ON THE K-THEORY OF TWISTED HIGHER-RANK-GRAPH C^*-ALGEBRAS

ALEX KUMJIAN, DAVID PASK, AND AIDAN SIMS

Abstract. We investigate the K-theory of twisted higher-rank-graph algebras by adapting parts of Elliott's computation of the K-theory of the rotation algebras. We show that each 2-cocycle on a higher-rank graph taking values in an abelian group determines a continuous bundle of twisted higher-rank graph algebras over the dual group. We use this to show that for a circle-valued 2-cocycle on a higher-rank graph obtained by exponentiating a real-valued cocycle, the K-theory of the twisted higher-rank graph algebra coincides with that of the untwisted one.

1. Introduction and preliminaries

Higher-rank graphs, or k-graphs, and their C^*-algebras were introduced by the first two authors in [12] as a common generalisation of the graph algebras of [13] and the higher-rank Cuntz-Krieger algebras of [22]. Since then these C^*-algebras have been studied by a number of authors (see for example [5, 7, 9, 25]).

In [14, 15] we introduced homology and cohomology theories for k-graphs, and showed how a $\mathbb{T}$-valued 2-cocycle c on a k-graph Λ can be incorporated into the relations defining its C^*-algebra to obtain a twisted k-graph C^*-algebra $C^*(\Lambda, c)$. We showed that elementary examples of this construction include all noncommutative tori and the twisted Heegaard-type quantum spheres of [2].

In this paper we consider the K-theory of twisted k-graph C^*-algebras, following the approach of Elliott to the computation of K-theory for the noncommutative tori [6]. Rather than computing the K-theory of a twisted k-graph C^*-algebra directly, we show that in many instances the problem can be reduced to that of computing the K-theory of the untwisted C^*-algebra of the same k-graph. In many examples, the latter is known or readily computable (see [7]). In order to follow Elliott's method, we must first construct a suitable C^*-algebra bundle from the data used to build a twisted k-graph C^*-algebra.

In Section 2 we consider an abelian group A, and an A-valued 2-cocycle on a k-graph Λ. To this data we associate a C^*-algebra $C^*(\Lambda, A, c)$. We show in Proposition 2.5 that $C^*(\Lambda, A, c)$ is a $C_0(\widehat{A})$-algebra whose fibre over a character χ of A is the twisted k-graph C^*-algebra $C^*(\Lambda, \chi \circ c)$. In particular, $C^*(\Lambda, A, c)$ is the section algebra of an upper semicontinuous C^*-bundle over $\widehat{A}$. In Section 3, we adapt an argument of Rieffel to show that this bundle is continuous (Corollary 3.3). We demonstrate that examples of this construction include continuous fields of noncommutative tori and of Heegaard-type quantum spheres.

Date: November 8, 2012.
2010 *Mathematics Subject Classification.* Primary 46L05; secondary 46L80.
Key words and phrases. C^*-algebra; Graph algebra; k-graph; K-theory.
This research was supported by the Australian Research Council.

In Section 4 we consider k-graphs Λ for which the degree map is a coboundary. We show that for every A-valued 2-cocycle on such a k-graph, we have $C^*(\Lambda, A, c) \cong C^*(A) \otimes C^*(\Lambda)$. If $A = \mathbb{R}$ then $\widehat{A} \cong \mathbb{R}$ and so $C^*(\Lambda, \mathbb{R}, c)$ is a $C_0(\mathbb{R})$-algebra. We deduce in Corollary 4.3 that the "restriction" $C^*(\Lambda, \mathbb{R}, c)|_{[0,1]}$ is isomorphic to $C([0, 1]) \otimes C^*(\Lambda)$, and so the maps $C^*(\Lambda, \mathbb{R}, c)|_{[0,1]} \to C^*(\Lambda, \mathbb{R}, c)_t \cong C^*(\Lambda)$ all induce isomorphisms in K-theory.

