GRADED K-THEORY AND K-HOMOLOGY OF RELATIVE CUNTZ–PIMSNER ALGEBRAS AND GRAPH C^*-ALGEBRAS

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Abstract. We establish exact sequences in KK-theory for graded relative Cuntz–Pimsner algebras associated to nondegenerate C^*-correspondences. We use this to calculate the graded K-theory and K-homology of relative Cuntz–Krieger algebras of directed graphs for gradings induced by $\{0,1\}$–valued labellings of their edge sets.

Introduction

In the study of C^*-algebras, operator K-theory, which generalises topological K-theory via Gelfand duality, has long been a key invariant. Indeed, it is a knee-jerk reaction for C^*-algebraist these days, when presented with a new example, to try to compute its K-theory; and the K-theory frequently reflects key structural properties. For example, the ordered K-theory of an irrational rotation algebra recovers the angle of rotation up to a minus sign [11, 12], while the K-theory of a Cuntz–Krieger algebra recovers the Bowen–Franks group of the associated shift space [6, 7]. Cuntz proved that the K-theory groups of the C^*-algebra of a finite directed graph E with no sources and with $\{0,1\}$-valued adjacency matrix A_E are the cokernel and kernel of the matrix $1 - A_E^t$ regarded as an endomorphism of the free abelian group $\mathbb{Z}E^0$ (see [7, Proposition 3.1]). This was generalised to row-finite directed graphs E with no sources in [24], and later (with appropriate adjustments made to the domain and the codomain of A_E^t in each case) to all row-finite directed graphs E in [28], and to arbitrary graphs in [1, 10] (see also [13, 27, 32, 33, 36]).

Dual to K-theory is the K-homology theory that emerged in the pioneering work of Brown–Douglas–Fillmore [3, 4]. It is less of an automatic reaction to compute K-homology for C^*-algebras, but, for example, Cuntz and Krieger computed (in [6, Theorem 5.3]) the Ext-group (that is, the odd K-homology group) of the Cuntz–Krieger algebra $\mathcal{O}_A$ of $A \in M_n(\mathbb{Z}_+)$ as the cokernel of $1 - A$ regarded as an endomorphism of $\mathbb{Z}^n$. The computation was later generalised to graph C^*-algebras in [10, 35] (see also [5, 36]).

Both K-theory and K-homology are unified in Kasparov's KK-theory [18, 19]: the K-theory and K-homology groups of A are isomorphic to the Kasparov groups $KK_*(\mathbb{C}, A)$ and $KK_*(A, \mathbb{C})$ respectively. So Kasparov's theory provides a unified approach to calculating K-theory and K-homology. Pimsner exploited this in [26], developing two exact sequences in KK-theory for the C^*-algebra $\mathcal{O}_X$ associated to a A–A-correspondence X, and using one of them to compute the K-theory of $\mathcal{O}_X$ in terms of that of A. Every graph determines an associated Hilbert module, and while the Pimsner algebra of this module only agrees with the graph C^*-algebra when the graph has no sources, Muhly and

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Tomforde developed a modified C^*-correspondence [23] whose Pimsner algebra always contains the graph C^*-algebra as a full corner. In particular, combining these results provides a new means of computing the K-theory and K-homology of graph C^*-algebras.

Kasparov's KK-theory is most naturally a theory for graded C^*-algebras, and the results described above are obtained by endowing the C^*-algebras involved with the trivial grading. However, graph C^*-algebras admit many natural gradings: by the universal property of $C^*(E)$, every binary labelling $\delta : E^1 \to \{0,1\}$ of the edges of E induces a grading automorphism that sends the generator s_e associated to an edge e to $(-1)^{\delta(e)} s_e$. More generally, every grading of a C^*-correspondence induces a grading of the associated Pimsner algebra.

In [20], Kumjian, Pask and Sims investigated graded K-theory and K-homology, defined in terms of Kasparov theory (other approaches to graded K-theory are investigated in, for example, [8, 9, 16]) of graded graph C^*-algebras, extending earlier results of Haag [14, 15] for the Cuntz algebras $\mathcal{O}_n$. By extending Pimsner's arguments to C^*-algebras of graded C^*-correspondences with injective left actions by compacts, they computed the graded K-theory of the C^*-algebras of row-finite graphs with no sources. They showed (in [20, Collollary 8.3]) that if E is a row-finite directed graph with no sources, α_δ is the grading associated with a given function $\delta : E^1 \to \{0,1\}$, and A^δ_E is the $E^0 \times E^0$ matrix with entries $A^\delta_E(v,w) = \sum_{e \in vE^1w} \delta(e)$, then the graded K-theory groups are isomorphic to the cokernel and kernel of $1 - (A^\delta_E)^t$ regarded as an endomorphism of $\mathbb{Z}E^0$.

In this paper we compute both the graded K-theory and the graded K-homology of relative Cuntz–Krieger algebras of arbitrary graphs: Let V be any subset of *regular vertices* E^0_{rg} (those which receive a nonzero finite set of edges). The relative Cuntz–Krieger algebra $C^*(E;V)$ is the universal C^*-algebra in which the Cuntz–Krieger relation is only imposed at vertices in V. In particular $C^*(E) = C^*(E; E^0_{\mathrm{rg}})$. Let A^δ_V be the $V \times E^0$ matrix with entries given by the same formula as A^δ_E, regarded as a homomorphism from $\mathbb{Z}E^0$ to $\mathbb{Z}V$. Write $\widetilde{A}^\delta_V$ for the dual homomorphism from $\mathbb{Z}^{E^0}$ to $\mathbb{Z}^V$. Let $\iota : \mathbb{Z}V \to \mathbb{Z}E^0$ be the inclusion map, and let $\pi : \mathbb{Z}^{E^0} \to \mathbb{Z}^V$ be the projection map. Our theorem states that the graded K-theory groups and K-homology groups of the relative Cuntz–Krieger algebra are given by

$$K_0^{\mathrm{gr}}(C^*(E;V),\alpha_\delta) \cong \operatorname{coker}(\iota - A^\delta_V)^t, \qquad K_1^{\mathrm{gr}}(C^*(E;V),\alpha_\delta) \cong \ker(\iota - A^\delta_V)^t,$$
$$K^0_{\mathrm{gr}}(C^*(E;V),\alpha_\delta) \cong \ker(\pi - \widetilde{A}^\delta_V), \qquad K^1_{\mathrm{gr}}(C^*(E;V),\alpha_\delta) \cong \operatorname{coker}(\pi - \widetilde{A}^\delta_V).$$

To prove this, we use that $C^*(E;V)$ may be realised as a relative Cuntz–Pimsner algebra of a graph module $X(E)$. We verify that the two assumptions (namely injectivity and compactness) imposed on the left actions of A on a Hilbert module X in the arguments of [20, 26] are not needed. As a result we obtain exact sequences in KK-theory analogous to those of [20] for relative Cuntz–Pimsner algebras. By calculating the KK-groups and the maps between them in the situation where X is the graph module $X(E)$, we obtain the desired calculations of graded K-theory and K-homology for relative graph C^*-algebras, substantially generalising the results in [20].

We then present an alternative calculation using Muhly and Tomforde's adding-tails construction. In this approach, Pimsner's exact sequences are needed only for modules where the homomorphism implementing the left action is injective. Given an arbitrary nondegenerate C^*-correspondence X, we add an infinite direct sum of copies of the Katsura ideal J_X to both the coefficient algebra and the module X to obtain a new module

Y over a new algebra B which acts injectively on the left. We then recover the exact sequences for $\mathcal{O}_X$ from the ones we already have for $\mathcal{O}_Y$ using countable additivity in KK-theory. This is automatic in the first variable, so we obtain a complete generalisation of the contravariant exact sequence of [20] for $\mathcal{O}_X$; but, KK-theory is not in general countably additive in the second variable. However $KK(\mathbb{C}, \cdot)$ is countably additive for graded C^*-algebras (we could not find a reference, so we give the details) so we recover the exact sequence describing the graded K-theory $KK_i(\mathbb{C}, \mathcal{O}_X)$.

We begin in Section 1 with some background on KK-theory, mostly to establish notation. Detailed background on KK-theory can be found in [20], and of course in Blackadar's book [2], which is our primary reference. We assume the reader is familiar with Hilbert modules and graph C^*-algebras. A convenient summary of the requisite background appears in [20], and more details can be found in [21, 26, 27, 29]. We also provide a little background on the relative Cuntz–Pimsner algebras of Muhly and Solel [22], of which Katsura's Katsura–Pimsner algebras are a special case. In Section 2 we show how to generalise the results of [20] to relative Cuntz–Pimsner algebras of arbitrary essential graded Hilbert modules. In Section 3 we apply these results to compute the graded K-theory and K-homology of relative Cuntz–Krieger algebras of arbitrary graphs. In Section 4, we show how Muhly and Tomforde's adding-tails construction for Hilbert modules can be adapted to graded modules, and reconcile this with our K-theory and K-homology results for the graded Katsura–Pimsner algebra of a nondegenerate Hilbert module.

1. Background material

In this section we provide some background on KK-theory, relative graph C^*-algebras and terminology used in the later sections.

1.1. **Direct sums and products of groups.** Let S be any countable set. We let $\mathbb{Z}S$ denote the direct sum $\bigoplus_{s\in S}\mathbb{Z}$ of copies of $\mathbb{Z}$ (the group of all finitely supported functions from S to $\mathbb{Z}$), while $\mathbb{Z}^S$ denotes the direct product $\prod_{s\in S}\mathbb{Z}$ of copies of $\mathbb{Z}$ (the group of all functions from S to $\mathbb{Z}$). More generally we write $\prod_{n=1}^{\infty} G_n$ for the infinite product of a sequence of groups G_n, and $\bigoplus_{n=1}^{\infty} G_n$ for the subgroup generated by the G_n.

1.2. **Hilbert modules.** Given a C^*-algebra B and a right Hilbert B-module X, we write $\mathcal{L}(X)$ for the adjointable operators on X, we write $\mathcal{K}(X)$ for the generalised compact operators on X, and given $\xi, \eta \in X$ we write $\Theta_{\xi,\eta}$ for the compact operator $\zeta \mapsto \xi \cdot \langle \eta, \zeta\rangle_B$. If $\phi : A \to \mathcal{L}(X)$ is a C^*-homomorphism so that X is an A–B-correspondence, we say that the left action is *injective* if ϕ is injective and that the left action is *by compact operators* if $\phi(A) \subseteq \mathcal{K}(X)$.

Let I be an ideal of a C^*-algebra A, and X a right Hilbert A-module. Following [17], we define $XI := \{x \in X : \langle x, x\rangle \in I\}$. This XI is a right Hilbert I-module under the same operations as Y, and $XI = X \cdot I := \{x \cdot i : x \in X, i \in I\}$, justifying the notation.

1.3. **Gradings.** A *grading* of a C^*-algebra is a self-inverse automorphism α of A, and decomposes A into direct summands $A_0 = \{a : \alpha(a) = a\}$ and $A_1 = \{a : \alpha(a) = -a\}$. We write $\partial(a) = i$ if $a \in A_i$. If a, b are homogeneous, then their graded commutator is $[a, b]^{\mathrm{gr}} := ab - (-1)^{\partial(a)\partial(b)}ba$, and we extend this formula bilinearly to arbitrary $a, b \in A$. Elements of $A_0 \cup A_1$ are called *homogeneous*, elements of A_0 are *even* and elements of A_1 are *odd*. A homomorphism of graded C^*-algebras is a graded homomorphism if it intertwines the grading automorphisms.

A *grading* of a C^*-correspondence X over graded C^*-algebras (A, α_A) and (B, α_B) is a map $\alpha_X : X \to X$ such that $\alpha_X^2 = \mathrm{id}$, $\alpha_X(a \cdot x \cdot b) = \alpha_A(a) \cdot \alpha_X(a) \cdot \alpha_B(b)$, and $\alpha_B(\langle x, y\rangle_B) = \langle \alpha_X(x), \alpha_X(y)\rangle_B$. There is a grading $\tilde{\alpha}_X$ on $\mathcal{L}(X)$ given by $\tilde{\alpha}_X(T) = \alpha_X \circ T \circ \alpha_X$. Again, X decomposes as the direct sum of $X_0 = \{\xi : \alpha_X(\xi) = \xi\}$ and $X_1 = \{\xi : \alpha_X(\xi) = -\xi\}$; we call elements of the X_i homogeneous, odd and even as above.

For a graded C^*-algebra B set $\mathbb{I}B := C([0, 1]) \mathbin{\widehat{\otimes}} B$ with the trivial grading on $C([0, 1])$. For each $t \in [0, 1]$ the homomorphism $\epsilon_t : \mathbb{I}B \to B$ given by $\epsilon_t(f \mathbin{\widehat{\otimes}} b) = f(t)b$ is graded. It follows that B may be regarded as a graded $\mathbb{I}B$–B-correspondence (ϵ_t, B_B).

1.4. **Graded tensor products.** The *graded tensor product* of graded C^*-algebras A and B is the minimal C^*-completion of the algebraic tensor product $A \odot B$ with involution satisfying $(a \mathbin{\widehat{\otimes}} b)^* = (-1)^{\partial a \cdot \partial b} a^* \mathbin{\widehat{\otimes}} b^*$ and multiplication satisfying $(a \mathbin{\widehat{\otimes}} b)(a' \mathbin{\widehat{\otimes}} b') = (-1)^{\partial b \cdot \partial a'} aa' \mathbin{\widehat{\otimes}} bb'$ for homogeneous elements $a, a' \in A$ and $b, b' \in B$. There is a grading $\alpha_A \mathbin{\widehat{\otimes}} \alpha_B$ on $A \mathbin{\widehat{\otimes}} B$ such that $(\alpha_A \mathbin{\widehat{\otimes}} \alpha_B)(a \mathbin{\widehat{\otimes}} b) = \alpha_A(a) \mathbin{\widehat{\otimes}} \alpha_B(b)$. The balanced tensor product $X \otimes_B Y$ of graded C^*-correspondences admits a grading $\alpha_X \mathbin{\widehat{\otimes}} \alpha_Y$ on $X \otimes_\psi Y$ such that $(\alpha_X \mathbin{\widehat{\otimes}} \alpha_Y)(x \mathbin{\widehat{\otimes}} y) = \alpha_X(x) \mathbin{\widehat{\otimes}} \alpha_Y(y)$.

1.5. **Relative Cuntz–Pimsner algebras.** Let X be a graded A–A-correspondence, and write $\varphi : A \to \mathcal{L}(X)$ for the homomorphism inducing the left action.

A *representation* of X in a C^*-algebra B is a pair (π, ψ) consisting of a homomorphism $\pi : A \to B$ and a linear map $\psi : X \to B$ such that $\pi(a)\psi(\xi)\pi(b) = \psi(a \cdot x \cdot b)$ and $\pi(\langle \xi, \eta\rangle_A) = \psi(\xi)^*\psi(\eta)$ for all $a, b \in A$ and $\xi, \eta \in X$. The *Toeplitz algebra* $\mathcal{T}_X$ of X is the universal C^*-algebra generated by a representation of X. Such a representation induces a homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ satisfying $\psi^{(1)}(\Theta_{\xi,\eta}) = \psi(\xi)\psi(\eta)^*$ for all ξ, η. The universal property of $\mathcal{T}_X$ gives a grading $\alpha_{\mathcal{T}}$ of $\mathcal{T}_X$ such that if (π, ψ) is the universal representation of X in $\mathcal{T}_X$, then $\alpha_{\mathcal{T}} \circ \pi = \pi \circ \alpha_A$, and $\alpha_{\mathcal{T}} \circ \psi = \psi \circ \alpha_X$.

