal{O}_X$.

More recently [12], Fowler introduced compactly aligned product systems of Hilbert A–A bimodules over the positive cones in quasi-lattice ordered groups (G,P), and studied associated C^*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$, and an interpolating quotient $\mathcal{NT}_X$ (Fowler denoted it by $\mathcal{T}_{\rm cov}(X)$, but we follow the notation of [3]). When $(G,P)=(\mathbb{Z},\mathbb{N})$, $\mathcal{T}_X=\mathcal{NT}_X$ agrees with Pimsner's Toeplitz algebra, and $\mathcal{O}_X$ with Pimsner's Cuntz-Pimsner algebra. But even for $(\mathbb{Z}^2,\mathbb{N}^2)$ the situation is more complicated. The algebra $\mathcal{NT}_X$ is essentially universal for the relations encoded by the natural Fock representation of X, so it is a natural analogue of Pimsner's Toeplitz algebra. But the quotient by the ideal of compact operators on the Fock space is much too large to behave like an analogue of Pimsner's $\mathcal{O}_X$. (This is analogous to the fact that $C^*(\mathbb{Z})$ is the quotient of $C^*(\mathbb{N})$ by the compact operators on $\ell^2(\mathbb{N})$, but $C^*(\mathbb{Z}^2)$ is much smaller than the quotient of $C^*(\mathbb{N}^2)$ by the compact operators on $\ell^2(\mathbb{N}^2)$.) Fowler also lacked an analogue of X_∞; the direct limit should be taken over P, but P is typically not directed. So Fowler's approach to defining $\mathcal{O}_X$ was to mimic Pimsner's second alternative presentation of $\mathcal{O}_X$: identify a natural covariance relation and *define* $\mathcal{O}_X$ as the universal quotient of $\mathcal{T}_X$ determined by this relation. Subsequent papers [26, 4] have modified Fowler's definition to accommodate various levels of

2010 *Mathematics Subject Classification.* 46L05.

Key words and phrases. Groupoid; C^*-correspondence; Product system; Fell bundle.

additional generality, but have taken the same fundamental approach of defining $\mathcal{NO}_X$ as the universal C^*-algebra determined by a representation of $\mathcal{T}_X$ satisfying some additional essentially ad hoc relations. Nevertheless, there is strong evidence [12, 4] that the resulting C^*-algebra $\mathcal{NO}_X$ can profitably be regarded as a generalised crossed product of the coefficient algebra A by the group G. In particular, in the case that $(G, P) = (\mathbb{Z}^k, \mathbb{N}^k)$ and X is the product system arising from an action α of $\mathbb{N}^k$ on A by endomorphisms, a new characterisation and analysis of $\mathcal{NO}_X$, closely related to Pimsner's dilation approach, is achieved in [6] using the powerful machinery of Arveson envelopes of non-self-adjoint operator algebras. The authors answer in the affirmative a question raised in [4] about whether $\mathcal{NO}_X$ can be recovered using Arveson's approach, and use this to show, amongst other things, that $\mathcal{NO}_X$ is Morita equivalent (in fact isomorphic in the case that the α_p are all injective) to a genuine crossed-product by $\mathbb{Z}^k$.

In this paper we provide an analogue of Pimsner's first representation of $\mathcal{O}_X$ that is applicable to compactly aligned product systems over quasi-lattice ordered groups, under the additional hypothesis that the left A-actions are implemented by nondegenerate injective homomorphisms $\phi_p : A \to \mathcal{L}(X_p)$. Our approach is to use a natural groupoid $\mathcal{G}$ associated to (G, P) [17], and construct a Fell bundle over $\mathcal{G}$ whose cross-sectional C^*-algebra coincides with $\mathcal{NT}_X$. The groupoid $\mathcal{G}$ has a natural boundary, which is a closed subgroupoid (see [5]), and the restriction of our Fell bundle to this boundary subgroupoid has cross-sectional algebra isomorphic to the algebra $\mathcal{NO}_X$ of [26]. This is strong evidence that the relations recorded in [26] are the right ones, at least for nondegenerate product systems with injective left actions. As practical upshots of our results, we deduce that if the groupoid $\mathcal{G}$ is amenable, then: (1) each of $\mathcal{NT}_X$ and $\mathcal{NO}_X$ is nuclear whenever the coefficient algebra A is nuclear, and (2) $\mathcal{NO}_X$ coincides with its co-universal quotient $\mathcal{NO}_X^r$ as in [4]. This improves on previous results along these lines, which assume that the group G is amenable, a stronger hypothesis than amenability of $\mathcal{G}$.

We mention that the work of Kwasniewski and Szymański in [15], is related to our construction. There the authors consider product systems over semigroups P that satisfy the Ore condition but are not necessarily part of a quasi-lattice ordered pair, and assume that the left actions in the product system are by compact operators. Here, by contrast, we insist that P is quasi-lattice ordered, but do not require compact actions. Both approaches use the machinery of Fell bundles, but Kwasniewski and Szymański construct Fell bundles over the enveloping group G of P, whereas we construct a bundle over the associated groupoid $\mathcal{G}$; as mentioned above, an advantage of the latter is that $\mathcal{G}$ can be amenable even when G is not.

2. Preliminaries

2.1. **Product systems over quasi-lattice ordered groups.**

Let G be a discrete group and let P be a subsemigroup of G satisfying $P \cap P^{-1} = \{e\}$. Define a partial order $\leq$ on G by

$$g \leq h \iff g^{-1}h \in P.$$

We call the pair (G, P) a *quasi-lattice ordered group* if, whenever two elements $g, h \in G$ have a common upper bound in G, they have a least common upper bound $g \vee h$ in G. We write $g \vee h < \infty$ if two elements $g, h \in G$ have a common upper bound and $g \vee h = \infty$ otherwise.

A *product system* over a quasi-lattice ordered group (G, P) is a semigroup X equipped with a semigroup homomorphism $d : X \to P$ such that the following hold. For each $p \in P$, let $X_p = d^{-1}(p)$. Then we require that $A = X_e$ is a C^*-algebra, thought of as a right-Hilbert module over itself in the usual way, and that each X_p is a right-Hilbert A-module together with a left action of A by adjointable operators denoted $\varphi_p : A \to \mathcal{L}(X_p)$. We require that φ_e is given by left multiplication. Furthermore, for each $p, q \in P$ with $p \neq e$, we require that multiplication in X determines a Hilbert bimodule isomorphism $X_p \otimes_A X_q \to X_{pq}$ satisfying $x_p \otimes x_q \mapsto x_p x_q$. The product system is *nondegenerate* if multiplication $X_e \times X_p \to X_p$ also determines an isomorphism $X_e \otimes_a X_p \to X_p$ for each p; that is, if each X_p is nondegenerate as a left A-module. Every right Hilbert module is automatically nondegenerate as a right A-module by the Hewitt-Cohen factorisation theorem.

If $p, q \in P$ satisfy $e \neq p \leq q$, then there is a homomorphism $i_{p^{-1}q} : \mathcal{L}(X_p) \to \mathcal{L}(X_q)$ characterised by
$$i_{p^{-1}q}(S)(xy) = (Sx)y \text{ for all } x \in X_p,\ y \in X_{p^{-1}q}.$$
If we identify A with $\mathcal{K}(X_e)$ in the usual way then the corresponding map $i_p : \mathcal{K}(X_e) \to \mathcal{L}(X_p)$ is $i_p = \varphi_p$. We say that a product system X is *compactly aligned* if, whenever $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$ and $p \vee q < \infty$ we have
$$i_{p^{-1}(p\vee q)}(S) i_{q^{-1}(p\vee q)}(T) \in \mathcal{K}(X_{p\vee q}).$$
If $g \in G \setminus P$ we define i_g to be 0.

Example 2.1. The pair $(\mathbb{Z}, \mathbb{N})$ is a quasi-lattice ordered group, where $\leq$ agrees with the usual ordering on $\mathbb{Z}$. Let A be a C^*-algebra and let E be an A-correspondence; i.e. E is a right Hilbert A-module with a left action $A \to \mathcal{L}(E)$. Let $X_0 := A$ and for each $n \in \mathbb{N} \setminus \{0\}$ let $X_n := E^{\otimes n}$. Then
$$X := \textstyle\bigcup_{n\in\mathbb{N}} X_n$$
is a product system over $(\mathbb{Z}, \mathbb{N})$. With multiplication given by $\xi\eta := \xi \otimes \eta$.

Example 2.2. For each $k \geq 1$, the pair $(\mathbb{Z}^k, \mathbb{N}^k)$ is a quasi lattice ordered group where, for $m, n \in \mathbb{Z}^k$ and $1 \leq i \leq k$
$$(m \vee n)_i = \max\{m_i, n_i\}.$$
Suppose that (Λ, d) is a k-graph. For each $n \in \mathbb{N}^k$, $C_c(d^{-1}(n))$ is a pre-Hilbert $A = C_0(\Lambda^0)$ module. Let $X_n = \overline{C_c(d^{-1}(n))}$. Then
$$X = \textstyle\bigcup_{n\in\mathbb{N}^k} X_n$$
is a product system over $(\mathbb{Z}^k, \mathbb{N}^k)$. (See [23].)

2.2. **Representations of product systems.** For details of the following, see [4, 12, 26].