In Section 5 we execute Elliott's argument. We consider an $\mathbb{R}$-valued 2-cocycle c_0 on a k-graph Λ. The degree functor on $\Lambda \times_d \mathbb{Z}^k$ is a coboundary, and we exploit this to show that $C^*(\Lambda, \mathbb{R}, c_0)_{[0,1]}$ embeds as a full corner of a crossed product of $C([0, 1]) \otimes C^*(\Lambda \times_d \mathbb{Z}^k)$ by a fibre-preserving $\mathbb{Z}^k$-action. An induction on k shows that the homomorphisms of this crossed product onto its fibres induce isomorphisms in K-theory. Hence the K-theory of $C^*(\Lambda, \mathbb{R}, c_0)_1 \cong C^*(\Lambda, e^{\imath t c_0})$ is identical to that of $C^*(\Lambda, \mathbb{R}, c_0)_0 \cong C^*(\Lambda)$ (Theorem 5.4). We apply our results to the noncommutative tori and to the twisted Heegaard-type quantum spheres of [2], recovering the existing formulae for their K-theory. We also discuss the implications of our results for Kirchberg algebras associated to k-graphs.

Background and notation. Throughout the paper, we regard $\mathbb{N}^k$ as a monoid under addition, with identity 0 and generators $e_1, \dots, e_k$. For $m, n \in \mathbb{N}^k$, we write m_i for the i^{th} coordinate of m, and define $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max\{m_i, n_i\}$.

Let Λ be a countable small category and $d : \Lambda \to \mathbb{N}^k$ be a functor. Write $\Lambda^n := d^{-1}(n)$ for $n \in \mathbb{N}^k$. Recall that Λ is a k-graph if d satisfies the *factorisation property*: $(\mu, \nu) \mapsto \mu\nu$ is a bijection of $\{(\mu, \nu) \in \Lambda^m \times \Lambda^n : s(\mu) = r(\nu)\}$ onto Λ^{m+n} for each $m, n \in \mathbb{N}^k$. We then have $\Lambda^0 = \{\mathrm{id}_o : o \in \mathrm{Obj}(\Lambda)\}$, and so we regard the domain and codomain maps as maps $s, r : \Lambda \to \Lambda^0$. Recall from [18] that for $v, w \in \Lambda^0$ and $X \subseteq \Lambda$, we write

$$vX := \{\lambda \in X : r(\lambda) = v\}, \quad Xw := \{\lambda \in X : s(\lambda) = w\}, \quad \text{and} \quad vXw = vX \cap Xw.$$

A k-graph is *row-finite with no sources* if $0 < |v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Given $\mu, \nu \in \Lambda$, we write $\mathrm{MCE}(\mu, \nu)$ for the set $\{\lambda \in \Lambda^{d(\mu)\vee d(\nu)} : \lambda = \mu\mu' = \nu\nu' \text{ for some } \mu', \nu'\}$. See [12] for further details regarding the basic structure of k-graphs. Let $\Lambda^{*2} = \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = r(\mu)\}$ denote the collection of composable pairs in Λ.

Given an abelian group A and a k-graph Λ, an A-valued 2-cocycle[1] c on Λ is a map $c : \Lambda^{*2} \to A$ such that $c(\lambda, s(\lambda)) = c(r(\lambda), \lambda) = 0$ for all λ and $c(\mu, \nu) + c(\lambda, \mu\nu) = c(\lambda, \mu) + c(\lambda\mu, \nu)$ for each composable triple (λ, μ, ν). The group of all such 2-cocycles is denoted $\underline{Z}^2(\Lambda, A)$. Given $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, the twisted k-graph C^*-algebra $C^*(\Lambda, c)$ is the universal C^*-algebra generated by elements s_λ, $\lambda \in \Lambda$ such that: (1) the s_v, $v \in \Lambda^0$ are mutually orthogonal projections; (2) $s_\mu s_\nu = c(\mu, \nu)s_{\mu\nu}$ when μ, ν are composable; (3) $s_\mu^* s_\mu = s_{s(\mu)}$ for all μ; and (4) $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. See [15] for further details regarding twisted k-graph C^*-algebras. We denote by $\mathcal{Z}(B)$, $\mathcal{M}(B)$ and $\mathcal{U}(B)$ the centre, multiplier algebra and unitary group of a C^*-algebra B.