Let $C := \varphi^{-1}(\mathcal{K}(X))$; observe that if (π, ψ) is a representation of X, then both π and $\psi^{(1)} \circ \varphi$ are homomorphisms from C to $\mathcal{T}_X$. Given an ideal $I \subseteq C$, the relative Cuntz–Pimsner algebra $\mathcal{O}_{X,I}$ is defined to be the universal C^*-algebra generated by a representation (π, ψ) that is I-*covariant* in the sense that $\pi|_I = (\psi^{(1)} \circ \varphi)|_I$. If I is invariant under the grading α_A of A, then the universal property of $\mathcal{O}_{X,I}$ shows that the grading $\alpha_{\mathcal{T}}$ of $\mathcal{T}_X$ descends to a grading $\alpha_{\mathcal{O}}$ of $\mathcal{O}_{X,I}$.

The *Fock space* $\mathcal{F}_X$ is the internal direct sum $\mathcal{F}_X := \bigoplus_{n=0}^\infty X^{\widehat{\otimes} n}$, with the convention that $X^{\widehat{\otimes} 0} = {}_A A_A$. There is a representation (ℓ_0, ℓ_1) of X in $\mathcal{L}(\mathcal{F}_X)$ such that $\ell_0(a)\xi = a \cdot \xi$ and such that $\ell_1(\xi)\eta = \xi \otimes_A \eta$. The induced homomorphism $\pi_0 : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X)$ is injective.

The ideal $C = \varphi^{-1}(\mathcal{K}(X))$ of A induces the submodule $\mathcal{F}_{X,C} := \mathcal{F}_X C$. The subalgebra $\mathcal{K}(\mathcal{F}_{X,C}) := \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in \mathcal{F}_{X,C}\} \subseteq \mathcal{L}(\mathcal{F}_X)$ is contained in $\pi_0(\mathcal{T}_X)$ (see [22, Lemma 2.17]). Since $\pi_0 : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X)$ is injective, we obtain an inclusion $j : \mathcal{K}(\mathcal{F}_{X,C}) \to \mathcal{T}_X$. In particular, for any ideal $I \subseteq C$, this j restricts to a graded inclusion $\mathcal{K}(\mathcal{F}_{X,I}) \hookrightarrow \mathcal{T}_X$. Theorem 2.19 in [22] and an application of universal properties show that the quotient map $\mathcal{T}_X \to \mathcal{O}_{X,I}$ induces an isomorphism $\mathcal{T}_X / j(\mathcal{K}(\mathcal{F}_{X,I})) \cong \mathcal{O}_{X,I}$.

1.6. **Kasparov modules.** If (A, α_A) and (B, α_B) are separable graded C^*-algebras, then a *Kasparov A–B-module* is a quadruple (X, ϕ, F, α_X) where (ϕ, X) is a countably generated A–B-correspondence, α_X is a grading of X, and $F \in \mathcal{L}(X)$ is an odd element with respect to the grading $\tilde{\alpha}_X$ on $\mathcal{L}(X)$ such that for all $a \in A$ the elements $(F - F^*)\phi(a)$,

$(F^2-1)\phi(a)$, and $[F,\phi(a)]^{\mathrm{gr}}$ are compact. When these elements are all zero we call (X,ϕ,F,α_X) a *degenerate* Kasparov module.

Kasparov A–B-modules $(X_0,\phi_0,F_0,\alpha_{X_0})$ and $(X_1,\phi_1,F_1,\alpha_{X_1})$ are *unitarily equivalent* if there is a unitary $U\in\mathcal{L}(X_0,X_1)$ that intertwines ϕ_0 and ϕ_1, F_0 and F_1, and α_0 and α_1. They are *homotopy equivalent* if there is Kasparov A–$\mathbb{I}B$-module (X,ϕ,F,α_X) such that, $(X\mathbin{\widehat{\otimes}_{\epsilon_i}}B_B,\tilde{\epsilon}_i\circ\phi,\tilde{\epsilon}_i(F),\alpha_X\mathbin{\widehat{\otimes}}\alpha_B)$ is unitarily equivalent to $(X_i,\phi_i,F_i,\alpha_{X_i})$ for each of $i=0,1$. Homotopy equivalence is denoted $\sim_h$, and is an equivalence relation. The Kasparov group $KK(A,B)$ is the collection of all homotopy classes of Kasparov A–B-modules, which is a group under the operation induced by taking direct sums of Kasparov modules. Given a graded homomorphism $\psi:A\to B$ of C^*-algebras, and a Kasparov B–C-module (X,ϕ,F,α_X), we obtain a new Kasparov A–C-module $(X,\phi\circ\psi,F,\alpha_X)$, whose class we denote $\psi^*[X]$. For a graded homomorphism $\psi\colon B\to C\cong\mathcal{K}(C_C)$ we let $[\psi]:=[C_C,\psi,0,\alpha_C]\in KK(B,C)$. If $\phi\colon A\to B$ is a graded homomorphism and (Y,ψ,G,α_Y) is a Kasparov C–A-module, then $(Y\mathbin{\widehat{\otimes}_\phi}B_B,\psi\mathbin{\widehat{\otimes}}1,G\mathbin{\widehat{\otimes}}1,\alpha_Y\mathbin{\widehat{\otimes}}\alpha_B)$ is a Kasparov C–B-module whose class we denote by $\phi_*[Y,\psi,G,\alpha_Y]$. We write $KK_0(A,B):=KK(A,B)$ and $KK_1(A,B):=KK(A\widehat{\otimes}\mathbb{C}\mathrm{liff}_1,B)$ where $\mathbb{C}\mathrm{liff}_1$ is the first complex Clifford algebra. We let $KK_*(A,B)$ represent either $KK_0(A,B)$ or $KK_1(A,B)$. More generally each $KK_n(A,B),n\in\mathbb{N}$ is defined with periodicity $KK_n(A,B)\cong KK_{n\pmod 2}(A,B)$.

We will need to use that graded Morita equivalence implies KK-equivalence. Suppose that (A,α_A) and (B,α_B) are graded C^*-algebras, and (X,α_X) is a graded imprimitivity A–B-module. Then the left action of A on X is implemented by a homomorphism $\varphi:A\to\mathcal{K}(X)$, and the left action of B on the dual module X^* is implemented by a homomorphism $\psi:B\to\mathcal{K}(X^*)$. So we obtain KK-classes $[X]:=[X,\varphi,0,\alpha_X]\in KK(A,B)$ and $[X^*]:=[X^*,\psi,0,\alpha_{X^*}]\in KK(B,A)$. Since the Fredholm operators in both of these Kasparov modules are zero, we have $[X]\mathbin{\widehat{\otimes}_B}[X^*]=[X\otimes_B X^*,\phi\otimes 1,0,\alpha_X\otimes\alpha_{X^*}]$. Since X is an imprimitivity module, we have $X\otimes X^*\cong A$ via $\xi\otimes\eta\mapsto {}_A\langle\xi,\eta\rangle$, and it is routine to check that this isomorphism intertwines the canonical left action $\tilde{\varphi}$ of A on $X\otimes X^*$ with the left action L_A of A on itself by left multiplication, and intertwines $\alpha_X\otimes\alpha_{X^*}$ with α_A. So $[X]\mathbin{\widehat{\otimes}_B}[X^*]=[A,L_A,0,\alpha_A]=[\mathrm{id}_A]$. Similarly $[X^*]\mathbin{\widehat{\otimes}_A}[X]=[\mathrm{id}_B]$, and so (A,α_A) and (B,α_B) are KK-equivalent.

2. Graded K-theory of relative Cuntz–Pimsner algebras

In this section we generalise the main results in Section 4 of [20] establishing exact sequences in KK-theory for graded relative Cuntz–Pimsner algebras associated to an essential graded A–A-correspondence. We do not assume the action is injective, nor compact nor that X is full, but we do assume that X is *essential* (or *nondegenerate*) in the sense that $\overline{\varphi(A)X}=X$.

Set-up. *Throughout this section we fix a graded, separable, nuclear C^*-algebra A and a graded countably generated essential A–A-correspondence X with a left action φ and we fix an ideal $I\subseteq\varphi^{-1}(\mathcal{K}(X))$.*

For readers interested in the Katsura–Pimsner algebra $\mathcal{O}_X$ ([17, Definition 3.5]) we recall that it coincides with the relative Cuntz–Pimsner algebra $\mathcal{O}_{X,I}$ for $I=\varphi^{-1}(\mathcal{K}(X))\cap\ker\varphi^\perp$, where $\ker\varphi^\perp=\{b\in A: b\ker\varphi=\{0\}\}$.

We present terminology of [20] relevant to Lemma 2.1. Let $\mathcal{F}_X$ be the Fock space of X, let α_X^∞ be the diagonal grading on $\mathcal{F}_X$, and let $\varphi^\infty:A\to\mathcal{L}(\mathcal{F}_X)$ be the diagonal left action of A on $\mathcal{F}_X$. Recall that $\mathcal{T}_X$ is the Toeplitz algebra associated to X, generated by

$i_A(a) = \varphi^\infty(a)$ and $i_X(\xi) = T_\xi$, and $\alpha_{\mathcal{T}}$ is the restriction of $\tilde{\alpha}_X^\infty$ to $\mathcal{T}_X$. Let $\pi_i : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X)$ be the representations determined by

$$\pi_0(T_\xi)\rho = \begin{cases} \xi \mathbin{\widehat{\otimes}} \rho, & \rho \in X^{\widehat{\otimes} n}, n \geq 1, \\ \xi \cdot \rho, & \rho \in A, \end{cases}, \quad \pi_1(T_\xi)\rho = \begin{cases} \xi \mathbin{\widehat{\otimes}} \rho, & \rho \in X^{\widehat{\otimes} n}, n \geq 1, \\ 0, & \rho \in A. \end{cases}$$

As presented in [20, Section 4] there is a Kasparov $\mathcal{T}_X$–A-module given by

$$M = \left(\mathcal{F}_X \oplus \mathcal{F}_X, \pi_0 \oplus (\pi_1 \circ \alpha_{\mathcal{T}}), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{smallmatrix}\right)\right).$$

Recall, for the canonical inclusion $i_A : A \hookrightarrow \mathcal{T}_X$ and a graded C^*-algebra B we have

$$[i_A] = [(\mathcal{T}_X, i_A, 0, \alpha_{\mathcal{T}})] \in KK(A, \mathcal{T}_X), \text{ and } \quad [\mathrm{id}_B] = [B, \mathrm{id}_B, 0, \alpha_B] \in KK(B, B).$$

By [20, Theorem 4.2], if φ is injective and by compact operators, then the Kasparov classes $[i_A]$ and $[M]$ are mutually inverse in the sense that $[i_A] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = [\mathrm{id}_A]$ and $[M] \mathbin{\widehat{\otimes}}_A [i_A] = [\mathrm{id}_{\mathcal{T}_X}]$. We prove that this result remais true without assuming φ is injective or by compact operators.

Lemma 2.1 (cf. [20, Theorem 4.2])**.** *With notation as above, the pair $[i_A]$ and $[M]$ are mutually inverse. In particular, (A, α_A) and $(\mathcal{T}_X, \alpha_{\mathcal{T}})$ are KK-equivalent.*

Proof. The argument in the proof of [20, Theorem 4.2] showing that $[i_A] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] = [\mathrm{id}_A]$ does not require injectivity nor compactness of the left action φ.

To show $[M] \mathbin{\widehat{\otimes}}_A [i_A] = [\mathrm{id}_{\mathcal{T}_X}]$, we adjust the proof of [20, Theorem 4.2]. Let $\pi_0' := \pi_0 \mathbin{\widehat{\otimes}} 1_{\mathcal{T}}$ and $\pi_1' := (\pi_1 \circ \alpha_{\mathcal{T}}) \mathbin{\widehat{\otimes}} 1_{\mathcal{T}}$. By [2, Proposition 18.7.2(a)], identifying $(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$ with $(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X) \oplus (\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$,

$$[M] \mathbin{\widehat{\otimes}}_A [i_A] = (i_A)_*[M] = \left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, \pi_0' \oplus \pi_1', \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{smallmatrix}\right)\right].$$

Since X is essential we have $A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X \cong \mathcal{T}_X$ as graded A–$\mathcal{T}_X$-correspondences, and so $\alpha_{\mathcal{T}}$ defines a left action of $\mathcal{T}_X$ on $A \mathbin{\widehat{\otimes}} \mathcal{T}_X$. Extending by zero, we get an action τ of $\mathcal{T}_X$ on $\mathcal{F}_X \mathbin{\widehat{\otimes}} \mathcal{T}_X$. The proof of [20, Theorem 4.2] shows that

$$\left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, 0 \oplus \tau, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{smallmatrix}\right)\right] = -[\mathrm{id}_{\mathcal{T}_X}],$$

and hence

$$(i_A)_*[M] - [\mathrm{id}_{\mathcal{T}_X}] = \left[(\mathcal{F}_X \oplus \mathcal{F}_X) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X, \pi_0' \oplus (\pi_1' + \tau), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} & 0 \\ 0 & -\alpha_X^\infty \widehat{\otimes} \alpha_{\mathcal{T}} \end{smallmatrix}\right)\right]. \tag{2.1}$$

We claim (2.1) is the class of a degenerate Kasparov module. To show this, for each $t \in [0,1]$ define $\psi_t : X \to \mathcal{L}(\mathcal{F}_X \mathbin{\widehat{\otimes}} \mathcal{T}_X)$ by

$$\psi_t(\xi) = \cos(t\pi/2)(\pi_0'(\alpha_{\mathcal{T}}(T_\xi)) - \pi_1'(T_\xi)) + \sin(t\pi/2)\tau(\xi) + \pi_1'(T_\xi).$$

With $\tilde{\varphi}^\infty := \varphi^\infty \mathbin{\widehat{\otimes}} 1_{\mathcal{T}_X} : A \to \mathcal{L}(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$ we have for each $t \in [0,1]$ a Toeplitz representation $(\tilde{\varphi}^\infty \circ \alpha_A, \psi_t)$ of X. Hence for each $t \in [0,1]$ there is a homomorphism $\pi_t' : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$ such that $\pi_t'(T_\xi) = \psi_t(\xi)$ and $\pi_t'(a) = \tilde{\varphi}^\infty \circ \alpha_A(a)$ for all $\xi \in X$ and $a \in A$. We claim that $K_{t,\xi} := (\pi_t' - \pi_1')(T_\xi)$ is compact for each $\xi \in X$. To see this, note that $K_{t,\xi}$ vanishes on $(\mathcal{F}_X \ominus A) \mathbin{\widehat{\otimes}}_A \mathcal{T}_X = (\mathcal{F}_X \mathbin{\widehat{\otimes}}_A \mathcal{T}_X) \ominus (A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X)$, and has range contained in the subspace $A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$. Thus we need only need to show that $K_{t,\xi}$ is compact on $A \mathbin{\widehat{\otimes}}_A \mathcal{T}_X$. To show this recall that for an A–B-correspondence Y with left action $\psi : A \to \mathcal{L}(Y)$, putting $J := \psi^{-1}(\mathcal{K}(Y))$, for each $k \in \mathcal{K}(AJ)$ the operator $k \mathbin{\widehat{\otimes}} 1_Y$ is compact on $A \mathbin{\widehat{\otimes}}_\psi Y$, see [21, Proposition 4.7]. With $Y := \mathcal{T}_X$, $\psi := i_A = \varphi^\infty$ and