Definition 2.3. Let X be a compactly aligned product system over a quasi-lattice ordered group (G, P). A *Toeplitz representation* of X in a C^*-algebra B is a map $\psi : X \to B$ satisfying

(T1) $\psi_p := \psi|_{X_p} : X_p \to B$ is linear for all $p \in P$ and ψ_e is a homomorphism,
(T2) $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in X$, and
(T3) for any $p \in P$ and $x, y \in X_p$, $\psi(\langle x, y\rangle) = \psi(x)^*\psi(y)$.

Given a Toeplitz respresentation $\psi : X \to B$, for each $p \in P$ there is a homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to B$ satisfying
$$\psi^{(p)}(\theta_{x,y}) = \psi_p(x)\psi_p(y)^*.$$
We call a Toeplitz representation $\psi : X \to B$ *Nica covariant* if

(N) for all $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$ we have
$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{p\vee q}\left(i_{p^{-1}(p\vee q)}(S) i_{q^{-1}(p\vee q)}(T)\right) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Following [3], we will write $\mathcal{NT}_X$ for the universal C^*-algebra generated by a Nica-covariant Toeplitz representation i_X of X. (Fowler shows that such a C^*-algebra exists in [12], but denotes it $\mathcal{T}_{\mathrm{cov}}(X)$.)

Given a predicate $\mathcal{P}$ on P, we say $\mathcal{P}$ is true *for large* s if for every $q \in P$, there exists an $r \geq q$ such that $\mathcal{P}(s)$ is true whenever $s \geq r$.

We now present the definition of the Cuntz–Nica–Pimsner algebra $\mathcal{NO}_X$ of a product system X under the assumption that the left action on each fibre is implemented by an injective homomorphism φ_p. This hypothesis is not needed for $\mathcal{NO}_X$ to make sense (see [26]); but if the left actions are not implemented by injective homomorphisms, then the relation (CNP) as described below does not hold in $\mathcal{NO}_X$. In particular, this hypothesis will be necessary in all statements that involve Cuntz-Nica-Pimsner covariance and representations of $\mathcal{NO}_X$: Proposition 4.2, Theorem 5.2, and the results in Section 6

Definition 2.4. Let X be a compactly aligned product system over a quasi-lattice ordered group (G, P) and suppose that for each $p \in P$ the left action $\phi_p : A \to \mathcal{L}(X_p)$ is injective. We say a Nica covariant Toeplitz representation $\psi : X \to B$ is *Cuntz–Nica–Pimsner covariant* if it satisfies the following property:

(CNP) for each finite $F \subset P$ and collection of elements $T_p \in \mathcal{K}(X_p)$, $p \in F$,

$$\text{if } \textstyle\sum_{p\in F} i_{p^{-1}q}(T_p) = 0 \text{ for large } q, \qquad \text{then} \qquad \textstyle\sum_{p\in F} \psi^{(p)}(T_p) = 0.$$

We write $\mathcal{NO}_X$ for the universal C^*-algebra generated by a Cuntz–Nica–Pimsner covariant representation j_X of X.

2.3. **Fell bundles over groupoids.** We say that a groupoid $\mathcal{G}$ is a *topological groupoid* if $\mathcal{G}$ is a topological space and the multiplication and inversion are continuous functions. We call a topological groupoid $\mathcal{G}$ *étale* if the unit space $\mathcal{G}^{(0)}$ is locally compact and Hausdorff, and the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism. It follows that the source map s is also a local homeomorphism. A *bisection* of G is an open subset $U \subseteq G$ such that $r|_U$ and $s|_U$ are homeomorphisms; the topology of a Hausdorff étale groupoid admits a basis consisting of bisections. See [9] for an overview of étale groupoids.

Given a Hausdorff étale groupoid $\mathcal{G}$, a *Fell bundle* over $\mathcal{G}$ is an upper-semicontinuous Banach bundle $p : \mathscr{E} \to \mathcal{G}$ with a multiplication

$$\mathscr{E}^{(2)} = \{(e, f) \in \mathscr{E} \times \mathscr{E} : (p(e), p(f)) \in \mathcal{G}^{(2)}\} \to \mathscr{E}$$

and an involution

$$* : \mathscr{E} \to \mathscr{E}, \ e \mapsto e^*$$

satisfying the following properties:

(1) the multiplication is associative and bilinear, whenever it makes sense;
(2) $p(ef) = p(e)p(f)$ for all $(e, f) \in \mathscr{E}^{(2)}$;
(3) multiplication is continuous in the relative topology on $\mathscr{E}^{(2)} \subseteq \mathscr{E} \times \mathscr{E}$;
(4) $\|ef\| \leq \|e\|\|f\|$ for all $(e, f) \in \mathscr{E}^{(2)}$;
(5) $p(e^*) = p(e)^{-1}$ for all $e \in \mathscr{E}$, and involution is continuous and conjugate linear;
(6) $(e^*)^* = e$, $\|e^*\| = \|e\|$ and $(ef)^* = f^*e^*$ for all $(e, f) \in \mathscr{E}^{(2)}$;
(7) $\|e^*e\| = \|e\|^2$ for all $e \in \mathscr{E}$;
(8) $e^*e \geq 0$ as an element of $p^{-1}(s(p(e)))$—which is a C^*-algebra by (1)–(7)—for all $e \in \mathscr{E}$.

We denote by E_γ the fibre $p^{-1}(\gamma) \subset \mathscr{E}$.

Given a Fell bundle $\mathscr{E}$ over a locally compact Hausdorff étale groupoid, we write $\Gamma_c(\mathcal{G}; \mathscr{E})$ for the vector space of continuous, compactly supported sections $\xi : \mathcal{G} \to \mathscr{E}$. If $\mathcal{H} \subseteq \mathcal{G}$ is a closed subset, we will write $\Gamma_c(\mathcal{H}; \mathscr{E})$ for the compactly supported sections of the restriction of $\mathscr{E}$ to $\mathcal{H}$; that is, $\Gamma_c(\mathcal{H}; \mathscr{E}) := \Gamma_c(\mathcal{H}; \mathscr{E}|_{\mathcal{H}})$.

There are a convolution and involution on $\Gamma_c(\mathcal{G}; \mathscr{E})$ such that for $\xi, \eta \in \Gamma_c(\mathcal{G}; \mathscr{E})$,

$$(\xi * \eta)(\gamma) = \sum_{\alpha\beta=\gamma} \xi(\alpha)\eta(\beta) \quad \text{and} \quad \xi^*(\gamma) = \xi(\gamma^{-1})^*.$$

This gives $\Gamma_c(\mathcal{G}; \mathscr{E})$ the structure of a $*$-algebra. The *I-norm* on $\Gamma_c(\mathcal{G}; \mathscr{E})$ is given by

$$\|f\|_I := \sup_{u\in\mathcal{G}^{(0)}} \Big(\max\Big(\sum_{s(\gamma)=u} \|f(\gamma)\|, \sum_{r(\gamma)=u} \|f(\gamma)\| \Big)\Big).$$

A $*$-homomorphism $L : \Gamma_c(\mathcal{G}; \mathscr{E}) \to \mathcal{B}(\mathcal{H}_L)$ is called a *bounded representation* if $\|L(f)\| \leq \|f\|_I$ for all $f \in \Gamma_c(\mathcal{G}; \mathscr{E})$. It is *nondegenerate* if $\overline{\operatorname{span}}\{L(f)\xi : f \in \Gamma_c(\mathcal{G}; \mathscr{E}), \xi \in \mathcal{H}_L\} = \mathcal{H}_L$ is dense. The *universal C^*-norm* on $\Gamma_c(\mathcal{G}; \mathscr{E})$ is

$$\|f\| := \sup\{\|L(f)\| : L \text{ is an bounded representation}\}.$$

We define the *cross-sectional algebra* $C^*(\mathcal{G}, \mathscr{E})$ to be the completion of $\Gamma_c(\mathcal{G}; \mathscr{E})$ with respect to the universal C^*-norm. If $\mathcal{H} \subseteq \mathcal{G}$ is a closed subgroupoid, then we write $C^*(\mathcal{H}, \mathscr{E})$ for the completion of $\Gamma_c(\mathcal{H}, \mathscr{E})$ in the universal norm on $\Gamma_c(\mathcal{H}, \mathscr{E})$.

3. From a product system to a Fell bundle

In this section, given a product system X over a quasi-lattice ordered group (G, P), we construct a groupoid $\mathcal{G}$ and a Fell bundle $\mathscr{E}$ over $\mathcal{G}$. We will show in Section 5 that the C^*-algebra of this Fell bundle coincides with the Nica–Toeplitz algebra of X, and has a natural quotient that coincides with the Cuntz–Nica–Pimsner algebra.

Standing notation: We fix, for the duration of Section 3, a quasi-lattice ordered group (G, P), and a nondegenerate compactly aligned product system X over P. For the time being, we do not require that the left actions on the fibres of X are implemented by injective homomorphisms; as mentioned before, this additional hypothesis will be needed only in Proposition 4.2, Theorem 5.2, and the results of Section 6.

3.1. **The groupoid.** We first construct a groupoid from (G, P). This construction is by no means new—for example, it appears in the work of Muhly and Renault [17] in the context of Weiner-Hopf algebras. Fix a quasi-lattice ordered group (G, P). We say that $\omega \subset G$ is *directed* if

$$g, h \in \omega \implies \infty \neq g \vee h \in \omega$$

and *hereditary* if

$$h \in \omega \text{ and } g \leq h \implies g \in \omega.$$

Let $\Omega = \{\omega \subset G : \omega \text{ is directed and hereditary}\}$. With the relative product topology induced by identifying Ω with a subset of $\{0, 1\}^G$ in the usual way, Ω is a totally disconnected compact Hausdorff space: the sets

$$\mathcal{Z}(A_0, A_1) := \{\omega \in \Omega : g \in A_i \implies \chi_\omega(g) = i\},$$

indexed by pairs A_0, A_1 of finite subsets of G constitute a basis of compact open sets.