$J := i_A^{-1}(\mathcal{K}(\mathcal{T}_X)) = A$ it follows that each $\tilde{\varphi}^\infty(a)|_{A\widehat{\otimes}_A\mathcal{T}_X}$ is compact (because $\varphi^\infty(a)|_A$ is compact). Use [29, Proposition 2.31] to express $\xi = y \cdot \langle y, y\rangle$. We compute

$$K_{t,\xi} = \pi'_t(T_\xi) - \pi'_1(T_\xi) = (\pi'_t(T_y) - \pi'_1(T_y))\tilde{\varphi}^\infty(\langle y, y\rangle),$$

which is compact since $\tilde{\varphi}^\infty(\langle y, y\rangle)|_{A\widehat{\otimes}_A\mathcal{T}_X}$ is compact. As in [20], $\psi_0(\xi) = \pi'_0 \circ \alpha_{\mathcal{T}}(T_\xi)$ and $\psi_1(\xi) = (\pi'_1 + \tau)(T_\xi)$, so we can replace $\pi'_1 + \tau$ with $\pi'_0 \circ \alpha_{\mathcal{T}}$ in the expression (2.1) for $(i_A)_*[M] - [\mathrm{id}_{\mathcal{T}_X}]$ without changing the class. The latter is the class of a degenerate Kasparov module, proving the claim. Thus $[M] \widehat{\otimes}_A [i_A] = [\mathrm{id}_{\mathcal{T}_X}]$. □

Let $I \subseteq \varphi^{-1}(\mathcal{K}(X))$ be the ideal of A consisting of elements that act as compact operators on the left of X. Let $\mathcal{F}_{X,I}$ denote the right Hilbert I-module $\mathcal{F}_X I := \{\xi \in \mathcal{F}_X : \langle \xi, \xi\rangle_A \in I\}$. Let $\iota_I : I \hookrightarrow A$ and $\iota_{\mathcal{F}I} : \mathcal{K}(\mathcal{F}_{X,I}) \hookrightarrow \mathcal{L}(\mathcal{F}_X)$ be the inclusion maps. As discussed in Section 1, $\iota_{\mathcal{F}I}(\mathcal{K}(\mathcal{F}_{X,I}))$ is contained in the image of $\pi_0(\mathcal{T}_X) \subseteq \mathcal{L}(\mathcal{F}_X)$ of the Toeplitz algebra under its canonical representation on the Fock space. Thus there is a graded embedding $j : \mathcal{K}(\mathcal{F}_{X,I}) \hookrightarrow \mathcal{T}_X$ such that $\pi_0 \circ j = \iota_{\mathcal{F}I}$. We have the induced Kasparov classes

$$[\iota_I] = [A_A, \iota_I, 0, \alpha_A] \in KK(I, A), \qquad [j] = [\mathcal{T}_X, j, 0, \alpha_{\mathcal{T}}] \in KK(\mathcal{K}(\mathcal{F}_{X,I}), \mathcal{T}_X).$$

Writing $P : \mathcal{F}_X \to \mathcal{F}_X \ominus A = \bigoplus_{n=1}^\infty X^{\widehat{\otimes} n}$ for the projection onto the orthogonal complement of the 0^{th} summand, we have

$$[M] = \left[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), \pi_0 \oplus \pi_1 \circ \alpha_{\mathcal{T}}, \left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{pmatrix}\right] \in KK(\mathcal{T}_X, A).$$

We denote by $[{}_I X]$ the class $[X, \varphi|_I, 0, \alpha_X] \in KK(I, A)$ of the module X with the left action restricted to I. Note that $\iota_{\mathcal{F}I}$ induces the Kasparov class $[\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \in KK(\mathcal{K}(\mathcal{F}_{X,I}), I)$, where α_X^∞ and each $\iota_{\mathcal{F}I}(a)$ are regarded as operators on $\mathcal{F}_{X,I} \subseteq \mathcal{F}_X$.

Lemma 2.2 (cf. [20, Lemma 4.3])**.** *With notation as above,*

$$[j] \widehat{\otimes}_{\mathcal{T}_X} [M] = [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \widehat{\otimes}_I ([\iota_I] - [{}_I X]).$$

Proof. Using [2, Proposition 18.7.2(b)] we can express $[j] \widehat{\otimes}_{\mathcal{T}_X} [M]$ as

$$[j] \widehat{\otimes}_{\mathcal{T}_X} [M] = \left[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), (\pi_0 \oplus (\pi_1 \circ \alpha_{\mathcal{T}})) \circ j, \left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right), \begin{pmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{pmatrix}\right].$$

Let $\alpha_{\mathcal{K}}$ be the restriction of $\tilde{\alpha}_X^\infty$ to $\mathcal{K}(\mathcal{F}_{X,I})$. Since $\pi_0 \circ j = \iota_{\mathcal{F}I}$ and $\pi_1 \circ \alpha_{\mathcal{T}} \circ j = \pi_1 \circ j \circ \alpha_{\mathcal{K}}$ take values in $\mathcal{K}(\mathcal{F}_{X,I})$, the straight line path from $\left(\begin{smallmatrix} 0 & 1 \\ P & 0 \end{smallmatrix}\right)$ to 0 determines an operator homotopy of Kasparov modules. Hence we may write

$$\begin{aligned}[j] \widehat{\otimes}_{\mathcal{T}_X} [M] &= \left[\mathcal{F}_X \oplus (\mathcal{F}_X \ominus A), (\pi_0 \oplus (\pi_1 \circ \alpha_{\mathcal{T}})) \circ j, 0, \begin{pmatrix} \alpha_X^\infty & 0 \\ 0 & -\alpha_X^\infty \end{pmatrix}\right] \\ &= [\mathcal{F}_X, \pi_0 \circ j, 0, \alpha_X^\infty] + [\mathcal{F}_X \ominus A, \pi_1 \circ j \circ \alpha_{\mathcal{K}}, 0, -\alpha_X^\infty] \\ &= [\mathcal{F}_X, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] - [\mathcal{F}_X \ominus A, \pi_1 \circ j, 0, \alpha_X^\infty]. \qquad (2.2)\end{aligned}$$

Since $\overline{\mathcal{K}(\mathcal{F}_{X,I})\mathcal{F}_X} = \mathcal{F}_{X,I}$ and $\overline{\mathcal{K}(\mathcal{F}_{X,I})(\mathcal{F}_X \ominus A)} = \mathcal{F}_{X,I} \ominus A$ we may (see [2, Section 17.5]) replace each module in (2.2) with its essential submodule without changing the Kasparov classes, to obtain

$$[j] \widehat{\otimes}_{\mathcal{T}_X} [M] = [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] - [\mathcal{F}_{X,I} \ominus A, \pi_1 \circ j, 0, \alpha_X^\infty]. \qquad (2.3)$$

The map $(\xi \otimes a) \mapsto \xi \cdot a$ implements a unitary equivalence

$$(\mathcal{F}_{X,I} \widehat{\otimes}_I A_A, \iota_{\mathcal{F}I} \widehat{\otimes} 1, 0, \alpha_X^\infty \widehat{\otimes} \alpha_A) \cong (\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty).$$

So, writing $[\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty]_A$ for the class in $KK(\mathcal{T}_X, A)$ obtained by regarding $\mathcal{F}_{X,I}$ as a right A-module, and $[\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty]_I$ for the class obtained by regarding it as a right I-module, and recalling that $[\iota_I] \in KK(I, A)$ is the class of the inclusion of I in A, we have

$$[\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty]_I \mathbin{\widehat{\otimes}}_I [\iota_I] = [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty]_A.$$

Since X is essential, the map that sends $i \cdot \xi$ to $i \otimes \xi$ for $i \in I$ and $\xi \in X$, and sends $\xi_1 \otimes \cdots \otimes \xi_n$ to $(\xi_1 \otimes \cdots \otimes \xi_{n-1}) \otimes \xi_n$ for $\xi_1, \dots, \xi_n \in X$ is a unitary equivalence

$$(\mathcal{F}_{X,I} \ominus A, \pi_1 \circ j, 0, \alpha_X^\infty) \cong (\mathcal{F}_{X,I} \mathbin{\widehat{\otimes}}_I X, (\pi_0 \circ j) \mathbin{\widehat{\otimes}} 1, 0, \alpha_X^\infty \mathbin{\widehat{\otimes}} \alpha_X).$$

By [2, Proposition 18.10.1], we have

$$[\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}} [{}_I X] = [\mathcal{F}_{X,I} \mathbin{\widehat{\otimes}}_I X, \iota_{\mathcal{F}I} \mathbin{\widehat{\otimes}} 1, 0, \alpha_X^\infty \mathbin{\widehat{\otimes}} \alpha_X]$$

provided that $(\iota_{\mathcal{F}I} \mathbin{\widehat{\otimes}} 1)(\mathcal{K}(\mathcal{F}_{X,I})) \subseteq \mathcal{K}(\mathcal{F}_{X,I} \mathbin{\widehat{\otimes}}_I X)$. This containment is a direct consequence of [21, Proposition 4.7] since I consists of elements that act as compact operators on the left of X. It follows that

$$[\mathcal{F}_{X,I} \ominus A, \pi_1 \circ j, 0, \alpha_X^\infty] = [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}} [{}_I X].$$

Substituting both of these equalities into (2.3) and using distributivity of the Kasparov product gives

$$\begin{aligned}
[j] \mathbin{\widehat{\otimes}}_{\mathcal{T}_X} [M] &= [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] - [\mathcal{F}_{X,I} \ominus A, \pi_1 \circ j, 0, \alpha_X^\infty] \\
&= [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}}_I [\iota_I] + [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}} [{}_I X] \\
&= [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty] \mathbin{\widehat{\otimes}}_I ([\iota_I] - [{}_I X]).
\end{aligned}$$

□

Our next result appears in the first author's honours thesis [25, Theorem 7.0.3] for $I = \varphi^{-1}(\mathcal{K}(X))$, in the context where φ is injective. For the definition of the relative Cuntz–Pimsner algebra $\mathcal{O}_{X,I}$ see Section 1.

Theorem 2.3. *Let $(A, \alpha_A), (B, \alpha_B)$ be graded separable C^*-algebras and suppose that A is nuclear. Let X be a countably generated essential A–A-correspondence with left action φ, and let $I \subseteq \varphi^{-1}(\mathcal{K}(X))$ be a graded ideal of A. Let $\iota_I : I \to A$ be the inclusion map. Then we have six-term exact sequences*

$$\begin{array}{ccccc}
KK_0(B, I) & \xrightarrow{\widehat{\otimes}_A([\iota_I]-[{}_I X])} & KK_0(B, A) & \xrightarrow{i_*} & KK_0(B, \mathcal{O}_{X,I}) \\
\uparrow & & & & \downarrow \\
KK_1(B, \mathcal{O}_{X,I}) & \xleftarrow{i_*} & KK_1(B, A) & \xleftarrow{\widehat{\otimes}_A([\iota_I]-[{}_I X])} & KK_1(B, I)
\end{array}$$

and

$$\begin{array}{ccccc}
KK_0(I, B) & \xleftarrow{([\iota_I]-[{}_I X])\widehat{\otimes}_A} & KK_0(A, B) & \xleftarrow{i^*} & KK_0(\mathcal{O}_{X,I}, B) \\
\downarrow & & & & \uparrow \\
KK_1(\mathcal{O}_{X,I}, B) & \xrightarrow{i^*} & KK_1(A, B) & \xrightarrow{([\iota_I]-[{}_I X])\widehat{\otimes}_A} & KK_1(I, B)
\end{array}$$

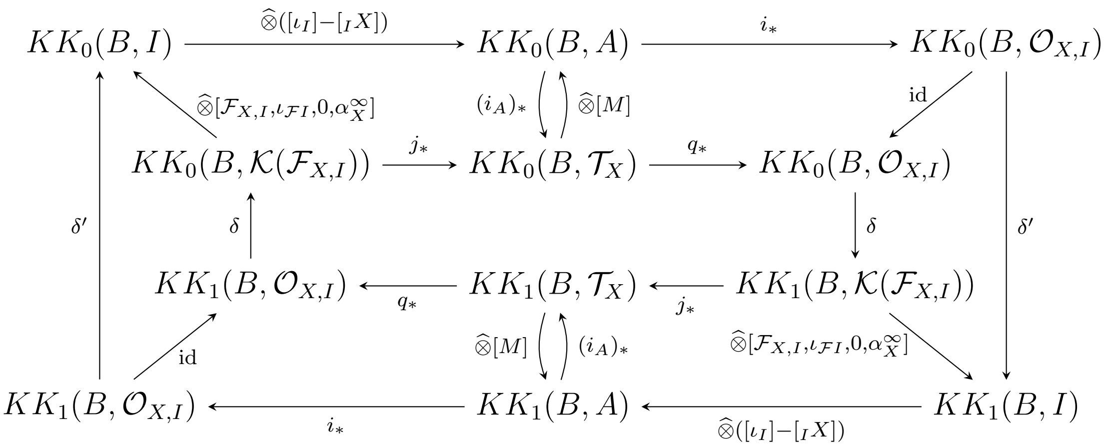
Proof. As in [20, Theorem 4.4], we shall prove exactness of the first diagram. Exactness of the second follows from a similar argument. Since A is nuclear, so is $\mathcal{T}_X$ by [30, Theorem 6.3]. Hence the quotient map $q : \mathcal{T}_X \to \mathcal{O}_{X,I} \cong \mathcal{T}_X / j(\mathcal{K}(\mathcal{F}_{X,I}))$ admits a completely positive splitting, see [2, Example 19.5.2]. Applying [31, Theorem 1.1] to the graded short exact sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_{X,I}) \xrightarrow{\ j\ } \mathcal{T}_X \xrightarrow{\ q\ } \mathcal{O}_{X,I} \longrightarrow 0$$

gives homomorphisms $\delta : KK_*(B, \mathcal{O}_{X,I}) \to KK_{*+1}(B, \mathcal{K}(\mathcal{F}_{X,I}))$ for which the following six-term sequence is exact

$$\begin{array}{ccccc} KK_0(B,\mathcal{K}(\mathcal{F}_{X,I})) & \xrightarrow{j_*} & KK_0(B,\mathcal{T}_X) & \xrightarrow{q_*} & KK_0(B,\mathcal{O}_{X,I}) \\ \delta\uparrow & & & & \downarrow\delta \\ KK_1(B,\mathcal{O}_{X,I}) & \xleftarrow{q_*} & KK_1(B,\mathcal{T}_X) & \xleftarrow{j_*} & KK_1(B,\mathcal{K}(\mathcal{F}_{X,I})) \end{array}$$

Define $\delta' : KK_*(B, \mathcal{O}_{X,I}) \to KK_{*+1}(B, A)$ by $\delta' = (\cdot \mathbin{\widehat{\otimes}} [\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty]) \circ \delta$ and let $i : A \to \mathcal{O}_{X,I}$ be the canonical homomorphism. Consider the following diagram.



By definition of δ', the left and right squares commute. Lemma 2.2 shows that the top left and bottom right squares commute. By definition we have $i = q \circ i_A$ as homomorphisms, so $i_* = (q \circ i_A)_* = q_* \circ (i_A)_*$. This shows that the top right and bottom left squares commute.

Lemma 2.1 implies that $(i_A)_*$ and $\widehat{\otimes}[M]$ are mutually inverse isomorphisms. Finally, the class of $[\mathcal{F}_{X,I}, \iota_{\mathcal{F}I}, 0, \alpha_X^\infty]$ is induced by the graded Morita equivalence bimodule $\mathcal{F}_{X,I}$ (see [29]), and so induces an isomorphism of KK-groups, so exactness of the interior six-term sequence gives the desired exactness of the exterior one. □

3. Graded K-theory and K-homology for relative Cuntz–Krieger algebras

In this section we use our results from the preceding section to calculate the graded K-theory and graded K-homology of a graded relative graph C^*-algebra.

We first recall the key elements of relative graph C^*-algebras that we will use here. Given a directed graph $E = (E^0, E^1, r, s)$, we denote by E^0_{rg} the set

$$E^0_{\mathrm{rg}} := \{v \in E^0 : vE^1 \text{ is finite and nonempty}\}$$

of *regular vertices* of E. Given a subset $V \subseteq E^0_{\mathrm{rg}}$, the *relative Cuntz–Krieger algebra* $C^*(E; V)$ of E is defined as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$, and such that $p_v = \sum_{e \in vE^1} s_e s_e^*$ for all $v \in V$. So $C^*(E;\emptyset)$ coincides with the usual Toeplitz algebra $\mathcal{T}C(E)$, and $C^*(E;E^0_{\mathrm{rg}})$ coincides with the graph C^*-algebra $C^*(E)$. Given a function $\delta : E^1 \to \{0,1\}$, the universal property of $C^*(E;V)$ yields a grading α^δ of $C^*(E;V)$ satisfying $\alpha^\delta(s_e) = (-1)^{\delta(e)} s_e$ for all $e \in E^1$, and $\alpha^\delta(p_v) = p_v$ for all $v \in E^0$.

There is a $C_0(E^0)$-valued inner-product on $C_c(E^1)$ given by

$$\langle \xi, \eta\rangle_{C_0(E^0)}(w) = \sum_{e \in E^1 w} \overline{\xi(e)}\eta(e).$$

The completion $X(E)$ of $C_c(E^1)$ in the resulting norm is a $C_0(E)$–$C_0(E)$-correspondence with actions determined by $(a \cdot \xi \cdot b)(e) = a(r(e))\xi(e)b(s(e))$. The ideal $C \subseteq C_0(E^0)$ of elements that act by compact operators on this module is $C_0(\{v : |vE^1| < \infty\})$. A routine argument using universal properties shows that for $V \subseteq E^0_{\mathrm{rg}}$, we have $\mathcal{O}_{X(E),C_0(V)} \cong C^*(E;V)$; in particular, $\mathcal{T}_{X(E)} \cong C^*(E;\emptyset) = \mathcal{T}C^*(E)$. Any map $\delta : E^1 \to \{0,1\}$ determines a grading α_X of $X(E)$ satisfying $\alpha_X(\xi)(e) = (-1)^{\delta(e)}\xi(e)$ for $\xi \in X(E)$ and $e \in E^1$, and then the induced grading on $\mathcal{O}_{X(E),C_0(V)}$ matches up with the grading α^δ on $C^*(E;V)$.

Our main result in this section is the following.

Theorem 3.1. *Let E be a directed graph. Fix a subset $V \subseteq E^0_{\mathrm{rg}}$ and a function $\delta : E^1 \to \mathbb{Z}_2$. Let $\alpha \in \mathrm{Aut}(C^*(E;V))$ be the grading such that $\alpha(s_e) = (-1)^{\delta(e)} s_e$ for all $e \in E^1$. Let A^δ_V denote the $V \times E^0$ matrix such that $A^\delta_V(v,w) = \sum_{e \in vE^1 w} (-1)^{\delta(e)}$ for all $v \in V$ and $w \in E^0$. Let $\iota : \mathbb{Z}V \to \mathbb{Z}E^0$ be the inclusion map. Regarding $(A^\delta_V)^t$ as a homomorphism from $\mathbb{Z}V$ to $\mathbb{Z}E^0$, the graded K-theory of $(C^*(E;V),\alpha)$ is given by*

$$K_0^{\mathrm{gr}}(C^*(E;V),\alpha) \cong \mathrm{coker}(\iota - (A^\delta_V))^t \quad \text{and} \quad K_1^{\mathrm{gr}}(C^*(E;V),\alpha) \cong \ker(\iota - (A^\delta_V))^t.$$

There is a homomorphism $(\widetilde{A}^\delta_V) : \mathbb{Z}^{E^0} \to \mathbb{Z}^V$ given by

$$(\widetilde{A}^\delta_V)(f)(v) = \textstyle\sum_{e \in vE^1} A^\delta_V(v,w) f(w) \quad \text{for all } v \in V \text{ and } f \in \mathbb{Z}^{E^0}.$$

Let $\pi : \mathbb{Z}^{E^0} \to \mathbb{Z}^V$ be the projection map. Then

$$K^0_{\mathrm{gr}}(C^*(E;V),\alpha) \cong \ker(\pi - \widetilde{A}^\delta_V) \quad \text{and} \quad K^1_{\mathrm{gr}}(C^*(E;V),\alpha) \cong \mathrm{coker}(\pi - \widetilde{A}^\delta_V).$$

Before proving the theorem, we state an immediate corollary about the graded K-theory and K-homology of graph C^*-algebras.

Corollary 3.2. *Let E be a directed graph. Fix a function $\delta : E^1 \to \mathbb{Z}_2$, and let α be the associated grading of $C^*(E)$. Let A^δ_{rg} denote the $E^0_{\mathrm{rg}} \times E^0$ matrix such that $A^\delta_{\mathrm{rg}}(v,w) = \sum_{e \in vE^1 w} (-1)^{\delta(e)}$ for all $v \in V$ and $w \in E^0$ regarded as a homomorphism from $\mathbb{Z}E^0_{\mathrm{rg}}$ to $\mathbb{Z}E^0$ and write $\widetilde{A}^\delta_{\mathrm{rg}}$ for the dual homomorphism $\mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0_{\mathrm{rg}}}$. Then*

$$K_0^{\mathrm{gr}}(C^*(E),\alpha) \cong \mathrm{coker}(\iota - (A^\delta_{\mathrm{rg}}))^t, \qquad K_1^{\mathrm{gr}}(C^*(E),\alpha) \cong \ker(\iota - (A^\delta_{\mathrm{rg}}))^t,$$
$$K^0_{\mathrm{gr}}(C^*(E),\alpha) \cong \ker(\pi - \widetilde{A}^\delta_{\mathrm{rg}}), \quad \text{and} \quad K^1_{\mathrm{gr}}(C^*(E),\alpha) \cong \mathrm{coker}(\pi - \widetilde{A}^\delta_{\mathrm{rg}}).$$

Proof. Apply Theorem 3.1 to $V = E^0_{\mathrm{rg}}$. □

To prove the K-homology formulas in the theorem, we need some preliminary results; the corresponding results for K-theory are established in [20].

Proposition 3.3. *Let E be a row-finite directed graph. For $\mathbf{n} \in \mathbb{Z}^{E^0}$ and $f \in C_0(E^0)$ let $\ell(f) \in \mathcal{L}\big(\bigoplus_{v\in E^0} \mathbb{C}^{|\mathbf{n}_v|}\big)$ be given by*

$$(\ell(f)x)_w = f(w)x_w \quad \text{for } w \in E^0.$$

Then there is an isomorphism $\mu : \mathbb{Z}^{E^0} \to KK_0(C_0(E^0), \mathbb{C})$ such that

$$\mu(\mathbf{n}) = \Big[\bigoplus_{v\in E^0} \mathbb{C}^{|\mathbf{n}_v|}, \ell, 0, \bigoplus_{v\in E^0} \operatorname{sign}(\mathbf{n}_v)\Big].$$

Proof. Since E^0 is discrete we can identify $C_0(E^0)$ with $\bigoplus_{v\in E^0} \mathbb{C}$. For each $w \in E^0$, let $g_w : \mathbb{C} \to \bigoplus_{v\in E^0} \mathbb{C}$ be the coordinate inclusion. Then Theorem 19.7.1 of [2] implies that $\theta := \prod_{v\in E^0} g_w^* : KK(\bigoplus_{v\in E^0} \mathbb{C}, \mathbb{C}) \to \prod_{v\in E^0} KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}^{E^0}$ is an isomorphism of groups, with inverse μ. □

Lemma 3.4. *Let E be a directed graph and let $\delta : E^1 \to \{0, 1\}$ be a function. Fix a subset $V \subseteq E^0_{\mathrm{rg}}$. Fix $v \in E^0$. For $f \in E^1 v$ and $a \in C_0(E^0)$, define $\psi^v(a) : \ell^2(E^1 v) \to \ell^2(E^1 v)$ on elementary basis vectors $\{e^f\} \subseteq \ell^2(E^1 v)$ by*

$$\psi^v(a)e^f = \begin{cases} a(r(f))e^f & \text{if } r(f) \in V \\ 0 & \text{otherwise.} \end{cases}$$

Define $\beta : \ell^2(E^1 v) \to \ell^2(E^1 v)$ on basis elements by

$$\beta(e^f) = (-1)^{\delta(f)} e^f.$$

Let $\phi_V : C_0(V) \to \mathcal{K}(X(E))$ be the restriction of the left action. Then $(\ell^2(E^1 v), \psi^v, 0, \beta)$ is a Kasparov $C_0(E^0)$–$\mathbb{C}$-module, and for each $\mathbf{n}_v \in \mathbb{Z}$ there is a unitary equivalence between the modules

$$(\ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}, \tilde{\psi}, 0, \beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v)\,\mathrm{id})$$

and

$$(X(E) \mathbin{\widehat{\otimes}_{\varepsilon_v}} \mathbb{C}^{|\mathbf{n}_v|}, \tilde{\phi}_V, 0, \alpha_X \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v)\,\mathrm{id}).$$

Proof. Throughout the proof, we write ψ for ψ^v. For each $w \in V$, the operator $\psi(\delta_w) = \sum_{f\in wE^1} \Theta_{e^f,e^f}$ is compact, and for $w \in E^0 \setminus V$, we have $\psi(\delta_w) = 0$. So each $\psi(a) = \bigoplus_{w\in V} a(w)\psi(\delta_w)$ is compact.

It is immediate that β is a grading, and it preserves the grading of the left action since $C_0(E^0)$ carries the trivial grading. So $(\ell^2(E^1 v), \psi, 0, \beta)$ is a Kasparov $C_0(V)$–$\mathbb{C}$-module.

Recall that for an edge $f \in E^1 v$ the element $\delta_f \in X(E)$ denotes the point mass at f. Further for $j \leq |\mathbf{n}_v|$ let e^j be an orthonormal basis for $\mathbb{C}^{|\mathbf{n}_v|}$. We claim there is a unitary equivalence $U : \ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|} \to X(E) \mathbin{\widehat{\otimes}_{\varepsilon_v}} \mathbb{C}^{|\mathbf{n}_v|}$ such that

$$U(e^f \mathbin{\widehat{\otimes}} e^j) = \delta_f \mathbin{\widehat{\otimes}} e^j \quad \text{for all } f \in E^1 v \text{ and } j \leq |\mathbf{n}_v|.$$

An elementary calculation shows that this formula preserves inner-products, so extends to a well defined isometry. To see that U is surjective, note that any function $x \in X(E)$ that is zero on $E^1 v$ satisfies $x \mathbin{\widehat{\otimes}} w = 0$ for any $w \in \mathbb{C}^{|\mathbf{n}_v|}$. Thus

$$\begin{aligned} X(E) \mathbin{\widehat{\otimes}_{\varepsilon_v}} \mathbb{C}^{|\mathbf{n}_v|} &= \overline{\operatorname{span}}\{e^f \mathbin{\widehat{\otimes}} e^j : f \in E^1 v, j \leq |\mathbf{n}_v|\} \\ &= \overline{\operatorname{span}}\{U(e^f \mathbin{\widehat{\otimes}} e^j) : f \in E^1 v, j \leq |\mathbf{n}_v|\} = U(\ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}). \end{aligned}$$

The definitions of ϕ_V and ψ^v show that U intertwines the left actions. To see that it preserves gradings, fix $f \in E^1$ and $j \leq |\mathbf{n}_v|$ and compute

$$\begin{aligned} U\big((\beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v)\operatorname{id})(e^f \otimes e^j)\big) &= \operatorname{sign}(\mathbf{n}_v)(-1)^{\delta(f)} U(e^f \otimes e^j) \\ &= \operatorname{sign}(\mathbf{n}_v)(-1)^{\delta(f)} \delta^f \mathbin{\widehat{\otimes}} e^j \\ &= (\alpha_X \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v)\operatorname{id})(\delta^f \mathbin{\widehat{\otimes}} e^j). \end{aligned}$$

□

Proposition 3.5. *Let E be a directed graph. Fix a subset $V \subseteq E^0_{\mathrm{rg}}$ and a function $\delta : E^1 \to \{0,1\}$. Let A^δ_V denote the $V \times E^0$ matrix such that $A^\delta_V(v,w) = \sum_{e \in vE^1w} (-1)^{\delta(e)}$ for all $v \in V$ and $w \in E^0$. Let $[X_{E,V}] = [X(E), \phi|_{C_0(V)}, 0, \alpha_X]$ be the Kasparov class of the graded graph module with left action restricted to $C_0(V)$. Let μ denote both the isomorphism $KK_0(C_0(E^0),\mathbb{C}) \to \mathbb{Z}^{E^0}$ and the isomorphism $KK_0(C_0(V),\mathbb{C}) \to \mathbb{Z}^V$ from Proposition 3.3. Let $[\iota] \in KK(J_X, A)$ be the class of the inclusion, and let $\pi : \mathbb{Z}^{E^0} \to \mathbb{Z}^V$ be the projection map. Then the following diagrams commute.*

$$\begin{array}{ccc} \mathbb{Z}^V & \xleftarrow{\;A^\delta_V\;} & \mathbb{Z}^{E^0} \\ \mu\uparrow & & \uparrow\mu \\ KK_0(C_0(V),\mathbb{C}) & \xleftarrow[{[X_{E,V}]\widehat{\otimes}}]{} & KK_0(C_0(E^0),\mathbb{C}) \end{array} \qquad \begin{array}{ccc} \mathbb{Z}^V & \xleftarrow{\;\pi\;} & \mathbb{Z}^{E^0} \\ \mu\uparrow & & \uparrow\mu \\ KK_0(C_0(V),\mathbb{C}) & \xleftarrow[{[\iota]\widehat{\otimes}}]{} & KK_0(C_0(E^0),\mathbb{C}) \end{array}$$