We say that $\omega \in \Omega$ is *maximal* if $\omega \subset \rho \in \Omega$ implies $\omega = \rho$. Let $\Omega_{\max} = \{\omega \in \Omega : \omega \text{ is maximal}\}$. Define the *boundary* of Ω to be

$$\partial\Omega := \overline{\Omega_{\max}} \subset \Omega.$$

Given $g \in G$ and $\omega \in \Omega$, let

$$g\omega := \{gh : h \in \omega\}.$$

For finite $A_0, A_1 \subseteq G$ and $g \in G$, we have $g^{-1}\mathcal{Z}(A_0, A_1) = \mathcal{Z}(g^{-1}A_0, g^{-1}A_1)$. Hence $g \cdot \omega := g\omega$ defines an action of G by homeomorphisms of Ω. Given $p \in P$, the set $\omega_p := \{g \in G : g \leq p\}$ belongs to Ω, so we can regard P as a subset of Ω.

Proposition 3.1. *The boundary $\partial\Omega$ is invariant under the action of G.*

Proof. By continuity of the G-action, it suffices to show that $\Omega_{\max}$ is invariant. Fix $\omega \in \Omega_{\max}$ and $g \in G$ and suppose that $g\omega \subset \rho$ for some $\rho \in \Omega$. Then $\omega \subset g^{-1}\rho$ and hence $\omega = g^{-1}\rho$, since ω is maximal. So $g\omega = gg^{-1}\rho = \rho$. □

The set

$$\mathcal{G} = \{(g, \omega) : P \cap \omega \neq \varnothing \text{ and } P \cap g\omega \neq \varnothing\}$$

becomes a groupoid when endowed with the operations

$$(g, h\omega)(h, \omega) = (gh, \omega) \quad \text{and} \quad (g, \omega)^{-1} = (g^{-1}, g\omega).$$

The unit space is $\{e\} \times \Omega$, which we identify with Ω, and the structure maps are

$$r(g, \omega) = (e, g\omega) \quad \text{and} \quad s(g, \omega) = (e, \omega).$$

One can check that $\mathcal{G}$ is equal to the restriction of the transformation groupoid $G \ltimes \Omega$ to the closure of the copy of P in Ω; in symbols, $\mathcal{G} = (G \ltimes \Omega)|_{\overline{P}}$. We write $\mathcal{G}|_{\partial\Omega}$ for the subgroupoid

$$\mathcal{G}|_{\partial\Omega} := \{(g, \omega) \in \mathcal{G} : \omega \in \partial\Omega\}.$$

3.2. **The fibres of the Fell bundle.** For a fixed $r \in P$ and any $p, q \in P$ there is a map

$$i_r : \mathcal{L}(X_p, X_q) \to \mathcal{L}(X_{pr}, X_{qr})$$

such that, for $x \in X_p$ and $y \in X_r$

$$i_r(S)(xy) = S(x)y.$$

There is no notational dependence on p and q, but this will not cause confusion—indeed, it is helpful to think of i_r as a map from $\bigoplus_{p,q\in P} \mathcal{L}(X_p, X_q)$ to $\bigoplus_{p,q\in P} \mathcal{L}(X_{pr}, X_{qr})$.

For $\omega \in \Omega$ and $p \in \omega$, we define $[p, \omega) := \{q \in \omega : p \leq q\}$. Given any $(g, \omega) \in \mathcal{G}$, we have $[e \vee g^{-1}, \omega) = \{p \in P \cap \omega : gp \in P\}$, and this set is directed (under the usual ordering on P). So we can form the Banach-space direct limit

$$\varinjlim_{p\in[e\vee g^{-1},\omega)} \mathcal{L}(X_p, X_{gp})$$

with respect to the maps $i_r : \mathcal{L}(X_p, X_{gp}) \to \mathcal{L}(X_{pr}, X_{gpr})$ where $pr, gpr \in \omega$. By definition of the direct limit, there are bounded linear maps $\mathcal{L}(X_p, X_{gp}) \to \varinjlim \mathcal{L}(X_p, X_{gp})$, $p \in [e \vee g^{-1}, \omega)$, that are compatible with the linking maps i_r. To lighten notation we regard all of these maps as components of a single map $i_{(g,\omega)} : \bigoplus_p \mathcal{L}(X_p, X_{gp}) \to \varinjlim \mathcal{L}(X_p, X_{gp})$. We define

$$E_{(g,\omega)} := \overline{\operatorname{span}} \bigcup_{p\in[e\vee g^{-1},\omega)} i_{(g,\omega)}(\mathcal{K}(X_p, X_{gp})) \subset \varinjlim \mathcal{L}(X_p, X_{gp}).$$

Lemma 3.2. *Each $A_\omega := E_{(e,\omega)}$ is a C^*-algebra and each $E_{(g,\omega)}$ is an $A_{g\omega}$–A_ω imprimitivity bimodule.*

Proof. By definition of the maps i_r, if $T \in \mathcal{L}(X_p, X_{p'})$ and $S \in \mathcal{L}(X_{p'}, X_{p''})$, then $i_r(T)i_r(S) = i_r(TS)$, and $i_r(T)^* = i_r(T^*)$. Using this, one checks that, identifying each $\mathcal{L}(X_p \oplus X_{gp})$ with the algebra of block-operator matrices $\begin{pmatrix} \mathcal{L}(X_p) & \mathcal{L}(X_{gp}, X_p) \\ \mathcal{L}(X_p, X_{gp}) & \mathcal{L}(X_{gp}) \end{pmatrix}$, the maps i_r determine a homomorphism $i_r : \mathcal{L}(X_p \oplus X_{gp}) \to \mathcal{L}(X_{pr} \oplus X_{gpr})$. In the same vein as above, we use the notation $\tilde{\imath}_{g,\omega}$ for all of the homomorphisms $\mathcal{L}(X_p \oplus X_{gp}) \to \varinjlim \mathcal{L}(X_p, X_{gp})$.

The following is adapted from the proof of [16, Lemma 4.1]. Since ω is directed, each finite subset $H \subseteq [e \vee g^{-1}, \omega)$ is contained in a finite $F \subseteq [e \vee g^{-1}, \omega)$ which is closed under $\vee$, and each such F has a maximum element p_F. For each such F, let

$$B_F := \sum_{s\in F} i_{s^{-1}p_F}(\mathcal{K}(X_s \oplus \mathcal{K}(X_{gs})) \subseteq \mathcal{L}(X_{p_F} \oplus X_{gp_F}).$$

If $F \subseteq \omega$ is finite with more than one element and $\vee$-closed, and if $q \in F$ is minimal, then $F' := F \setminus \{q\}$ is also $\vee$ closed, and $p_{F'} = p_F$. We have $B_F = i_{q^{-1}p_F}(\mathcal{K}(X_q \oplus X_{gq})) + B_{F'}$. Nica covariance and minimality of q ensures that

$$i_{q^{-1}p_F}(\mathcal{K}(X_q \oplus X_{gq}))i_{s^{-1}p_F}(\mathcal{K}(X_s \oplus X_{gs})) \subseteq i_{(q\vee s)^{-1}p_F}(\mathcal{K}(X_(q \vee s) \oplus X_{g(q\vee s)})) \subseteq B_{F'}$$

So $B_{F'}B_F, B_FB_{F'} \subseteq B_{F'}$. Assuming as an inductive hypothesis that $B_{F'}$ is a C^*-algebra, we deduce from [7, Corollary 1.8.4] that B_F is a C^*-algebra. Since each $B_{\{p\}} = \mathcal{K}(X_p \oplus X_{gp})$ is clearly a C^*-algebra, we conclude by induction that each B_F is a C^*-algebra. So

$$\overline{\operatorname{span}} \bigcup_{p\in[e\vee g^{-1},\omega)} \tilde{\imath}_{g,\omega}(\mathcal{K}(X_p \oplus X_{gp})) \subset \varinjlim \mathcal{L}(X_p \oplus X_{gp})$$

is canonically isometrically isomorphic to $L_{g,\omega} := \varinjlim_F \tilde{\imath}_{g,\omega}(B_F)$, so is a C^*-algebra. Put $p = e \vee g^{-1}$, so $p \in \omega \cap P$ and $gp \in g\omega \cap P$. Since X is nondegenerate, the spaces A_ω and $A_{g\omega}$ appear as the complementary full corners $\tilde{\imath}_{g,\omega}(1_{X_p})L_{g,\omega}\tilde{\imath}_{g,\omega}(1_{X_p})$ and $\tilde{\imath}_{g,\omega}(1_{X_{gp}})L_{g,\omega}\tilde{\imath}_{g,\omega}(1_{X_{gp}})$ of $L_{g,\omega}$, so they are C^*-algebras. Furthermore, $E_{(g,\omega)} = \tilde{\imath}_{g,\omega}(1_{X_{gp}})L_{g,\omega}\tilde{\imath}_{g,\omega}(1_{X_p})$, and so it is an $A_{g\omega}$–A_ω-imprimitivity bimodule. $\square$

3.3. **The operations on the Fell bundle.** Let

$$\mathscr{E} := \bigcup_{(g,\omega)\in\mathcal{G}} E_{(g,\omega)}.$$

Then $\mathscr{E}$ is a bundle over $\mathcal{G}$, with $\pi : \mathscr{E} \to \mathcal{G}$ defined by $\pi(E_{(g,\omega)}) = \{(g,\omega)\}$.