Proof. By surjectivity of the isomorphism μ each element of $KK_0(C_0(E^0),\mathbb{C})$ is of the form

$$\mu(\mathbf{n}) = \Big[\bigoplus_{v\in E^0} \mathbb{C}^{|\mathbf{n}_v|}, \ell, 0, \bigoplus_{v \in E^0} \operatorname{sign}(\mathbf{n}_v)\Big]$$

for some $\mathbf{n} \in \mathbb{Z}^{E^0}$. In what follows, we write ϕ_V for $\phi|_{C_0(V)}$. We compute using Lemma 3.4

$$\begin{aligned} [X(E), \phi_V, 0, \alpha_X] \mathbin{\widehat{\otimes}} \mu(\mathbf{n}) &= \Big[\bigoplus_{v\in E^0}(X(E) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}), \bigoplus_{v\in E^0} \phi \mathbin{\widehat{\otimes}} 1, 0, \bigoplus_{v\in E^0}(\alpha_X \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v))\Big] \\ &= \Big[\bigoplus_{v\in E^0} \ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}, \bigoplus_{v\in E^0} \psi^v \mathbin{\widehat{\otimes}} 1, 0, \bigoplus_{v\in E^0}(\beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v))\Big]. \end{aligned}$$

Let $g_v : \mathbb{C} \to C_0(E^0)$ be the coordinate inclusion corresponding to the vertex $v \in E^0$. Theorem 19.7.1 of [2] gives an isomorphism $\theta := \prod_{v\in E^0} g_v^* : KK(C_0(E^0),\mathbb{C}) \to \prod_{v\in E^0} KK(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}^{E^0}$. Applying θ to $[X_{E,V}] \mathbin{\widehat{\otimes}} \mu(\mathbf{n})$ gives

$$\begin{aligned} &\theta\Big(\Big[\bigoplus_{v\in E^0} \ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}, \bigoplus_{v\in E^0} \psi^v \mathbin{\widehat{\otimes}} 1, 0, \bigoplus_{v\in E^0}(\beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v))\Big]\Big) \\ &= \bigg(\Big[\bigoplus_{v\in E^0} \ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}, \Big(\bigoplus_{v\in E^0} \psi^v \mathbin{\widehat{\otimes}} 1\Big) \circ g_w, 0, \bigoplus_{v\in E^0}(\beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v))\Big]_w\bigg)_{w\in E^0}. \end{aligned} \tag{3.1}$$

Since the action $(\psi \mathbin{\widehat{\otimes}} 1) \circ g_w$ is zero on the submodule $\ell^2(E^1 v \setminus wE^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}$ for each $v, w \in E^0$, Equation (3.1) becomes

$$\begin{aligned}
&\Big(\Big[\bigoplus_{v\in E^0} \ell^2(E^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}, \Big(\bigoplus_{v\in E^0} \psi^v \mathbin{\widehat{\otimes}} 1\Big) \circ g_w, 0, \bigoplus_{v\in E^0} (\beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v))\Big]_w\Big)_{w\in E^0} \\
&\qquad = \Big(\Big[\bigoplus_{v\in E^0} \ell^2(wE^1 v) \mathbin{\widehat{\otimes}} \mathbb{C}^{|\mathbf{n}_v|}, \Big(\bigoplus_{v\in E^0} \psi^v \mathbin{\widehat{\otimes}} 1\Big) \circ g_w, 0, \bigoplus_{v\in E^0} (\beta \mathbin{\widehat{\otimes}} \operatorname{sign}(\mathbf{n}_v))\Big]_w\Big)_{w\in E^0} \\
&\qquad = \Big(\Big[\bigoplus_{v\in E^0} \mathbb{C}^{|A^\delta(w,v)\mathbf{n}_v|}, \Big(\bigoplus_{v\in E^0} \psi^v \mathbin{\widehat{\otimes}} 1\Big) \circ g_w, 0, \bigoplus_{v\in E^0} (\operatorname{sign}(A^\delta(w,v)\mathbf{n}_v))\Big]_w\Big)_{w\in E^0}. \qquad (3.2)
\end{aligned}$$

Since $\psi^v(\delta_w)$ is zero when $w \notin V$, we may pass to the essential submodule, so that (3.2) becomes

$$\begin{aligned}
&\Big(\Big[\bigoplus_{v\in E^0} \mathbb{C}^{|A^\delta(w,v)\mathbf{n}_v|}, \Big(\bigoplus_{v\in E^0} \psi \mathbin{\widehat{\otimes}} 1\Big) \circ g_w, 0, \bigoplus_{v\in E^0} (\operatorname{sign}(A^\delta(w,v)\mathbf{n}_v))\Big]_w\Big)_{w\in V} \\
&\qquad = \Big(\Big[\mathbb{C}^{|\sum_{v\in E^0} A^\delta(w,v)\mathbf{n}_v|}, \cdot, 0, \operatorname{sign}\Big(\sum_{v\in E^0} A^\delta(w,v)\mathbf{n}_v\Big)\Big]_w\Big)_{w\in V}.
\end{aligned}$$

This is exactly the representative of $A^\delta_V \mathbf{n} \in \mathbb{Z}^{E^0}$ as a module in $KK_0(\mathbb{C},\mathbb{C})^V$. Hence $[X_{E,V}] \mathbin{\widehat{\otimes}} \mu(\mathbf{n}) = \mu(A^\delta_V \mathbf{n})$. Thus the left-hand diagram commutes.

Commutativity of the right-hand diagram follows directly from the definition of μ. □

Proof of Theorem 3.1. For the $C_0(E)$–$C_0(E)$-correspondence $X(E)$, the coefficient algebra A is $C_0(E^0) = \bigoplus_{v\in E^0} \mathbb{C}$ and the ideal $I \subseteq C$ is $C_0(V) = \bigoplus_{v\in V} \mathbb{C}$. Countable additivity of K-theory (or Proposition 4.1) shows that $KK_0(\mathbb{C}, C_0(E^0)) \cong \mathbb{Z}E^0$ and $KK_0(\mathbb{C}, I) \cong \mathbb{Z}V$. The argument of [20, Lemma 8.2] shows that these isomorphisms intertwine the map $\cdot \mathbin{\widehat{\otimes}} [{}_I X(E)]$ from $KK_0(\mathbb{C}, I)$ to $KK(\mathbb{C}, A)$ with the map $(A^\delta_V)^t : \mathbb{Z}V \to \mathbb{Z}E^0$. Functoriality of $KK(\mathbb{C}, \cdot)$ shows that these isomorphisms also intertwine $[\iota]$ with the inclusion map $\iota : \mathbb{Z}V \to \mathbb{Z}E^0$. So the exact sequence of Theorem 4.4 induces the exact sequence

$$0 \to K_1^{\mathrm{gr}}(C^*(E;V), \alpha) \to \mathbb{Z}V \xrightarrow{1-(A^\delta_V)^t} \mathbb{Z}E^0 \to K_0^{\mathrm{gr}}(C^*(E;V), \alpha) \to 0,$$

and the first part of the theorem follows.

For the second statement, first note that Proposition 3.3 and Theorem 19.7.1 of [2] give isomorphisms $KK_0(A, \mathbb{C}) \cong \mathbb{Z}^{E^0}$ and $KK_0(I, \mathbb{C}) \cong \mathbb{Z}^V$, and show that $KK_1(A, \mathbb{C}) = KK(J_X, \mathbb{C}) = \{0\}$. Proposition 3.5 shows that these isomorphisms intertwine $(1 - [{}_I X(E)]) \mathbin{\widehat{\otimes}} \cdot$ with $(1 - \tilde{A}^\delta_V)$. So the exact sequence of Theorem 4.6 gives the exact sequence

$$0 \to K^0_{\mathrm{gr}}(C^*(E;V), \alpha) \to \mathbb{Z}^{E^0} \xrightarrow{1-A^\delta_V} \mathbb{Z}^V \to K^1_{\mathrm{gr}}(C^*(E;V), \alpha) \to 0. \qquad \square$$

4. Adding tails to graded correspondences

Inspired by the technique of 'adding tails' to directed graphs which transforms a directed graph into a graph without sources with a Morita equivalent C^*-algebra, Muhly and Tomforde proved that given an A–A correspondence X with non-injective left action, there is a correspondence Y over a C^*-algebra B such that the left action of B on Y is implemented by an injective homomorphism, and $\mathcal{O}_X$ is a full corner of $\mathcal{O}_Y$. If A and X are graded, these gradings extend naturally to gradings on B and Y, and the inclusion of $\mathcal{O}_X$ in $\mathcal{O}_Y$ is graded. In particular $(\mathcal{O}_X, \alpha_X)$ and $(\mathcal{O}_Y, \alpha_Y)$ are KK-equivalent, and in particular have isomorphic graded K-theory and K-homology.

In this section we show how to apply Muhly and M. Tomforde's construction to calculate graded K-theory. Specifically, we show how to recover Pimsner's six-term exact sequences for $KK(\cdot, B)$ and $KK(\mathbb{C}, \cdot)$ for any graded countably generated essential A–A-correspondence X from the corresponding sequences for graded correspondences with an injective left action (these sequences were established in the first author's honours thesis [25]). This yields the calculations of graded K-theory and K-homology of C^*-algebras of arbitrary graphs that first appeared in the first author's honours thesis [25] and the second author's honours thesis [34] respectively (see Corollary 3.2). It also provides a useful reality check for our more general results in the preceding sections; we thank Ralf Meyer for pointing us to the direct proof of Pimsner's sequences for modules with non-injective left actions employed there. To exploit the adding-tails technique, we need to know that graded K-theory and K-homology are each countably additive in the appropriate sense. We provide details below.

4.1. **Countable additivity.** In what follows, a direct sum of graded C^*-algebras is endowed with the natural direct-sum grading; so given a Kasparov class $[X] = [X, \phi, F, \alpha_X] \in KK(\bigoplus_i A_i, B)$, for each n, the inclusion map $\iota_{A_n} : A_n \hookrightarrow \bigoplus_i A_i$ is graded and induces the class $\iota^*_{A_n}[X] \in KK(A_n, B)$.

Theorem 19.7.1 of [2] shows that for separable B, $KK(\,\cdot\,, B)$ is countably additive in the following sense. If $A = \bigoplus_{n=1}^\infty A_n$ is a c_0-direct sum of separable C^*-algebras A_n, then there is an isomorphism $\overline{\zeta} : KK(A, B) \to \prod_{n=1}^\infty KK(A_n, B)$ that carries the class of a Kasparov A–B-module (X, ϕ, F, α_X) to the sequence whose nth term is the class of $(X, \phi \circ \iota_{A_n}, F, \alpha_X)$; that is, in the notation of Section 1.6, writing $[X]$ for the class of (X, ϕ, F, α_X), we have

$$\overline{\zeta}[X] = \big(\iota^*_{A_n}[X]\big)_{n=1}^\infty.$$

Identifying $KK(A, \mathbb{C})$ with $K^0_{\mathrm{gr}}(A)$, gives countable additivity of graded K-homology.

On the other hand, as discussed in [2, 19.7.2], the map $KK(B, \,\cdot\,)$ is typically not countably additive. There is a natural map ω from $\bigoplus_{i=1}^\infty KK(B, A_i)$ to $KK(B, \bigoplus_{i=1}^\infty A_i)$ such that

$$\omega(([X_i, \phi_i, F_i, \alpha_i])_{i=1}^n) = [\textstyle\bigoplus_{i=1}^n X_i, \bigoplus_{i=1}^n \phi_i, \bigoplus_{i=1}^n F_i, \bigoplus_{i=1}^n \alpha_i]. \tag{4.1}$$

This ω is always injective (see the proof of Proposition 4.1 below), but is typically not surjective. Since our focus is on graded K-theory and graded K-homology, we just need the well-known fact that $KK(\mathbb{C}, \,\cdot\,)$ is countably additive; as we do not know of a reference for this, we record a proof here. Of course, for ungraded C^*-algebras, this follows from countable additivity of K-theory since $KK(\mathbb{C}, A) \cong K_0(A)$.

Proposition 4.1. *Let $(A_i, \alpha_i)_{i=1}^\infty$ be a sequence of σ-unital graded C^*-algebras. Then the map $\omega : \bigoplus_{i=1}^\infty KK(\mathbb{C}, A_i) \to KK(\mathbb{C}, \bigoplus_{i=1}^\infty A_i)$ of (4.1) is an isomorphism.*

Proof. Write $A_\infty := \bigoplus_i A_i$. Let (X, ϕ, F, α_X) be a Kasparov $\mathbb{C}$–A_∞-module. Since each A_i is an ideal of A_∞, each $X_i := \{\xi : \langle \xi, \xi\rangle \in A_i\}$ is a right-Hilbert A_i-module. We identify X and $\bigoplus_i X_i$ as right-Hilbert A_∞-modules, so $X = \overline{\operatorname{span}}\{\bigcup_{i\geq 1} X_i\}$ and $\langle x, y\rangle_A = 0$ whenever $x \in X_i$ and $y \in X_j$ with $i \neq j$. Since F and $\phi(1)$ are adjointable, they leave each X_i invariant ($FX_i \subseteq X_i$ and $\phi(1)X_i \subseteq X_i$); so F and ϕ decompose as $F = \bigoplus F_i$ and $\phi = \bigoplus \phi_i$. Since the inclusion map $\iota_{A_i} : A_i \hookrightarrow A_\infty$ is graded and $X_i = X \cdot A_i$, the automorphism α_X also leaves each X_i invariant; so α_X decomposes as a direct sum $\alpha_X = \bigoplus \alpha_i$. Since each compact operator on $X = \bigoplus_i X_i$ restricts to a compact operator on X_i, each $(X_i, \phi_i, F_i, \alpha_i)$ is a Kasparov $\mathbb{C}$–A_i-module.

Applying the preceding paragraph with A_∞ replaced by $C([0,1], A_\infty)$ shows that any homotopy of Kasparov $\mathbb{C}$–A_∞ modules decomposes as a direct sum of homotopies of Kasparov $\mathbb{C}$–A_i-modules, and so ω is injective.

To show that ω is surjective, we claim that it suffices to show that if (X, ϕ, F, α) is a Kasparov $\mathbb{C}$–A_∞-module, then there exists N such that $[X_i, \phi_i, F_i, \alpha_i] = 0_{KK(\mathbb{C}, A_i)}$ for all $i \geq N$. To see why, suppose that $[X_i, \phi_i, F_i, \alpha_i] = 0_{KK(\mathbb{C}, A_i)}$ for all $i \geq N$. Let

$$[X_N^\infty] := \Big[0^{N-1} \oplus \bigoplus\nolimits_{i=N}^\infty X_i, 0^{N-1} \oplus \bigoplus\nolimits_{i=N}^\infty \phi_i, 0^{N-1} \oplus \bigoplus\nolimits_{i=N}^\infty F_i, 0^{N-1} \oplus \bigoplus\nolimits_{i=N}^\infty \alpha_i\Big].$$

We have

$$[X, \phi, F, \alpha] = \omega\big(\bigoplus\nolimits_{i=1}^{N-1} [X_i, \phi_i, F_i, \alpha_i]\big) + [X_N^\infty].$$

Since, for each $i \geq N$, the module $(X_i, \phi_i, F_i, \alpha_i)$ is a degenerate Kasparov module, so is $\big(\bigoplus_{i=N}^\infty X_i, \bigoplus_{i=N}^\infty \phi_i, \bigoplus_{i=N}^\infty F_i, \bigoplus_{i=N}^\infty \alpha_i\big)$. So $[X_N^\infty] = 0_{KK(\mathbb{C}, A_\infty)}$, and it follows that $[X, \phi, F, \alpha] = \omega(\bigoplus_{i=1}^{N-1} [X_i, \phi_i, F_i, \alpha_i])$ belongs to the range of ω.

To see that there exists N such that $[X_i, \phi_i, F_i, \alpha_i] = 0_{KK(\mathbb{C}, A_i)}$ for all $i \geq N$, first note that by [2, Propositions 17.4.2 and 18.3.6], we may assume that $\phi(1) = 1_X$, that $F = F^*$ and that $\|F\| \leq 1$. Using that $\mathcal{K}(X) = \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in \bigoplus_{i=1}^n X_i, n \in \mathbb{N}\}$ it follows that if $T \in \mathcal{K}(X)$, then $\|T|_{X_i}\| \to 0$. Since $\phi_i(1) = 1_{X_i}$ and $F^2 - 1 = (F^2 - 1)\phi(1) \in \mathcal{K}(X)$, we deduce that $\|F_i^2 - 1\| \to 0$. So there exists N large enough so that F_i^2 is invertible for $i \geq N$. Fix $i \geq N$; we will show that $(X_i, \phi_i, F_i, \alpha_i)$ is degenerate. Since $F_i^* = F_i$ we see that F_i is normal, and so $\sigma(F_i^2) = \sigma(F_i)^2$, and in particular, F_i is invertible. Since $\|F_i\| \leq 1$, we have $\sigma(F_i) \subseteq [-1, 0) \cup (0, 1]$, so for $t \in [0, 1]$ there is a continuous function $f_t \in C(\sigma(F_i))$ given by

$$f_t(x) = \begin{cases} (1-t)x + t & \text{if } x > 0 \\ (1-t)x - t & \text{if } x < 0. \end{cases}$$

Now the path $(F_t)_{t \in [0,1]}$ is a continous path from F_i to $f_1(F_i)$. Since $\sigma(f_1(F_i))) = f_1(\sigma(F_i)) = \{-1, 1\}$, we have $f_1(F_i)^2 = 1$.

We claim that for each t, the tuple $(X_i, \phi_i, f_t(F_i), \alpha_i)$ is a Kasparov module. To see this, first note that each $f_t(F_i)^* = f_t(F_i)$. Since $\phi_i(\mathbb{C}) = \operatorname{span} 1_{X_i}$ is even-graded and central, we have $[\phi_i(a), f_t(F_i)]^{\mathrm{gr}} = a[\phi_i(1), f_t(F_i)] = 0$ for all t, a. Since F_i is odd with respect to the grading on $\mathcal{L}(X_i)$, so is F_i^{2n+1} for every $n \geq 0$. So writing P_{odd} for the space $\big\{z \mapsto \sum_{n=0}^N a_n z^{2n+1} \mid N \geq 0 \text{ and } a_i \in \mathbb{C}\big\}$ of odd polynomials, $f(F_i)$ is odd for each $f \in P_{\mathrm{odd}}$. Since the f_t are all odd functions, they can be uniformly approximated by elements of P_{odd}, and we deduce that each $f_t(F_i)$ is odd with respect to $\tilde{\alpha}_i$. So to prove the claim, it remains to show that each $f_t(F_i)^2 - 1 \in \mathcal{K}(X_i)$. For this, observe that the functional-calculus isomorphism for F_i carries $F_i^2 - 1$ to the function $z \mapsto z^2 - 1$. Since this function vanishes only at 1 and -1, the ideal of $C^*(F_i)$ generated by $F_i^2 - 1$ is $\{f(F_i) : f \in C(\sigma(F_i)), f(1) = f(-1) = 0\}$. Since each $f_t^2(1) = 1 = f_t^2(-1)$, we deduce that each $f_t(F_i)^2 - 1$ belongs to the ideal generated by $F_i^2 - 1$, and since $F_i^2 - 1 \in \mathcal{K}(X_i)$ it follows that each $f_t(F_i)^2 - 1 \in \mathcal{K}(X_i)$. This proves the claim.

Hence $(X_i, \phi_i, f_t(F_i), \alpha_i)_{t \in [0,1]}$ is an operator homotopy, and it follows that

$$[X_i, \phi_i, F_i, \alpha_i] = [X_i, \phi_i, f_1(F_i), \alpha_i].$$

By construction, $f_1(F_i)^2 = 1$, and we already saw that $f_1(F_i)$ is odd, self-adjoint and commutes with $\phi_i(1)$. So $[X_i, \phi_i, f_1(F_i), \alpha_i] = 0_{KK(\mathbb{C},A_i)}$ as required. $\square$

Set-up. *Throughout the remainder of this section we fix a graded, separable, nuclear C^*-algebra A and a graded countably generated essential A–A-correspondence X with left action implemented by $\varphi : A \to \mathcal{L}(X)$.*

Recall that if I is an ideal of A, then $I^\perp$ is the ideal $\{b \in A : bI = \{0\}\}$. Following Katsura, we define $J_X := \varphi^{-1}(\mathcal{K}(X)) \cap \ker\varphi^\perp \lhd A$. Since $ja = aj = 0$ for all $j \in J_X$ and $a \in \ker\varphi$, the ideal $J_X + \ker\varphi \subseteq A$ is the internal direct sum $J_X \oplus \ker\varphi$. We sometimes identify it with the external direct sum via the map $j + a \mapsto (j, a)$.

Since J_X acts compactly on X, the quadruple $(X, \varphi|_{J_X}, 0, \alpha_X)$ is a Kasparov module, and we write $[X]$ for the corresponding class in $KK(J_X, A)$.

We define $K_\varphi^\infty := \bigoplus_{n=1}^\infty \ker\varphi$ as a graded C^*-algebra, and $T := (K_\varphi^\infty)_{K_\varphi^\infty}$ regarded as an $(A \oplus K_\varphi^\infty)$–$K_\varphi^\infty$-correspondence with left action $(a, f) \cdot g = (ag_1, f_1g_2, f_2g_3, \dots)$.

We define $Y := X \oplus T$ as a right-Hilbert $A \oplus K_\varphi^\infty$-module, so the right action of $A \oplus K_\varphi^\infty$ is given by

$$(x, f) \cdot (a, g) = (x \cdot a, fg)$$

and the inner product is given by

$$\langle (x, f), (y, g)\rangle = (\langle x, y\rangle, f^*g),$$

for $x, y \in X$, $a \in A$, and $f, g \in T$. This Y is an $(A \oplus K_\varphi^\infty)$–$(A \oplus K_\varphi^\infty)$-correspondence with left action

$$\varphi^Y(a, f)(x, g) = (\varphi(a)x, ag_1, f_1g_2, \cdots, f_ng_{n+1}, \dots),$$

and the homomorphism φ^Y is injective (see [23, Lemma 4.2]).

Proposition 4.2. *The ideal $(\varphi^Y)^{-1}(\mathcal{K}(Y))$ is equal to $J_X \oplus \ker\varphi \oplus K_\varphi^\infty \subseteq A \oplus K_\varphi^\infty$.*

Proof. As discussed just before the statement of the lemma, we have $J_X + \ker\varphi = J_X \oplus \ker\varphi$, so it suffices to show that $(\varphi^Y)^{-1}(\mathcal{K}(Y)) = (J_X + \ker\varphi) \oplus K_\varphi^\infty$

To prove that $(J_X + \ker\varphi) \oplus K_\varphi^\infty \subseteq (\varphi^Y)^{-1}(\mathcal{K}(Y))$, fix $j + a \in J_X + \ker\varphi$, and $f \in K_\varphi^\infty$. For $x \in X$ and $g \in T$, since $\varphi(a + j) = \varphi(j)$ and since $j \cdot T = 0$, we have

$$\varphi^Y(j + a, f)(x, g) = (\varphi(j)x, ag_1, f_1g_2, f_2g_3, \dots).$$

Since $j \in J_X$ we have $\varphi(j) \in \mathcal{K}(X)$. Further, letting $F = (a, f_1, f_2, \dots)$ and letting $L_F \in \mathcal{K}(T)$ denote the left multiplication operator by F we have

$$\varphi^Y(j + a, f)(x, g) = (\varphi(j)x, ag_1, f_1g_2, \dots) = (\varphi(j)x, 0, 0, \dots) + (0, L_Fg),$$

so $\varphi^Y(j + a, f) = (\varphi(j), L_F) \in \mathcal{K}(X) \oplus \mathcal{K}(T) \subset \mathcal{K}(X \oplus T)$.

To prove that $(\varphi^Y)^{-1}(\mathcal{K}(Y)) \subseteq J_X \oplus \ker\varphi \oplus K_\varphi^\infty$, first note that φ^Y decomposes as the (internal) direct sum $\varphi^A \oplus \varphi^T$ of the homomorphisms φ^A and φ^T given by

$$\varphi^A(a, f)(x, g_1, g_2, \dots) = (\varphi(a)x, ag_1, 0, 0 \dots), \text{ and } \varphi^T(a, f)(x, g) = (0, 0, f_1g_2, f_2g_3, \dots).$$

Suppose that $(a, f) \in (\varphi^Y)^{-1}(\mathcal{K}(Y))$; that is, $\varphi^Y(a, f)$ is compact. Since $\varphi^T(a, f)$ is left multiplication by $(0, 0, f_1, f_2, \dots)$, it is compact. Hence $\varphi^A = \varphi^Y - \varphi^T$ is also compact. In particular $\varphi(a)$ is compact as it is the restriction of $\varphi^A(a, f)$ to the first entry, and so $a \in \varphi^{-1}(\mathcal{K}(X))$. Since a also acts compactly on the first coordinate of T, which is the right-Hilbert module $(\ker\varphi)_{\ker\varphi}$, we see that the left-multiplication-by-a operator on $(\ker\varphi)_{\ker\varphi}$ agrees with left-multiplication by some $a' \in \ker\varphi$. In particular, $j := a - a' \in (\ker\varphi)^\perp$.

Since $\varphi(j) = \varphi(a) - \varphi(a') = \varphi(a)$, we have $j \in \varphi^{-1}(\mathcal{K}(X)) \cap (\ker\varphi)^\perp = J_X$. Hence $a = j + a' \in J_X + \ker\varphi$. □

Having identified $(\varphi^Y)^{-1}(\mathcal{K}(Y))$ with $I := J_X \oplus \ker\varphi \oplus K_\varphi^\infty$, since φ^Y is injective, we can apply the graded version of Pimsner's exact sequence [25, Theorem 7.0.3] to the $(A \oplus K_\varphi^\infty)$–$(A \oplus K_\varphi^\infty)$-correspondence Y. To avoid overly heavy notation in the resulting sequences (4.2) and (4.3), we employ the following slight abuses of notation:

Although $\iota : J_X \oplus \ker\varphi \to A$ is the inclusion map, we write $[\iota]$ for the class $[I, \iota \oplus \mathrm{id}_{K_\varphi^\infty}, 0, \alpha_I] \in KK(I, A \oplus K_\varphi^\infty)$; moreover, in these diagrams we write $[X]$ and $[T]$ for the classes $[X, \varphi|_{J_X + \ker\varphi}, 0, \alpha_X]$ and $[T, \varphi_T|_I, 0, \alpha_T]$ respectively. Both are elements of $KK(I, A \oplus K_\varphi^\infty)$ by letting K_φ^∞ act trivially on X (on the left and right) and letting A act trivially on T on the right (but not on the left). While this is slightly at odds with our usual use of, for example, the notation $[X]$ corresponding to a Hilbert module X, the ambient KK-groups in the diagrams should provide enough context to avoid confusion. We obtain the following diagrams:

$$\begin{array}{ccccc} KK_0(B, J_X \oplus \ker\varphi \oplus K_\varphi^\infty) & \xrightarrow{\hat\otimes([\iota]-[X]-[T])} & KK_0(B, A \oplus K_\varphi^\infty) & \xrightarrow{i_*} & KK_0(B, \mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ KK_1(B, \mathcal{O}_Y) & \xleftarrow{i_*} & KK_1(B, A \oplus K_\varphi^\infty) & \xleftarrow{\hat\otimes([\iota]-[X]-[T])} & KK_1(B, J_X \oplus \ker\varphi \oplus K_\varphi^\infty) \end{array} \tag{4.2}$$

$$\begin{array}{ccccc} KK_0(J_X \oplus \ker\varphi \oplus K_\varphi^\infty, B) & \xleftarrow{([\iota]-[X]-[T])\hat\otimes} & KK_0(A \oplus K_\varphi^\infty, B) & \xleftarrow{i^*} & KK_0(\mathcal{O}_X, B) \\ \downarrow & & & & \uparrow \\ KK_1(\mathcal{O}_Y, B) & \xrightarrow{i^*} & KK_1(A \oplus K_\varphi^\infty, B) & \xrightarrow{([\iota]-[X]-[T])\hat\otimes} & KK_1(J_X \oplus \ker\varphi \oplus K_\varphi^\infty, B) \end{array} \tag{4.3}$$

4.2. **The covariant exact sequence.** In this section we will use (4.2) to recover our exact sequence describing the graded K-theory $KK(\mathbb{C}, \mathcal{O}_X)$.

Proposition 4.3. *For any graded C^*-algebra B, define*

$$U : KK(B, J_X) \oplus KK(B, \ker\varphi) \oplus \big(\textstyle\bigoplus_{n=1}^\infty KK(B, \ker\varphi)\big) \to KK(B, A) \oplus \big(\textstyle\bigoplus_{n=1}^\infty KK(B, \ker\varphi)\big),$$

by $U(j, a, (f_1, f_2, \dots)) = (0, (a, f_1, f_2, \dots))$. Let $\omega : \bigoplus_{n=1}^\infty KK(B, \ker\varphi) \to KK(B, K_\varphi^\infty)$ be the canonical homomorphism of (4.1), and let $[T] = [T, \varphi^Y, 0, \alpha_T]$ be the Kasparov class of T. Then the following diagram commutes

$$\begin{array}{ccc} KK(B, J_X \oplus \ker\varphi \oplus K_\varphi^\infty) & \xrightarrow{\otimes[T]} & KK(B, A \oplus K_\varphi^\infty) \\ \uparrow{\scriptstyle\omega} & & \uparrow{\scriptstyle\omega} \\ KK(B, J_X) \oplus KK(B, \ker\varphi) \oplus \displaystyle\bigoplus_{n=1}^\infty KK(B, \ker\varphi) & \xrightarrow{U} & KK(B, A) \oplus \displaystyle\bigoplus_{n=1}^\infty KK(B, \ker\varphi) \end{array}$$

Proof. Denote the nth copy of $\ker\varphi$ in K_φ^∞ by K_n, and let K_0 be the copy of $\ker\varphi$ in A. Since classes of Kasaprov B–J_X-modules and Kasparov B–K_n-modules generate $KK(B, J_X \oplus K_0 \oplus K_\varphi^\infty)$ it suffices to show that the diagram commutes on such elements.