Lemma 3.3. *Fix $p, p', q, q' \in P$ with $p \vee q' < \infty$ and let $r = p^{-1}(p \vee q')$, and $r' = q'^{-1}(p \vee q')$. Then for any $S \in \mathcal{K}(X_p, X_{p'})$ and $T \in \mathcal{K}(X_q, X_{q'})$ we have*

$$i_r(S)i_{r'}(T) \in \mathcal{K}(X_{qr'}, X_{p'r}).$$

Proof. Since both the left and right actions are nondegenerate, it is enough to prove the result for SU and VT where $S \in \mathcal{K}(X_{p,p'}), U \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q, X_{q'}), V \in \mathcal{K}(X_{q'})$. We have

$$i_r(SU)i_{r'}(VT) = i_r(S)i_r(U)i_{r'}(V)i_{r'}(T).$$

Since X is compactly aligned, we have $i_r(U)i_{r'}(V) \in \mathcal{K}(X_{p\vee q'})$, and hence $i_r(SU)i_{r'}(VT) \in \mathcal{K}(X_{qr'}, X_{p'r})$ as claimed. □

Fix $((g, h\omega), (h, \omega)) \in \mathcal{G}^{(2)}$, $hp \in [e \vee g^{-1}, h\omega), q \in [e \vee h^{-1}, \omega)$ and $S \in \mathcal{K}(X_{hp}, X_{ghp})$, $T \in \mathcal{K}(X_q, X_{hq})$. Let $r = p^{-1}(p \vee q), r' = q^{-1}(p \vee q)$, and define

$$i_{(g,h\omega)}(S)i_{(h,\omega)}(T) := i_{(gh,\omega)}\left(i_r(S)i_{r'}(T)\right).$$

The right hand side makes sense by Lemma 3.3. This extends to a multiplication

$$\mathscr{E}^{(2)} := \{(e, f) \in \mathscr{E} \times \mathscr{E} : (\pi(e), \pi(f)) \in \mathcal{G}^{(2)}\} \to \mathscr{E}.$$

For $(g,\omega) \in \mathcal{G}$ and $p \in [e \vee g^{-1}, \omega)$, the usual adjoint operation $* : \mathcal{L}(X_p, X_{gp}) \to \mathcal{L}(X_{gp}, X_p) = \mathcal{L}(X_{gp}, X_{g^{-1}(gp)})$ is isometric. So for each (g,ω) it extends to an involution $\varinjlim \mathcal{L}(X_p, X_{gp}) \to \varinjlim \mathcal{L}(X_{gp}, X_p)$, which then restricts to an involution $E_{(g,\omega)} \to E_{(g^{-1},g\omega)}$.

3.4. **The topology on the Fell bundle.** Given $p, q \in P$ and $S \in \mathcal{L}(X_p, X_q)$ define $f^S : \mathcal{G} \to \bigcup_{(g,\omega)\in\mathcal{G}} \varinjlim_{p\in[e\vee g^{-1},\omega)} \mathcal{L}(X_p, X_{gp})$ by

$$f^S(g,\omega) = \begin{cases} i_{(qp^{-1},\omega)}(S) & \text{if } g = qp^{-1} \text{ and } p \in \omega \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.4. *For any $p, q \in P$ and any $S \in \mathcal{L}(X_p, X_q)$, the map*

$$(g,\omega) \mapsto \|f^S(g,\omega)\|$$

is upper semicontinuous.

Proof. Since $\|f^S(g,\omega)\| = \|f^{S^*S}(\omega)\|^{1/2}$ for any $(g,\omega) \in \mathcal{G}$, it is enough to check upper semicontinuity on the unit space $\mathcal{G}^{(0)} = \Omega$. Fix $p \in P$, $S \in \mathcal{L}(X_p)$ and $\alpha > 0$. We must show that the set

$$\{\omega : \|f^S(\omega)\| < \alpha\}$$

is open. Since $p \notin \omega$ implies that $f^S(\omega) = 0$, we see that

$$\{\omega : \|f^S(\omega)\| < \alpha\} = \mathcal{Z}(\{p\}, \varnothing) \cup \{\omega : p \in \omega \text{ and } \|i_\omega(S)\| < \alpha\}$$

and so it is enough to show that $\{\omega : p \in \omega \text{ and } \|f^S(\omega)\| < \alpha\}$ is open. Fix ω in this set. Since A_ω is a direct limit we have

$$\|f^S(\omega)\| = \|i_\omega(S)\| = \lim_{q\geq p} \|i_{qp^{-1}}(S)\| = \inf_{q\geq p} \|i_{qp^{-1}}(S)\|.$$

Therefore, there exists a $q \geq p$ such that $\|i_{qp^{-1}}(S)\| < \alpha$. Suppose that $\omega' \in \mathcal{Z}(\varnothing, \{q\})$. Then $p \in \omega'$, and so

$$\|f^S(\omega')\| = \|i_{\omega'}(S)\| \leq \|i_{qp^{-1}}(S)\| < \alpha.$$ □

Now let

$$\Gamma = \operatorname{span}\{f^S : p, q \in P,\ S \in \mathcal{K}(X_p, X_q)\}.$$

Given finitely many pairs $(p_1, q_1), \dots, (p_n, q_n)$ and operators $S_i \in \mathcal{K}(X_{p_i}, X_{q_i})$, there are finitely many maximal subsets $F_1, \dots, F_m$ of $\{p_1, \dots, p_n\}$ such that each F_j has an upper bound r_j in P. Putting $T_j := \sum_{p \in F_j} i_{p^{-1}r_j}(S_i)$ for each j, we have $T_j \in \mathcal{L}(X_{r_j})$ and

$$\textstyle\sum_{i=1}^n f^{S_i} = \sum_{j=1}^m f^{T_j},$$

where the f^{T_j} have mutually disjoint support. So Lemma 3.4 shows that the sections in Γ are upper semicontinuous.

Given $(g, \omega) \in \mathcal{G}$ we have

$$\begin{aligned}\{f(g,\omega) : f \in \Gamma\} &= \big\{i_{(g,\omega)}(S) :\ p \in [e \vee g^{-1}, \omega),\ S \in \mathcal{K}(X_p, X_{gp})\big\}\\ &= \textstyle\bigcup_{[e \vee g^{-1}, \omega)} i_{(g,\omega)}(\mathcal{K}(X_p, X_{gp}))\end{aligned}$$

which densely spans $E_{(g,\omega)}$. Hence [11, Section II.13.18] shows that there is a unique topology on $\mathscr{E}$ such that $(\mathscr{E}, \pi)$ is a Banach bundle and all the functions in Γ are continuous cross sections of $\mathscr{E}$; and $\mathscr{E}$ becomes a Fell-bundle over $\mathcal{G}$ in this topology.

4. Representing the product system

4.1. **Toeplitz representation.** Let (G, P) be a quasi-lattice ordered group, and X a nondegenerate compactly aligned product system over P. For $p \in P$, identify X_p with $\mathcal{K}(X_e, X_p)$ as usual: $x \in X_p$ is identified with the operator $a \mapsto x \cdot a$. We then write x^* for the operator $y \mapsto \langle x, y\rangle_{X_e}$ in $\mathcal{K}(X_p, X_e)$. Define $\psi_p : X_p \to C^*(\mathcal{G}, \mathscr{E})$ by $\psi_p(x) = f^x$.

Proposition 4.1. *Let (G, P) be a quasi-lattice ordered group, and X a nondegenerate compactly aligned product system over P. Let $\mathcal{G}$ and $\mathscr{E}$ be the groupoid and Fell bundle constructed in Section 3. The map $\psi : X \to C^*(\mathcal{G}, \mathscr{E})$ such that $\psi|_{X_p} = \psi_p$ is a Nica covariant Toeplitz representation of X, and for $S \in \mathcal{K}(X_p)$, we have $\psi^{(p)}(S) = f^S$.*

Proof. We need to check the conditions of Definition 2.3. For $x, y \in X_p$ and $a \in X_e$,

$$\begin{aligned}\psi_p(x)^*\psi_p(y)(g,\omega) &= [(f^{x^*}) * f^y](g,\omega) = \sum_{h\omega \cap P \neq \varnothing} f^x((gh^{-1}, h\omega)^{-1})^* f^y(h,\omega)\\ &= \sum_{h\omega \cap P \neq \varnothing} f^x(hg^{-1}, g\omega)^* f^y(h,\omega) = \delta_{g,e} f^x(p,\omega)^* f^y(p,\omega)\\ &= \delta_{g,e} i_{(p,\omega)}(x)^* i_{(p,\omega)}(y) = \delta_{g,e} i_{(p^{-1},p\omega)}(x^*) i_{(p,\omega)}(y)\\ &= \delta_{g,e} i_\omega(\langle x, y\rangle_A) = f^{\langle x,y\rangle_A}(g,\omega) = \psi_e(\langle x, y\rangle).\end{aligned}$$