Let $[J, \psi_J, F_j, \alpha_J]$ be a B–J_X-module. Then there is a Fredholm operator G such that

$$[J, \psi_J, F_j, \alpha_J] \mathbin{\widehat{\otimes}} [T, \varphi^Y, 0, \alpha_T] = [J \mathbin{\widehat{\otimes}} T, \psi_J \mathbin{\widehat{\otimes}} 1, G, \alpha_J \mathbin{\widehat{\otimes}} \alpha_T].$$

We claim that $J \mathbin{\widehat{\otimes}} T$ is the zero module. To see this, fix $j \in J$ and $f \in T$, and use [29, Proposition 2.31] to write $j = k \cdot \langle k, k\rangle$. Using that $\langle k, k\rangle \in \ker\varphi^{\perp}$ at the last equality, we compute

$$j \mathbin{\widehat{\otimes}} f = k \cdot \langle k, k\rangle \mathbin{\widehat{\otimes}} f = k \mathbin{\widehat{\otimes}} \varphi^Y(\langle k, k\rangle) f = k \mathbin{\widehat{\otimes}} (0, \langle k, k\rangle f_1, 0) = 0.$$

Since simple tensors span $J \widehat{\otimes} T$, it follows the tensor product is zero. Hence $\omega([J]) \widehat{\otimes} [T] = \omega \circ U([J]) = 0$.

Now consider a Kasparov B–K_n-module $(Z_n, \psi_n, F_n, \alpha_n)$. Since Z_n is a B–K_n-correspondence there exists $a_z \in \ker\varphi$ such that $\langle z, z\rangle_i = a_z\delta_{i,n}$ for all $z \in Z_n$. Hence for $z \in Z_n$ and $f \in T$ we have

$$\langle z \mathbin{\widehat{\otimes}} f, z \mathbin{\widehat{\otimes}} f\rangle_i = \langle f, \varphi^Y(\langle z, z\rangle) f\rangle_i = f_i^*\langle z, z\rangle_{i-1} f_i = f_i^* a_z \delta_{i-1,n} f_i,$$

which is non-zero only if $i = n+1$. Hence $\langle z \mathbin{\widehat{\otimes}} f, z \mathbin{\widehat{\otimes}} f\rangle \in K_{n+1}$. Thus $\langle y, y\rangle \in K_{n+1}$ for all $y \in Z_n \mathbin{\widehat{\otimes}} T$. With $j_n \colon KK(B, K_n) \to KK(B, J_X \oplus K_0 \oplus K_\varphi^\infty)$ denoting the canonical inclusion,

$$\omega \circ j_n([Z_n]) \mathbin{\widehat{\otimes}} [T] = \omega\big((\overbrace{0, \dots, 0}^{n \text{ terms}}, [Z_n \mathbin{\widehat{\otimes}} T], 0, \dots)\big) = \omega \circ j_{n+1}([Z_n \mathbin{\widehat{\otimes}} T]).$$

There is an isomorphism $Z_n \mathbin{\widehat{\otimes}} T \cong Z_n$ that carries an elementary tensor $z \mathbin{\widehat{\otimes}} f$ to $z \cdot f$. Thus, for $f_n = j_n[Z_n]$,

$$\omega(U(f_n)) = \omega(j_{n+1}[Z_n]) = \omega \circ j_n([Z_n]) \mathbin{\widehat{\otimes}} [T] = \omega(f_n) \mathbin{\widehat{\otimes}} [T]. \qquad \square$$

Theorem 4.4 (cf. Theorem 2.3)**.** *Let (A, α_A) be a graded separable nuclear C^*-algebra, and X an essential graded A–A-correspondence with left action φ. Let $J_X = \varphi^{-1}(\mathcal{K}(X)) \cap \ker\varphi^{\perp}$, and let $\iota_{J_X} : J_X \to A$ be the inclusion map. Then there is an exact sequence*

$$\begin{array}{ccccc}
KK_0(\mathbb{C}, J_X) & \xrightarrow{\widehat{\otimes}_A([\iota_{J_X}]-[X])} & KK_0(\mathbb{C}, A) & \xrightarrow{i_*} & KK_0(\mathbb{C}, \mathcal{O}_X) \\
\uparrow & & & & \downarrow \\
KK_1(\mathbb{C}, \mathcal{O}_X) & \xleftarrow{i_*} & KK_1(\mathbb{C}, A) & \xleftarrow{\widehat{\otimes}_A([\iota_{J_X}]-[X])} & KK_1(\mathbb{C}, J_X)
\end{array}$$

Proof. By [23, Lemma 4.2] the left action φ^Y on $Y = X \oplus T$ is injective. Let $I := (\varphi^Y)^{-1}(\mathcal{K}(Y))$ and let $\iota_I : I \hookrightarrow A \oplus K_\varphi^\infty$ be the inclusion map. Consider the resulting exact sequence (4.2). Let $P : KK(\mathbb{C}, (J_X + \ker\varphi) \oplus K_\varphi^\infty) \to KK(\mathbb{C}, J_X)$ be the projection map given by $P[Z, \psi, F, \alpha_Z] = [Z \cdot J_X, \psi, F, \alpha_Z]$, and let $\ell : KK(\mathbb{C}, A) \to KK(\mathbb{C}, A \oplus K_\varphi^\infty)$ be the inclusion map. Consider the following ten-term diagram with (4.2) as its central six-term rectangle.

$$\begin{array}{ccccc}
KK_0(\mathbb{C}, J_X) & \xrightarrow{\widehat{\otimes}_A([\iota_{J_X}]-[X])} & KK_0(\mathbb{C}, A) & & \\
\uparrow P & & \downarrow \ell & & \\
KK_0(\mathbb{C}, J_X \oplus \ker\varphi \oplus K_\varphi^\infty) & \xrightarrow{\widehat{\otimes}_A([\iota_I]-[X]-[T])} & KK_0(\mathbb{C}, A \oplus K_\varphi^\infty) & \xrightarrow{i_*} & KK_0(\mathbb{C}, \mathcal{O}_Y) \\
\uparrow \partial & & & & \downarrow \partial \\
KK_1(\mathbb{C}, \mathcal{O}_Y) & \xleftarrow{i_*} & KK_1(\mathbb{C}, A \oplus K_\varphi^\infty) & \xleftarrow{\widehat{\otimes}_A([\iota_I]-[X]-[T])} & KK_1(\mathbb{C}, J_X \oplus \ker\varphi \oplus K_\varphi^\infty) \\
 & & \uparrow \ell & & \downarrow P \\
 & & KK_1(\mathbb{C}, A) & \xleftarrow{\widehat{\otimes}_A([\iota_{J_X}]-[X])} & KK_1(\mathbb{C}, J_X)
\end{array}$$

We show that the sequence

$$\begin{array}{ccccc}
KK_0(\mathbb{C}, J_X) & \xrightarrow{\widehat{\otimes}_A([\iota_{J_X}]-[X])} & KK_0(\mathbb{C}, A) & \xrightarrow{i_*\circ\ell} & KK_0(\mathbb{C}, \mathcal{O}_Y) \\
\uparrow P\circ\partial & & & & \downarrow P\circ\partial \\
KK_1(\mathbb{C}, \mathcal{O}_Y) & \xleftarrow{i_*\circ\ell} & KK_1(\mathbb{C}, A) & \xleftarrow{\widehat{\otimes}_A([\iota_{J_X}]-[X])} & KK_1(\mathbb{C}, J_X)
\end{array} \tag{4.4}$$

obtained by traversing the outside of the ten-term diagram is exact. First we show that (4.4) is exact at $KK_*(\mathbb{C}, \mathcal{O}_Y)$. We claim that $\ker(P \circ \partial) = \ker \partial$. To see this, observe that $\mathrm{Img}(\partial) = \ker(\cdot \widehat{\otimes} ([\iota_I] - [X] - [T]))$ by exactness of (4.2). Identifying direct sums as in Proposition 4.1, suppose that $(j, a, f) \widehat{\otimes} ([\iota_I] - [X] - [T]) = 0$ for some $j \in KK_{*+1}(\mathbb{C}, J_X), a \in KK_{*+1}(\mathbb{C}, \ker\varphi)$, and $f = (f_1, f_2, \dots) \in \bigoplus_{n=1}^\infty KK_{*+1}(\mathbb{C}, \ker\varphi)$. Since ι_I is just the inclusion of I into $A \oplus K_\varphi^\infty$, the map $(\iota_I)_* := \cdot \widehat{\otimes} [\iota_I] : KK_{*+1}(\mathbb{C}, I) \to KK_{*+1}(\mathbb{C}, A \oplus K_\varphi^\infty)$ is the natural inclusion. Hence under the identification of Proposition 4.2, $j \widehat{\otimes} [\iota_I]$, $a \widehat{\otimes} [\iota_I]$, and $f \widehat{\otimes} [\iota_I]$ are the natural images of j, a and f in $KK_{*+1}(\mathbb{C}, A \oplus K_\varphi^\infty)$. Hence $(j, a, f) \widehat{\otimes} [\iota_I] = (j + a, f)$.

Since $\ker\varphi$ acts trivially on X, we have $a \widehat{\otimes} [X] = f \widehat{\otimes} [X] = 0$. Hence $(j, a, f) \widehat{\otimes} [X] = (j \widehat{\otimes} [X], 0, 0)$. Using Proposition 4.3, $(j, a, (f_1, f_2, \dots)) \widehat{\otimes} [T] = (0, (a, f_1, f_2, \dots))$, so

$$0 = (j, a, f) \widehat{\otimes} ([\iota_I] - [X] - [T]) = (j + a - j \widehat{\otimes} [X], f_1 - a, f_2 - f_1, \dots, f_n - f_{n-1}, \dots). \tag{4.5}$$

Proposition 4.1 shows that $f = (f_1, f_2, \dots) \in \bigoplus_{n=1}^\infty KK_{*+1}(\mathbb{C}, \ker\varphi)$, so there exists $N \in N$ such that $f_n = 0$ for all $n > N$. Hence (4.5) becomes

$$0 = (j + a - j \widehat{\otimes} [X], f_1 - a, f_2 - f_1, \dots, f_N - f_{N-1}, -f_N, 0, 0, \dots)$$

forcing $f_N = 0$. Continuing recursively down the sequence we have $f_k = 0$ for each $k \leq N$, and $a = 0$. Hence $(j, a, f) = (j, 0, 0)$. We conclude

$$\mathrm{Img}(\partial) \subseteq KK_{*+1}(\mathbb{C}, J_X) \oplus 0 \oplus 0, \tag{4.6}$$

so $P|_{\ker\partial}$ is injective, and therefore $\ker(P \circ \partial) = \ker(\partial)$.

To establish exactness of (4.4) at $KK_*(\mathbb{C}, \mathcal{O}_Y)$, it now suffices to show that $\mathrm{Img}(\iota_* \circ \ell) = \mathrm{Img}(\iota_*)$. For this, we first claim that

$$\mathrm{Img}(\widehat{\otimes}([\iota_I] - [X] - [T])|_{KK_*(\mathbb{C}, \ker\varphi \oplus K_\varphi^\infty)}) = \Big\{ \Big(-\sum f_j, f \Big) : f \in KK_*(\mathbb{C}, K_\varphi^\infty) \Big\}. \tag{4.7}$$

For all $a \in KK_*(\mathbb{C}, \ker\varphi)$ and $f \in KK_*(\mathbb{C}, K_\varphi^\infty)$, the product $(0, a, f)\widehat{\otimes}([\iota_I]-[X]-[T])$ is of the form $\big(-\sum g_j, g\big)$, where $g_j = f_j - f_{j-1}$ for $j > 1$, and $g_1 = f_1 - a$. Conversely, we have $\big(-\sum f_j, f\big) = (0, a, g)\widehat{\otimes}([\iota_I]-[X]-[T])$ for $g_j = -\sum_{k=0}^{n-j-1} f_{n-k}$ and $a = -\sum_{k=1}^{n} f_k$, where n is the index of the last non-zero entry of f. This proves (4.7). By exactness of the inner rectangle (4.2) of the ten-term diagram, we have $\operatorname{Img}(\widehat{\otimes}([\iota_I]-[X]-[T])\big|_{KK_*(\mathbb{C}, \ker\varphi \oplus K_\varphi^\infty)}) \subseteq \operatorname{Img}(\widehat{\otimes}([\iota_I] - [X] - [T])) = \ker i_*$, so we deduce that $i_*\big(-\sum f_j, f\big) = 0$ for all $f \in KK_*(\mathbb{C}, K_\varphi^\infty)$. Now suppose $i_*(a, f) \in \operatorname{Img}(i_*)$. Then

$$i_*(a, f) = i_*(a + \sum f_j, 0) + i_*(-\sum f_j, f) = i_*(a + \sum f_j, 0) = i_* \circ \ell(a + \sum f_j).$$

Hence $\operatorname{Img}(i_* \circ \ell) = \operatorname{Img}(i_*) = \ker(\partial)$ by exactness of (4.2). This completes the proof of exactness of (4.4) at $KK_*(\mathbb{C}, \mathcal{O}_Y)$.

We now establish exactness of (4.4) at $KK_*(\mathbb{C}, J_X)$. We have already demonstrated in (4.6) that $\ker(\widehat{\otimes}([\iota_I] - [X] - [T])) = \ker(\widehat{\otimes}([\iota_I] - [X] - [T])|_{KK_*(\mathbb{C}, J_X)})$, and further by Proposition 4.3 we have that $\cdot \widehat{\otimes} ([\iota_I] - [X] - [T])|_{KK_*(B, J_X)} = \cdot \widehat{\otimes} ([\iota_{J_X}] - [X])$. Hence

$$\operatorname{Img}(i_* \circ \ell) = \operatorname{Img}(i_*) = \ker(\widehat{\otimes}([\iota_I] - [X] - [T])) = \ker(\widehat{\otimes}([\iota_{J_X}] - [X])),$$

giving exactness at $KK_*(\mathbb{C}, J_X)$.