Likewise,

$$\begin{aligned}[\psi_e(a)\psi_p(x)](g,\omega) &= [f^a * f^x](g,\omega) = \sum_{h\omega \cap P \neq \varnothing} f^a(gh^{-1}, h\omega) f^x(h,\omega)\\ &= \delta_{g,p} i_{p\omega}(a) i_{(p,\omega)}(x) = \delta_{g,p} i_{(p,\omega)}(ax) = f^{ax}(g,\omega) = \psi_p(ax)\end{aligned}$$

and

$$\begin{aligned}[\psi_p(x)\psi_e(a)](g,\omega) &= [f^x * f^a](g,\omega) = \sum_{h\omega \cap P \neq \varnothing} f^x(gh^{-1}, h\omega) f^a(h,\omega)\\ &= \delta_{g,p} i_{(p,\omega)}(x) i_\omega(a) = \delta_{g,p} i_{(p,\omega)}(xa) = f^{xa}(g,\omega) = \psi_p(xa).\end{aligned}$$

To see that each $\psi^{(p)}(S) = f^S$, consider $S = \theta_{x,y}$ and calculate:

$$\begin{aligned}\psi^{(p)}(\theta_{x,y})(g,\omega) &= [\psi_p(x)\psi_p(y)^*](g,\omega) = [f^x * f^y](g,\omega) = \sum_{h\omega\cap P\neq\varnothing} f^x(gh^{-1}, h\omega) f^y((h,\omega)^{-1})^* \\ &= \sum_{h\omega\cap P\neq\varnothing} f^x(gh^{-1}, h\omega) f^y(h^{-1}, h\omega)^* = \delta_{g,p} i_{(p,p^{-1}\omega)}(x) i_{(p,p^{-1}\omega)}(y)^* \\ &= \delta_{g,p} i_{(p,p^{-1}\omega)}(x) i_{(p^{-1},\omega)}(y^*) = \delta_{g,p} i_\omega(\theta_{x,y}) = f^{\theta_{x,y}}(g,\omega).\end{aligned}$$

So continuity and linearity give $\psi^{(p)}(S) = f^S$ for all $S \in \mathcal{K}(X_p)$. Fix $p, q \in P$ with $p \vee q < \infty$ and $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$. Then

$$\begin{aligned}[\psi^{(p)}(S)\psi^{(q)}(T)](g,\omega) &= [f^S * f^T](g,\omega) = \sum_{h\omega\cap P\neq\varnothing} f^S(gh^{-1}, h\omega) f^T(h,\omega) \\ &= \delta_{g,e} i_\omega(S) i_\omega(T) = \delta_{g,e} i_\omega(i_{p^{-1}(p\vee q)}(S) i_{q^{-1}(p\vee q)}(T)) \\ &= f^{i_{p^{-1}(p\vee q)}(S) i_{q^{-1}(p\vee q)}(T)}(g,\omega) = [\psi^{(p\vee q)}(i_{p^{-1}(p\vee q)}(S) i_{q^{-1}(p\vee q)}(T))](g,\omega).\end{aligned}$$

Thus all the conditions of Definition 2.3 are satisfied. □

4.2. **Restriction of the representation to the boundary groupoid.** Consider $\pi_p : X_p \to C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$ satisfying

$$\pi_p(x) = f^x|_{\mathcal{G}|_{\partial\Omega}}$$

Define $\pi : X \to C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$ by $\pi|_{X_p} = \pi_p$.

Proposition 4.2. *Let (G,P) be a quasi-lattice ordered group, and X a nondegenerate compactly aligned product system over P. Suppose that the homomorphisms $\phi_p : A \to \mathcal{L}(X_p)$ implementing the left actions are all injective. Let $\mathcal{G}$ and $\mathscr{E}$ be the groupoid and Fell bundle constructed in Section 3. The map $\pi : X \to C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$ is a Cuntz–Nica–Pimsner covariant Toeplitz representation.*

Before we prove this, we need two lemmas.

Lemma 4.3. *Suppose that $\omega \in \partial\Omega$ and $q \in P$ satisfy $q \vee p < \infty$ for all $p \in \omega$. Then $q \in \omega$.*

Proof. Consider the set

$$q \vee \omega := \{q \vee p : p \in \omega\}$$

If $q \vee p_1, q \vee p_2 \in q \vee \omega$ we have

$$(q \vee p_1) \vee (q \vee p_2) = q \vee (p_1 \vee p_2) \in q \vee \omega$$

since $p_1 \vee p_2 \in \omega$. So $q \vee \omega$ is directed. Let $\mathrm{Her}(q\vee\omega)$ denote the hereditary closure $\mathrm{Her}(q\vee\omega) = \{g \in G : g \leq p \text{ for some } p \in q \vee \omega\}$ of $q \vee \omega$. Notice that $q = q \vee e \in \mathrm{Her}(q \vee \omega)$. For any $p \in \omega$,

$$p \leq q \vee p \in q \vee \omega$$

and hence $p \in \mathrm{Her}(q \vee \omega)$. So $\omega \subset \mathrm{Her}(q \vee \omega)$ and hence $\omega = \mathrm{Her}(q \vee \omega)$ because $\omega \in \partial\Omega$. So $q \in \omega$. □

Lemma 4.4. *Fix a sequence $(\omega_n)_{n=1}^\infty \subset \Omega$ with $p \in \omega_n$ for all n, and suppose that $\omega_n \to \omega$. Then $p \in \omega$, and for $T \in \mathcal{K}(X_p)$,*

$$i_{\omega_n}(T) \to i_\omega(T) \text{ in } E \text{ as } n \to \infty.$$

Proof. We know that the set $\mathcal{Z}(\varnothing, \{p\})$ is closed and $\omega_n \in \mathcal{Z}(\varnothing, \{p\})$ for all n. Hence $\omega \in \mathcal{Z}(\varnothing, \{p\})$ and so $p \in \omega$.

Now, fix $T \in \mathcal{K}(X_p)$ and $U \subset \mathscr{E}$ open with $i_\omega(T) \in U$. By definition of the topology on $\mathscr{E}$, the function f^T is continuous, so $(f^T)^{-1}(U) \subset \mathcal{G}$ is open. Since $\omega_n \to \omega$ and $\mathcal{G}$ has the relative product topology, $(e,\omega_n) \to (e,\omega)$ in $\mathcal{G}$. We have $f^T(e,\omega) = i_\omega(T) \in U$, and hence $(e,\omega) \in (f^T)^{-1}(U)$. Thus there exists N such that $(e,\omega_n) \in (f^T)^{-1}(U)$ for all $n > N$, and so

$$f^T(e,\omega_n) = i_{\omega_n}(T) \in U \text{ for all } n > N,$$

giving $i_{\omega_n}(T) \to i_\omega(T)$. $\square$

Proof of Proposition 4.2. Replacing an $\omega \in \Omega$ with $\omega \in \partial\Omega$ in the proof of Proposition 4.1 shows that π is a Nica covariant Toeplitz representation. Since all the left actions are by injective homomorphisms, the representation π is Cuntz-Nica-Pimsner covariant if it satisfies relation (CNP) of Definition 2.4.

Fix a finite set $F \subset P$ and elements $T_p \in \mathcal{K}(X_p)$, $p \in F$ such that

$$\textstyle\sum_{p\in F} i_{qp^{-1}}(T_p) = 0$$

for large q. We must show that $\sum_{p\in F} \pi^{(p)}(T_p) = 0$. So, since each $\pi^{(p)}(T) = \psi^{(p)}(T)|_{\partial\Omega}$, we have to check that

$$\textstyle\sum_{p\in F} f^{T_p}(g,\omega) = 0$$

for all $(g,\omega) \in \mathcal{G}|_{\partial\Omega}$. Fix $(g,\omega) \in \mathcal{G}|_{\partial\Omega}$ with $\omega \in \Omega_{\max}$, and observe that

$$\textstyle\sum_{p\in F} f^{T_p}(g,\omega) = \delta_{g,e} \sum_{p\in F\cap\omega} i_\omega(T_p).$$

Since $F \cap \omega \subset P$ is finite and ω is directed, the element

$$r := \textstyle\bigvee_{p\in F\cap\omega} p$$

belongs to ω, and

$$\textstyle\sum_{p\in F\cap\omega} i_\omega(T_p) = i_\omega\Big(\sum_{p\in F\cap\omega} i_{p^{-1}r}(T_p)\Big).$$

Since ω is directed and countable we can choose a sequence $(r_n)_{n=1}^\infty \subset \omega$ satisfying

- $r_1 \geq r$,
- $r_{n+1} \geq r_n$ for all n
- for all $q \in \omega$, there exists n with $r_n \geq q$.

For each n, choose $q_n \geq r_n$ and $\omega_n \in \partial\Omega$ with $q_n \in \omega_n$ (and hence $r_n \in \omega_n$) such that

$$\textstyle\sum_{p\in F} i_{p^{-1}q_n}(T_p) = 0.$$

Then in particular,

$$\textstyle\sum_{p\in F\cap\omega_n} i_{p^{-1}q_n}(T_p) = \sum_{p\in F} i_{p^{-1}q_n}(T_p) = 0 \tag{4.1}$$

since $p \in F \setminus \omega_n$ implies $p \not\leq q_n$ and so $i_p^{p^{-1}q_n} = 0$. We claim that $\omega_n \to \omega$ as $n \to \infty$. To see this fix $\mathcal{Z}(A_0, A_1)$ containing ω. Since $A_1 \subset \omega$, A_1 is directed. Let

$$s = \textstyle\bigvee_{p\in A_1} p.$$

By definition of $(r_n)_{n=1}^\infty$ there is an n_1 with $r_{n_1} \geq s$. Then $A_1 \subset \omega_{r_n}$ for any $n \geq n_1$.

For each $q \in A_0$, let $N_q := \max\{n : q \in \omega_n\}$. Suppose for contradiction that $q \in A_0$ satisfies $N_q = \infty$. For any $p \in \omega$ we can find $r_j \geq p$. Since $N_q = \infty$ we can find $k \geq j$ with $q \in \omega_k$. But then

$$q \vee r_k < \infty \implies q \vee r_j < \infty \implies q \vee p < \infty.$$

Since $p \in \omega$ was arbitrary we deduce that $q \vee p < \infty$ for all $p \in \omega$ and hence $q \in \omega$ by Lemma 4.3. This contradicts $\omega \in \mathcal{Z}(A_0, A_1)$. Therefore N_q is finite for every $q \in A_0$. Now put

$$N := \max\big\{n_1, \max\nolimits_{q\in A_0} N_q\big\} < \infty.$$

Then $\omega_n \in \mathcal{Z}(A_0, A_1)$ for any $n > N$ and $\omega_n \to \omega$ as claimed. Since F is finite, there exists N_F such that $n \geq N_F$ implies $F \cap \omega_n = F \cap \omega$.

Hence, using Lemma 4.4 at the third equality and (4.1) at the last one, we have

$$\begin{aligned}\sum_{p\in F} f^{T_p}(g,\omega) &= \delta_{g,e} \sum_{p\in F\cap\omega} i_\omega(T_p) = \delta_{g,e} i_\omega\left(\sum_{p\in F\cap\omega} i_{p^{-1}r}(T_p)\right) = \delta_{g,e} \lim_{n\to\infty} i_{\omega_n}\left(\sum_{p\in F\cap\omega} i_{p^{-1}r}(T_p)\right)\\ &= \delta_{g,e} \lim_{n\to\infty} i_{\omega_n}\left(\sum_{p\in F\cap\omega} i_{p^{-1}q_n}(T_p)\right) = \delta_{g,e} \lim_{n\to\infty} i_{\omega_n}\left(\sum_{p\in F\cap\omega_n} i_{p^{-1}q_n}(T_p)\right) = 0.\end{aligned}$$

Since $\Omega_{\max}$ is dense in $\partial\Omega$ and $\sum_{p\in F}\pi^{(p)}(T_p)$ is a continuous section of $\mathscr{E}$, we deduce that $\sum_{p\in F}\pi^{(p)}(T_p)=0$. □

5. The isomorphisms

In this section, we prove our main results: that the C^*-algebra of the Fell bundle $\mathscr{E}$ constructed in Section 3 is isomorphic to the Nica–Toeplitz algebra $\mathcal{NT}_X$ and, under the hypothesis that the left actions of A on the X_p are implemented by injective homomorphisms, that the C^*-algebra of the restriction of $\mathscr{E}$ to the boundary groupoid $\mathcal{G}|_{\partial\Omega}$ is isomorphic to the Cuntz–Nica–Pimsner algebra $\mathcal{NO}_X$.

Theorem 5.1. *Let X be a compactly aligned product system over a quasi-lattice ordered group (G,P). Let $\mathcal{G}$ and $\mathscr{E}$ be the groupoid and Fell bundle constructed in Section 3. Then the homomorphism $\Psi:\mathcal{NT}_X\to C^*(\mathcal{G},\mathscr{E})$ induced by the Toeplitz representation ψ of Proposition 4.1 is an isomorphism.*

Proof. We begin by showing that Ψ is surjective. By definition of the topology on $\mathscr{E}$, it suffices to show that $f^S\in\operatorname{Im}\Psi$ for all $S\in\mathcal{K}(X_p,X_q)$. If $S,T\in\mathcal{K}(X_p,X_q)$ then $f^S+f^T=f^{S+T}$, so it suffices to show that $f^{\theta_{y,x}}\in\operatorname{Im}\Psi$ for all $x\in X_p$ and $y\in X_q$. Given $(g,\omega)\in\Omega$ we have

$$\begin{aligned}[]
[\psi_q(y)\psi_p(x)^*](g,\omega)=[f^y*f^{x*}](g,\omega)&=\sum_{h\omega\cap P\neq\varnothing}f^y(gh^{-1},h\omega)f^x(h^{-1},h\omega)^*\\
&=\delta_{g,qp^{-1}}f^y(q,p^{-1}\omega)f^x(p,p^{-1}\omega)^*=\delta_{g,qp^{-1}}i_{(q,p^{-1}\omega)}(x)i_{(p,p^{-1}\omega)}(y)^*\\
&=\delta_{g,qp^{-1}}i_{(q,p^{-1}\omega)}(x)i_{(p^{-1},\omega)}(y^*)=\delta_{g,qp^{-1}}i_{(qp^{-1},\omega)}(xy^*)=f^{\theta_{x,y}}(g,\omega)
\end{aligned}$$

as required. To see that Ψ is injective, we construct an inverse. We begin by showing that there is a well-defined map $\Phi:\operatorname{span}\{f^S:S\in\mathcal{K}(X_p,X_q)\}\to\mathcal{NT}_X$ satisfying

$$\Phi(f^{\theta_{y,x}})=i_X(y)i_X(x)^*\quad\text{for all } x\in X_p \text{ and } y\in X_q. \tag{5.1}$$

To see that such a map exists, suppose that

$$\textstyle\sum_{j=1}^n f^{\theta_{y_j,x_j}}=0\in\Gamma_c(\mathcal{G};\mathscr{E}).$$

It suffices to show that

$$\textstyle\sum_{j=1}^n i_X(y_j)i_X(x_j)^*=0\in\mathcal{NT}_X.$$

Since the Fock representation $l:X\to\mathcal{L}(F(X))$ is isometric [12, page 340], this is equivalent to

$$\textstyle\sum_{j=1}^n l(y_j)l(x_j)^*=0\in\mathcal{L}(F(X)).$$

To see this, fix $z\in X_r$ and $a\in A$. For any $p\in P$ we have

$$\left(\sum_{j=1}^n l(y_j)l(x_j)^*(z\cdot a)\right)(p)=\sum_{\substack{p_j\le r\\ q_jp_j^{-1}r=p}}y_j\left(i_{p_j^{-1}r}(x_j)^*(z\cdot a)\right).$$

Hence $\left(\sum_{j=1}^n f^{\theta_{y_j,x_j}}\right)*f^{\theta_{z,a}}=0$, and so

$$\begin{aligned}
0&=\left(\left(\sum_{j=1}^n f^{\theta_{y_j,x_j}}\right)*f^{\theta_{z,a}}\right)(p,[e])=\sum_{\substack{q_jp_j^{-1}r=p\\ p_j\in[r]}}i_{(q_jp_j^{-1},[r])}(\theta_{y_j,x_j})i_{(r,[e])}(\theta_{(z,a)})\\
&=\sum_{\substack{p_j\le r\\ q_jp_j^{-1}r=p}}i_{p_j^{-1}r}(\theta_{y_j,x_j})i_e(\theta_{z,a})=\sum_{\substack{p_j\le r\\ q_jp_j^{-1}r=p}}y_j\left(i_{p_j^{-1}r}(x_j)^*(z\cdot a)\right).
\end{aligned}$$

Hence

$$\left(\textstyle\sum_{j=1}^n l(y_j)l(x_j)^*(z\cdot a)\right)(p)=0.$$

Since $z \cdot a$ and p were arbitrary, we see that there is a well-defined linear map satisfying (5.1).

We now show that Φ in continuous in the inductive limit topology. Suppose that $f_i \to f$ in $\Gamma_c(\mathcal{G};\mathscr{E})$. Fix a compact subset $K \subset \mathcal{G}$ such that f and each of the f_i vanishes off K. Write $f = \sum_{j=1}^n f^{S_j}$ where each $S_j \in \mathcal{K}(X_{p_j}, X_{q_j})$. Inductively define

$$A_1 = \operatorname{supp}(f^{S_1}) \qquad \text{and} \qquad A_{k+1} = \operatorname{supp}(f^{S_{k+1}}) \setminus \Big(\bigcup_{j=1}^k A_k\Big)$$

for $1 \leq k \leq n$. Then each $A_k \subset \mathcal{G}$ is a bisection, so that $\|(f_i - f)|_{A_k}\|_{C^*(\mathcal{G},\mathscr{E})} = \|(f_i - f)|_{A_k}\|_\infty$ for all i. Define the set

$$A_{n+1} = K \setminus \Big(\bigcup_{j=1}^n A_k\Big).$$

Without loss of generality, we may assume that A_{n+1} is also a bisection. Then there exists $N \geq 1$ such that for all $i \geq N$ and $1 \leq k \leq n$

$$\|(f_i - f)|_{A_k}\|_\infty < \frac{\varepsilon}{n+1}.$$

So for $i \geq N$

$$\begin{aligned}\|\Phi(f_i) - \Phi(f)\| &= \Big\|\sum_{j=1}^n \Phi(f_i - f^{S_j})\Big\| = \Big\|\sum_{j=1}^n \sum_{k=1}^{n+1} \Phi((f_i - f^{S_j})|_{A_k})\Big\| \\ &\leq \sum_{j=1}^n \sum_{k=1}^{n+1} \|\Phi((f_i - f^{S_j})|_{A_k})\| \leq \sum_{j=1}^n \sum_{k=1}^{n+1} \|(f_i - f^{S_j})|_{A_k}\|_\infty < \varepsilon.\end{aligned}$$