Next, we establish exactness of (4.4) at $KK_*(\mathbb{C}, A)$. By definition we have

$$\ker(i_* \circ \ell) = \{a \in KK_*(\mathbb{C}, A), i_*(a, 0) = 0\} = \ker(i_*) \cap (KK_*(\mathbb{C}, A) \oplus \{0\}).$$

Suppose that for $j \in KK_*(\mathbb{C}, J_X), a \in KK_*(\mathbb{C}, \ker\varphi)$ and $f \in \bigoplus_{n=1}^\infty KK_*(\mathbb{C}, \ker\varphi)$ we have $(j, a, f) \,\widehat{\otimes}\, ([\iota_I] - [X] - [T]) = (b, 0)$ for some $b \in KK_*(\mathbb{C}, A)$. Then

$$\begin{aligned}(j, a, f) \,\widehat{\otimes}\, ([\iota_I] - [X] - [T]) &= (j + a - j \,\widehat{\otimes}\, [X], f_1 - a, f_2 - f_1, \dots, -f_N, 0, \dots) \\ &= (b, 0, \dots),\end{aligned}$$

where $N \in \mathbb{N}$ is the index of the last non-zero component of f. We have $f_N = 0$, and so recursively, $f_j = 0$ for each $j > 0$, and $a = 0$. Hence $(j, a, f) \,\widehat{\otimes}\, ([\iota_I] - [X] - [T]) = (j, 0) \,\widehat{\otimes}\, ([\iota_{J_X}] - [X]) \in \operatorname{Img}(\widehat{\otimes}([\iota_{J_X}] - [X]))$. Thus $\operatorname{Img}(\widehat{\otimes}([\iota_I] - [X] - [T])) \cap (KK_*(\mathbb{C}, A) \oplus \{0\}) = \operatorname{Img}([\iota_{J_X}] - [X])$, and so by exactness of (4.2), we deduce that $\operatorname{Img}(\widehat{\otimes}([\iota_{J_X}] - [X])) = \ker(i_* \circ \ell)$. This proves exactness of (4.4).

The inclusion $i : A \oplus K_\varphi^\infty \to \mathcal{O}_Y$ is nondegenerate and so extends to a homomorphism $\tilde{i} : \mathcal{M}(A \oplus K_\varphi^\infty) \to \mathcal{M}(\mathcal{O}_Y)$. Theorem 4.3 of [23] shows that $Q := \tilde{i}(1_{\mathcal{M}(A)}) \in \mathcal{M}(\mathcal{O}_Y)$ is a full projection and that $Q\mathcal{O}_Y Q \cong \mathcal{O}_X$. Since Q is trivially graded with respect to the grading on $A \oplus K_\varphi^\infty$, the space $\mathcal{O}_Y Q$ is a graded imprimitivity $\mathcal{O}_Y$–$\mathcal{O}_X$-module. So $(\mathcal{O}_Y, \alpha_{\mathcal{O}_Y})$ and $(\mathcal{O}_X, \alpha_{\mathcal{O}_X})$ are KK-equivalent as discussed in Section 1.6. In particular, $KK_*(\mathbb{C}, \mathcal{O}_X) \cong KK_*(\mathbb{C}, \mathcal{O}_Y)$. Now $i_* \circ \ell \colon KK_*(\mathbb{C}, A) \to KK_*(\mathbb{C}, \mathcal{O}_Y)$ restricts to $i_* \colon KK_*(\mathbb{C}, A) \to KK_*(\mathbb{C}, \mathcal{O}_X)$, giving the desired sequence. □

4.3. **The contravariant exact sequence.** We now use similar techniques to those used in the preceding subsection to obtain an exact sequence describing $KK_*(\mathcal{O}_X, B)$ using the contravariant exact sequence (4.3).

Proposition 4.5. *Define*

$$\overline{U} : KK(A, B) \oplus \left(\prod_{n=1}^{\infty} KK(\ker\varphi, B) \right) \\ \to KK(J_X, B) \oplus KK(\ker\varphi, B) \oplus \left(\prod_{n=1}^{\infty} KK(\ker\varphi, B) \right)$$

by $\overline{U}(a, (f_1, f_2, \dots)) = (0, f_1, (f_2, \dots))$*. Let* $\overline{\zeta} : KK(K_\varphi^\infty, B) \to \prod_{n=1}^{\infty} KK(\ker\varphi, B)$ *be the isomorphism discussed at the beginning of Section 4.1, and let* $[T] = [T, \varphi^Y, 0, \alpha_T]$ *be the class of the module associated to* T*. Then the following diagram commutes.*

$$\begin{array}{ccc} KK((J_X \oplus \ker\varphi) \oplus K_\varphi^\infty, B) & \xleftarrow{\;[T]\widehat{\otimes}\;} & KK(A \oplus K_\varphi^\infty, B) \\ \zeta \downarrow & & \downarrow \overline{\zeta} \\ KK(J_X, B) \oplus KK(\ker\varphi, B) \oplus \displaystyle\prod_{n=1}^{\infty} KK(B, \ker\varphi) & \xleftarrow[\overline{U}]{} & KK(A, B) \oplus \displaystyle\prod_{n=1}^{\infty} KK(\ker\varphi, B) \end{array}$$

Proof. As discussed at the beginning of Section 4.1, if $\iota_i : \ker\varphi \to K_\varphi^\infty$ is the inclusion into the ith coordinate, then the right-hand map ζ carries the class $[Z]$ of a Kasparov $(A \oplus K_\varphi^\infty)$–$B$-module to $\big([A \cdot Z], ([\iota_i(\ker\varphi) \cdot Z])_{i=1}^{\infty}\big)$. Likewise, the right-hand map takes $[W]$ to $\big([J_X \cdot W], [\ker\varphi \cdot W], ([\iota_i(\ker\varphi) \cdot W])_{i=1}^{\infty}\big)$.

Regard T as a right $(J_X \oplus K_\varphi^\infty)$-module, and take $W = T \mathbin{\widehat{\otimes}} Z$. The left action of K_φ^∞ on Z is given by the inclusion $K_\varphi^\infty \hookrightarrow A \oplus K_\varphi^\infty$, and so $T \mathbin{\widehat{\otimes}} (A \oplus 0) \cdot Z = 0$. For each $i \geq 1$, we have $T \mathbin{\widehat{\otimes}} (\iota_i(\ker\varphi)) \cdot Z \cong \ker\varphi \otimes_{\ker\varphi} (\iota_i(\ker\varphi)) \cdot Z \cong \iota_i(\ker\varphi) \cdot Z$ as a right module. The left action of $J_X \oplus \ker\varphi$ on $T \mathbin{\widehat{\otimes}} (0 \oplus \iota_1(\ker\varphi)) \cdot Z$ restricts to the zero action of J_X because $J_X \subseteq \ker\varphi^\perp$, and restricts to the standard action of $\ker\varphi$. So we see that W is isomorphic to $0 \oplus \bigoplus_{i=1}^{\infty} \iota_i(\ker\varphi) \cdot Z$ as a right-Hilbert $J_X \oplus \ker\varphi \oplus K_\varphi^\infty$-module. This isomorphism preserves gradings because ι is a graded homomorphism.

In particular, $\iota_{J_X} \cdot W = 0$, each $\iota_{K_\varphi^\infty}(\iota_i(\ker\varphi)) \cdot W \cong \iota_{i+1}(\ker\varphi) \cdot W$, and $\iota_{\ker\varphi} \cdot W \cong \iota_1(\ker\varphi) \cdot Z$. Thus $\zeta([W]) = (0, [\iota_1(\ker\varphi) \cdot Z], [\iota_2(\ker\varphi) \cdot Z], \cdots)$. Since $\zeta([Z]) = ([A \cdot Z], [\iota_1(\ker\varphi) \cdot Z], [\iota_2(\ker\varphi) \cdot Z], \cdots)$, the result follows. □

Theorem 4.6 (cf. Theorem 2.3)**.** *Let* $(A, \alpha_A), (B, \alpha_B)$ *be a graded separable* C^**-algebras, and suppose* A *is nuclear. Let* X *be an essential graded* A–A*-correspondence with left action* φ*. Let* $J_X = \varphi^{-1}(\mathcal{K}(X)) \cap \ker\varphi^\perp$*, and let* $\iota_{J_X} : J_X \to A$ *be the inclusion map. Then there is an exact sequence*

$$\begin{array}{ccccc} KK_0(J_X, B) & \xleftarrow{([\iota_{J_X}] - [X]) \widehat{\otimes}_A} & KK_0(A, B) & \xleftarrow{\;i^*\;} & KK_0(\mathcal{O}_X, B) \\ \downarrow & & & & \uparrow \\ KK_1(\mathcal{O}_X, B) & \xrightarrow{\;i^*\;} & KK_1(A, B) & \xrightarrow{([\iota_{J_X}] - [X]) \widehat{\otimes}_A} & KK_1(J_X, B) \end{array}$$

Proof. The argument is similar to that of Theorem 4.4. Since the left action φ^Y on Y is injective, if we write $I := (\varphi^Y)^{-1}(\mathcal{K}(Y))$ and $\iota_I : I \to A \oplus K_\varphi^\infty$ for the inclusion, then the exact sequence (4.3) fits as the central rectangle in the following diagram whose top and bottom rectangles commute:

$$\begin{array}{ccccc}
KK_0(J_X,B) & \xleftarrow{([\iota_{J_X}]-[X])\widehat{\otimes}_A} & KK_0(A,B) & & \\
\ell\downarrow & & \uparrow P & & \\
KK_0(J_X\oplus\ker\varphi\oplus K_\varphi^\infty,B) & \xleftarrow{([\iota_I]-[X]-[T])\widehat{\otimes}_A} & KK_0(A\oplus K_\varphi^\infty,B) & \xleftarrow{i^*} & KK_0(\mathcal{O}_Y,B) \\
\partial\downarrow & & & & \uparrow\partial \\
KK_1(\mathcal{O}_Y,B) & \xrightarrow{i^*} & KK_1(A\oplus K_\varphi^\infty,B) & \xrightarrow{([\iota_I]-[X]-[T])\widehat{\otimes}_A} & KK_1(J_X\oplus\ker\varphi\oplus K_\varphi^\infty,B) \\
 & & P\downarrow & & \uparrow\ell \\
 & & KK_1(A,B) & \xrightarrow{([\iota_{J_X}]-[X])\widehat{\otimes}_A} & KK_1(J_X,B)
\end{array}$$

To prove the result, we show that the six-term sequence consisting of the six extreme points of this diagram is exact; the result will again follow from the graded Morita equivalence of $\mathcal{O}_X$ and $\mathcal{O}_Y$.

Throughout this proof, without further comment, we identify $KK_*(J_X\oplus\ker\varphi\oplus K_\varphi^\infty,B)$ with $KK_*(J_X,B)\oplus KK_*(\ker\varphi,B)\oplus\prod_{i=1}^\infty KK_*(\ker\varphi,B)$ and we identify $KK_*(A\oplus K_\varphi^\infty,B)$ with $KK_*(A,B)\oplus\prod_{i=1}^\infty KK_*(\ker\varphi,B)$ as discussed at the beginning of Section 4.1.

For exactness at $KK_*(A,B)$, observe that $\operatorname{Img}(P\circ i^*) = P\big(\ker(([\iota_I]-[X]-[T])\,\widehat{\otimes}\,\cdot)\big)$. By Proposition 4.5, and using that $\ker\varphi$ annihilates X, we see that for any $(a,j_1,j_2,\cdots)\in KK_*(A\oplus K_\varphi^\infty,B)$, we have

$$([\iota_I]-[X]-[T])\,\widehat{\otimes}_A\,(a,j_1,j_2,\dots) = (([\iota_{J_X}]-[X])\,\widehat{\otimes}\,a, [\ker\varphi]\,\widehat{\otimes}\,a - j_1, j_1-j_2,\dots).$$

So $\ker(([\iota_I]-[X]-[T])\widehat{\otimes}\cdot)$ is the set of sequences $(a,j,j,j,\dots)$ such that $([\iota_{J_X}]-[X])\widehat{\otimes}_A a = 0$ and $j=[\ker\varphi]\,\widehat{\otimes}\,a$. In particular, $P\big(\ker(([\iota_I]-[X]-[T])\,\widehat{\otimes}\,\cdot)\big) = \ker([\iota_{J_X}]-[X])$ as required.

For exactness at $KK_*(J_X,B)$ first observe that ℓ is given by $\ell([j]) = ([j],0,0,\dots)$. So

$$\begin{aligned}\ker(\partial\circ\ell) &= \{[j] : ([j],0,0,\dots)\in\ker(\partial)\}\\ &= \{([j],0,\dots) : [j]\in\operatorname{Img}(([\iota_I]-[X]-[T])\,\widehat{\otimes}_A\,\cdot)\}.\end{aligned}$$

By the description of the map $([\iota_I]-[X]-[T])\,\widehat{\otimes}\,\cdot$ in the preceding paragraph, we see that if $([j],0,0,\cdots) = ([\iota_I]-[X]-[T])\,\widehat{\otimes}_A\,(a,j_1,j_2,\cdots)$, then $[j] = ([\iota_{J_X}]-[X])\,\widehat{\otimes}_A\,a$. Conversely, given $a\in KK_*(A,B)$, since the rectangles involving P and ℓ commute and since the maps P are surjective, $\ell(([\iota_{J_X}]-[X])\widehat{\otimes}_A a)\in\operatorname{Img}(([\iota_I]-[X]-[T])\widehat{\otimes}_A\cdot) = \ker(\partial)$, and so the image of $([\iota_{J_X}]-[X])\widehat{\otimes}_A$ is contained in the kernel of $\partial\circ\ell$.

It remains to establish exactness at $KK_*(\mathcal{O}_Y,B)$. By exactness of (4.3), $\operatorname{Img}(i^*) = \ker\big(([\iota_I]-[X]-[T])\widehat{\otimes}\cdot\big)$. As we saw earler, this is the collection of sequences $(a,j,j,j,\dots)$ such that $([\iota_{J_X}]-[X])\,\widehat{\otimes}_A\,a = 0$ and $j=[\ker\varphi]\,\widehat{\otimes}\,a$. In particular, if $(P\circ i^*)(x)=0$, then $i^*(x) = (0,j,j,\cdots)$ with $j=[\ker\varphi]\,\widehat{\otimes}\,0 = 0$, and hence $i^*(x)=0$. That is, $\ker(P\circ i^*) = \ker(i^*)$. Since (4.3) is exact, it now suffices to show that $\operatorname{Img}(\partial\circ\ell) = \operatorname{Img}\partial$. We clearly have $\operatorname{Img}(\partial\circ\ell)\subseteq\operatorname{Img}\partial$, so it suffices to show the reverse containment. For this, fix $\theta = (j_X,j_0,j_1,j_2,\dots)\in KK_{*-1}(J_X\oplus\ker\varphi\oplus K_\varphi^\infty,B)$, so that $\partial(\theta)$ is a typical element of $\operatorname{Img}(\partial)$. Consider the element $\eta := (0,-j_0,-j_0-j_1,-j_0-j_1-j_2,\cdots)\in KK_{*-1}(A\oplus K_\varphi^\infty,B)$. We have $([\iota_I]-[X]-[T])\,\widehat{\otimes}\,\eta = (0,j_0,j_1,j_2\cdots)$. In particular,

$\theta = ([\iota_I] - [X] - [T]) \mathbin{\widehat{\otimes}} \eta + (j_X, 0, 0, 0 \cdots) = ([\iota_I] - [X] - [T]) \mathbin{\widehat{\otimes}} \eta + \ell(j_X)$. Since (4.3) is exact, $\partial(\theta) = \partial(([\iota_I] - [X] - [T]) \mathbin{\widehat{\otimes}} \eta) + \partial(\ell(j_X)) = \partial \circ \ell(j_X)$. So $\partial(\theta) \in \mathrm{Img}(\partial \circ \ell)$ as required. □

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