So $\Phi(f_i) \to \Phi(f)$. Since the inductive limit topology on $\Gamma_c(\mathcal{G};\mathscr{E})$ is weaker than the norm topology, we see that Φ is bounded in norm. Since $\Gamma_c(\mathcal{G};\mathscr{E})$ is norm dense in $C^*(\mathcal{G},\mathscr{E})$, Φ extends to a $*$-homomorphism

$$\Phi : C^*(\mathcal{G},\mathscr{E}) \to \mathcal{NT}_X$$

which is, by construction, an inverse for Ψ. So $C^*(\mathcal{G},\mathscr{E}) \cong \mathcal{NT}_X$. □

Theorem 5.2. *Let X be a nondegenerate compactly aligned product system over a quasi-lattice ordered group (G,P). Suppose that the homomorphisms $\phi_p : A \to \mathcal{L}(X_p)$ implementing the left actions are all injective. Let $\mathcal{G}$ and $\mathscr{E}$ be the groupoid and Fell bundle constructed in Section 3. Then the homomorphism $\Pi : \mathcal{NO}_X \to C^*(\mathcal{G}|_{\partial\Omega},\mathscr{E})$, induced by the Cuntz–Nica–Pimsner covariant representation π of Proposition 4.2, is an isomorphism.*

Before we prove Theorem 5.2, we need to do some background work on coactions. The first lemma that we need is a general statement about coactions of discrete groups. The following brief summary of discrete coactions is based on [8, §A.3]. Given a discrete group G, the universal property of $C^*(G)$ shows that there is a homomorphism $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ whose extension to $\mathcal{M}C^*(G)$ satisfies $\delta_g(i_G(g)) = i_G(g) \otimes i_G(g)$. A coaction of a discrete group G on a C^*-algebra A is a nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$ which satisfies the coaction identity

$$(\delta \otimes 1_{C^*(G)}) \circ \delta = (1 \otimes \delta_G) \circ \delta.$$

The coaction δ is coaction-nondegenerate if $\overline{\operatorname{span}}\, \delta(A)(1_{\mathcal{M}(A)} \otimes C^*(G)) = A \otimes C^*(G)$.

It is claimed at the beginning of Section 1 of [22] that, in our setting of discrete groups G, every coaction of a discrete group is coaction-nondegenerate. This assertion was used in results of [4] that we in turn will want to use in the proof of Theorem 5.2. However, this assertion in [22] depends on [21, Proposition 2.5], and a gap has recently been identified in the proof of this result [14]. The following simple lemma is well known, but hard to find in the literature. We will use it first to show that the coactions used in [4] are indeed coaction-nondegenerate (so the results of [4] are not affected by the issue identified in [14]), and then again in the proof of Lemma 5.5 below.

Recall that if $\delta : A \to A \otimes C^*(G)$ is a coaction of a discrete group, then for each $g \in G$, we write A_g for the *spectral subspace* $\{a \in A : \delta(a) = a \otimes i_G(g)\}$.

Lemma 5.3. *Let A be a C^*-algebra and G a discrete group. Suppose that $\delta : A \to A \otimes C^*(G)$ is a coaction. Then δ is coaction-nondegenerate if and only if $A = \overline{\operatorname{span}} \bigcup_{g \in G} A_g$.*

Proof. First suppose that δ is coaction-nondegenerate. Then [8, Proposition A.31] shows that A is densely spanned by its spectral subspaces. Now suppose that A is densely spanned by its spectral subspaces. Fix a typical spanning element $a \otimes i_G(G)$ of $A \otimes C^*(G)$. Fix ε and choose finitely many $g_i \in G$ and $a_i \in A_{g_i}$ such that $\|a - \sum_i a_i\| < \varepsilon$. Then

$$\Big\| \sum_i \delta(a_i)(1 \otimes i_G(g_i^{-1} g)) - a \otimes i_G(g) \Big\| = \Big\| \Big(\sum_i a_i - a \Big) \otimes i_G(g) \Big\| < \varepsilon. \qquad \square$$

Corollary 5.4. *The coactions of G on $\mathcal{NT}_X$ and $\mathcal{NO}_X$ used in* [4] *are coaction-nondegenerate.*

Proof. By construction (see [12]), the algebra $\mathcal{NT}_X$ is the closure of the span of the elements $i_X(x)i_X(y)^*$ where $x, y \in X$. Hence $\mathcal{NO}_X$ is densely spanned by the corresponding elements $j_X(x)j_X(y)^*$. The coactions of [4] are given by $\delta(i_X(x)) = i_X(x) \otimes i_G(g)$ and $\delta(j_X(x)) = j_X(x) \otimes i_G(g)$ whenever $x \in X_g$. So each spanning element of $\mathcal{NT}_X$ and of $\mathcal{NO}_X$ belongs to a spectral subspace for δ. Hence $\mathcal{NT}_X$ and $\mathcal{NO}_X$ are spanned by their spectral subspaces. Thus Lemma 5.3 shows that the coactions δ are coaction-nondegenerate. $\square$

The second lemma that we need establishes that the C^*-algebra of the Fell bundle of Section 3 carries a coaction of G that is compatible with the gauge coactions on $\mathcal{NT}_X$ and $\mathcal{NO}_X$.

Lemma 5.5. *Let c be a continuous grading of a Hausdorff étale groupoid $\mathcal{G}$ by a discrete group G, and let $\mathscr{E}$ be a Fell bundle over $\mathcal{G}$. Let $i_G : G \to C^*(G)$ denote the universal representation of G. There is a coaction-nondegenerate coaction δ of G on $C^*(\mathscr{E}, \mathcal{G})$ satisfying*

$$\delta(f) = f \otimes i_G(g)$$

whenever $g \in G$ and $f \in \Gamma_c(\mathcal{G}; \mathscr{E})$ satisfies $\operatorname{supp}(f) \subset c^{-1}(\{g\})$.

Proof. As a vector space, $\Gamma_c(\mathcal{G}; \mathscr{E})$ is equal to the algebraic direct sum $\bigoplus_{g \in G} \Gamma_c(c^{-1}(g); \mathscr{E})$. So there is a linear map $\delta : \Gamma_c(\mathcal{G}; \mathscr{E}) \to \Gamma_c(\mathcal{G}; \mathscr{E}) \otimes C^*(G)$ such that $\delta(f) = f \otimes i_G$ whenever $f \in \Gamma_c(c^{-1(g)}; \mathscr{E})$. It is routine to check that this map is continuous in the inductive-limit topology, and therefore extends to a homomorphism $\delta : C^*(\mathcal{G}, \mathscr{E}) \to C^*(\mathcal{G}, \mathscr{E}) \otimes C^*(G)$. An elementary calculation checks the coaction identity on $f \in \Gamma_c(c^{-1}(g); \mathscr{E})$, which suffices by linearity and continuity. To check that δ is coaction-nondegenerate, observe that the spectral subspaces $C^*(\mathcal{G}, \mathscr{E})_g$ are precisely the spaces $\overline{\Gamma_c(c^{-1}(g)); \mathscr{E})}$. By definition, $C^*(G, \mathscr{E})$ is the closure of $\Gamma_c(\mathcal{G}; \mathscr{E})$, which is spanned by the spaces $\Gamma_c(c^{-1}(g)); \mathscr{E})$. It follows that $C^*(\mathcal{G}, \mathscr{E})$ is densely spanned by its spectral subspaces, and so δ is coaction-nondegenerate by Lemma 5.3. $\square$

Recall that the Cuntz–Nica–Pimsner algebra $\mathcal{NO}_X$ has a quotient $\mathcal{NO}^r_X$ that possesses a co-universal property described in [4, Theorem 4.1].

Proof of Theorem 5.2. To show that Π is an isomorphism, it is enough to show that the homomorphism $\Phi = \Psi^{-1}$ of (5.1) factors through the quotient map

$$\rho : C^*(\mathcal{G}, \mathscr{E}) \to C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$$

defined on $\Gamma_C(\mathcal{G}; \mathscr{E})$ by

$$\rho(f) = f|_{\mathcal{G}|_{\partial\Omega}}.$$

To see this we use the co-universal property of $\mathcal{NO}^r_X$. Since $\mathcal{G}|_{\partial\Omega}$ is G-graded via $(g, \omega) \mapsto g$, Lemma 5.5 gives a coaction $\beta : C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E}) \to C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E}) \otimes C^*(G)$ such that

$$\beta(f^S) = f^S \otimes i_G(qp^{-1}) \quad \text{ for all } X \in \mathcal{K}(X_p, X_q).$$

For any $x \in X_p$, we have

$$\beta(\pi(x)) = \beta(f^x) = f^x \otimes i_G(p) = ((\pi \otimes 1) \circ \delta)(j_X(x)),$$

where $j_X : X \to \mathcal{NO}_X$ is the universal representation. So π is gauge-compatible in the sense of [4]. We aim to apply [4, Theorem 4.1] to π, so we must show that $\pi_e : A \to C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$

is injective. Since the ϕ_p are injective, the maps $i_r : \mathcal{L}(X_p) \to \mathcal{L}(X_{pr})$ appearing in the construction of the fibres A_ω, $\omega \in \mathcal{G}^{(0)}$ in Section 3.2 are all injective. Hence the canonical map $i_\omega : A = X_e \to X_\omega$ is injective for each unit ω. In particular, for each $a \in A$, the element $\pi_e(A) := f^a$ satisfies $f^a(\omega) = i_\omega(a) \neq 0$ for all ω, and π_e is injective.

Now, writing λ_r for the canonical quotient map from $\mathcal{NO}_X$ to $\mathcal{NO}^r_X$, [4, Theorem 4.1] yields a homomorphism

$$\phi : C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E}) \to \mathcal{NO}^r_X$$

that carries f^S to $\lambda_r(j^{(p)}_X(S)$ for $S \in \mathcal{K}(X_p)$.

Fix $f \in \ker(\rho)$. Without loss of generality, assume that $\mathrm{supp}(f) \subset \mathcal{G}$ is a bisection. Then $\phi(\rho(f)) = 0$ and hence $\phi(\rho(f^*f)) = 0$. So we have $\lambda_r(\rho(\Phi(f^*f))) = 0$. But $\rho(\Phi(f^*f)) \in (\mathcal{NO}_X)_e$ and $\lambda_r|_{(\mathcal{NO}_X)_e}$ is isometric because the reduction map for any coaction is isometric on each spectral subspace. Hence

$$\|q(\Phi(f))\|^2 = \|q(\Phi(f^*f))\| = 0$$

as required. □

6. Applications

Takeishi [27] has recently characterised nuclearity for C^*-algebras of Fell bundles over étale groupoids as follows.

Theorem 6.1 ([27, Theorem 4.1]). *Let $\mathscr{E}$ be a Fell bundle over an étale locally compact Hausdorff groupoid $\mathcal{G}$. If $\mathcal{G}$ is amenable, then the following conditions are equivalent*

(i) *The C^*-algebra $C^*_r(\mathscr{E})$ is nuclear.*
(ii) *The fibre E_x is nuclear for every $x \in G^{(0)}$.*
(iii) *The C^*-algebra $C_0(\mathscr{E}|_{G^{(0)}}, G^{(0)})$ is nuclear.*

For our example, the following lemma shows that (ii) holds whenever the coefficient algebra X_e of the product system X is nuclear.

Lemma 6.2. *Let (G, P) be a quasi-lattice ordered group, and let X be a nondegenerate finitely aligned product system over P. If the coefficient algebra X_e of the product system is nuclear, then the fibres A_ω, $\omega \in \Omega = \mathcal{G}^{(0)}$ are nuclear.*

Proof. Fix $\omega \in \Omega$. Arguing as in Lemma 3.2, for each finite $F \subseteq \omega$ that is closed under $\vee$, writing p_F for the maximum element of F the set $B_F = \sum_{p \in F} i_{p^{-1}p_F}(\mathcal{K}(X_p))$ is a C^*-algebra. If F is not a singleton and $q \in F$ is minimal, then $B_{F\setminus\{q\}}$ is an ideal of B_F and the quotient $B_F/B_{F\setminus\{q\}}$ is a quotient of $i_{q^{-1}p_F}(\mathcal{K}(X_q))$ and hence a quotient of $\mathcal{K}(X_q)$.

Each $\mathcal{K}(X_p)$ is nuclear because it is Morita equivalent to X_e via X_p, and nuclearity is preserved by Morita equivalence [13, Theorem 15]. Fix a finite $F \subseteq \omega$ and a minimal $q \in F$, and write $F' = F \setminus \{q\}$. Assume as an inductive hypothesis that $B_{F'}$ is nuclear. Since $B_F/B_{F'}$ is a quotient of the nuclear C^*-algebra $\mathcal{K}(X_q)$, it is nuclear. So B_F is an extension of a nuclear C^*-algebra by a nuclear C^*-algebra, so also nuclear [24, Proposition 2.1.2(iv)]. Now $A_\omega = \varinjlim_F B_F$ is nuclear because direct limits of nuclear C^*-algebras are nuclear. □

We therefore have the following theorem.

Theorem 6.3. *Let X be a nondegenerate finitely-aligned product system over a quasi-lattice ordered group (G, P), and suppose that the coefficient algebra X_e is nuclear. If the groupoid $\mathcal{G}$ of Section 3 is amenable, then $\mathcal{NT}_X$ and $\mathcal{NO}_X$ is nuclear. If $\mathcal{G}|_{\partial\Omega}$ is amenable and the homomorphisms $\phi_p : A \to \mathcal{L}(X_p)$ implementing the left actions in X are all injective, then $\mathcal{NO}_X$ is nuclear.*

Proof. If $\mathcal{G}$ is amenable, then $C^*(\mathcal{G}, \mathscr{E})$ is amenable by [27, Theorem 4.1] and Lemma 6.2. Since $\mathcal{NT}_X \cong C^*(\mathcal{G}, \mathscr{E})$ by Theorem 5.1, we have $\mathcal{NT}_X$ nuclear, and then $\mathcal{NO}_X$ (as defined in [26]) is nuclear because it is a quotient of $\mathcal{NT}_X$. If $\mathcal{G}|_{\partial\Omega}$ is amenable then $C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$ is nuclear by [27, Theorem 4.1] and Lemma 6.2. If the ϕ_p are injective, then Theorem 5.2 gives an isomorphism $\mathcal{NO}_X \cong C^*(\mathcal{G}_{\partial\Omega}, \mathscr{E})$, and so $\mathcal{NO}_X$ is nuclear. $\square$

We also obtain an improvement on [4, Corollary 4.2]. There it is proved that $\mathcal{NO}_X$ and $\mathcal{NO}^r_X$ coincide whenever the group G is amenable. But our results show that in fact $\mathcal{NO}_X = \mathcal{NO}^r_X$ whenever $\mathcal{G}|_{\partial\Omega}$ is amenable.

Proposition 6.4. *Let X be a nondegenerate finitely aligned product system over a quasi-lattice ordered group (G, P), and suppose that the homomorphism $\phi_p : X_e \to \mathcal{L}(X_p)$ implementing the left actions in X are all injective. If $\mathcal{G}|_{\partial\Omega}$ is amenable, then the quotient map $\lambda_r : \mathcal{NO}_X \to \mathcal{NO}^r_X$ is an isomorphism.*

Proof. Theorem 5.2 gives an isomorphism $\Pi^{-1} : C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E}) \to \mathcal{NO}_X$. Write $c : \mathcal{G} \to G$ for the continuous cocycle $c(g, \omega) = g$. Since $\operatorname{supp} \pi(x) \subseteq \{p\} \times \partial\Omega$ whenever $x \in X_p$, we see that $\Pi((\mathcal{NO}_X)_g) = \overline{\Gamma_c(c^{-1}(g); \mathscr{E})}$ for each g. In particular Π^{-1} restricts to an isomorphism of the closure of $\Gamma_c(c^{-1}(e); \mathscr{E}) \subseteq C^*(\mathcal{G}, \mathscr{E})$ with $(\mathcal{NO}_X)_e$. Since $c^{-1}(e) = \mathcal{G}^{(0)}$, the closure of $\Gamma_c(c^{-1}(e); \mathscr{E})$ is $\Gamma_0(\mathcal{G}^{(0)}; \mathscr{E}) \subseteq C^*(\mathcal{G}, \mathscr{E})$. It is standard that restriction of compactly supported sections to $\mathcal{G}^{(0)}$ extends to a faithful conditional expectation $C^*_r(\mathcal{G}, \mathscr{E}) \to \Gamma_0(\mathcal{G}^{(0)}; \mathscr{E})$. Theorem 1 of [25] implies that $C^*(\mathcal{G}|_{\partial\Omega}, \mathscr{E}) = C^*_r(\mathcal{G}|_{\partial\Omega}, \mathscr{E})$, so we obtain a faithful conditional expectation $R : C^*(\mathcal{G}, \mathscr{E}) \to \Gamma_0(\mathcal{G}^{(0)}; \mathscr{E})$ extending restriction of compactly supported sections. Lemma 1.3(a) of [22] shows that there is a conditional expectation $P : \mathcal{NO}_X \to (\mathcal{NO}_X)_e$ that annihilates $(\mathcal{NO}_X)_g$ for $g \neq e$, and it is routine to check that $\Pi \circ P = R \circ \Pi$. Since Π is an isomorphism and R is a faithful conditional expectation, it follows that P is a faithful conditional expectation as well. That is, the coaction ν on $\mathcal{NO}_X$ such that $\delta(j_X(x)) = j_X(x) \otimes i_G(p)$ for $x \in X_p$ is a normal coaction, and $(\mathcal{NO}_X, G, \nu)$ is a normal cosystem. Corollary 4.6 of [4] shows that $\mathcal{NO}^r_X$ is the C^*-algebra appearing in the normalisation of the cosystem $(\mathcal{NO}_X, G, \nu)$, and λ_r is the normalisation homomorphism. Since this cosystem is already normal, we conclude that λ_r is injective. $\square$

Remark 6.5. It is worth pointing out, in light of the results in this section, that it is not uncommon for the groupoid $\mathcal{G}|_{\partial\Omega}$ of Section 3 to be amenable, even when G is not amenable. For example, $\mathcal{G}|_{\partial\Omega}$ is amenable when G is a finitely generated free group—or more generally a finitely-generated right-angled Artin group—and P its natural positive cone.

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