2. C^*-bundles associated to 2-cocycles

Given a k-graph Λ and an abelian group A, we associate to each $c \in \underline{Z}^2(\Lambda, A)$ a C^*-algebra $C^*(\Lambda, A, c)$. We show that $C^*(\Lambda, A, c)$ is a $C_0(\widehat{A})$-algebra whose fibres are twisted k-graph C^*-algebras associated to Λ.

Throughout this paper, if A is an abelian group, then given a unitally extendible homomorphism $\pi : C^*(A) \to B$ we also write $\pi : \mathcal{M}(C^*(A)) \to \mathcal{M}(B)$ for its extension. We

[1] In [15] these were called *categorical cocycles*, in contradistinction to cubical cocycles.

identify elements of A with the corresponding unitary multipliers of $C^*(A)$. We denote general elements of $C^*(A)$ by the letters $f, g, \dots$ and often regard them as elements of $C_0(\widehat{A})$.

Definition 2.1. Let Λ be a row-finite k-graph with no sources, let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. A *c-representation* (ϕ, π) of (Λ, A) in a C^*-algebra B consists of a map $\phi : \Lambda \to \mathcal{M}(B)$ and a unitally extendible homomorphism $\pi : C^*(A) \to \mathcal{M}(B)$ such that $\phi(\lambda)\pi(f) \in B$ for all $\lambda \in \Lambda$ and $f \in C^*(A)$, and:

(R1) $\phi(\lambda)\pi(f) = \pi(f)\phi(\lambda)$ for all $\lambda \in \Lambda$ and $f \in C^*(A)$;
(R2) $\phi(v)$, $v \in \Lambda^0$ are mutually orthogonal projections and $\sum \phi(v) \to 1_{\mathcal{M}(B)}$ strictly;
(R3) $\phi(\lambda)\phi(\mu) = \pi(c(\lambda, \mu))\phi(\lambda\mu)$ whenever $s(\lambda) = r(\mu)$;
(R4) $\phi(\lambda)^*\phi(\lambda) = \phi(s(\lambda))$ for all $\lambda \in \Lambda$;
(R5) $\sum_{\lambda \in v\Lambda^n} \phi(\lambda)\phi(\lambda)^* = \phi(v)$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

We define $C^*(\phi, \pi) := C^*(\{\phi(\lambda)\pi(f) : \lambda \in \Lambda, f \in C^*(A)\} \subseteq B$.

Remark 2.2. If A is discrete, $c \in \underline{Z}^2(\Lambda, A)$, and (ϕ, π) is a c-representation of (Λ, A), then $\phi(\Lambda) \subseteq C^*(\phi, \pi)$. If $|\Lambda^0| < \infty$, then $C^*(\Lambda)$ is unital, so $\pi(C^*(A)) \subseteq C^*(\phi, \pi)$ too; moreover condition (R1) then reduces to $\pi(a) \in \mathcal{UZ}(C^*(\phi, \pi))$ for each $a \in A$.

Lemma 2.3. *Let Λ be a row-finite k-graph with no sources, let A be an abelian group, and suppose that $c \in \underline{Z}^2(\Lambda, A)$. There is a c-representation (i_Λ, i_A) of (Λ, A) in a C^*-algebra $C^*(\Lambda, A, c)$ that is universal in the sense that*

$$C^*(\Lambda, A, c) = \overline{\operatorname{span}}\{i_\Lambda(\lambda)i_A(f)i_\Lambda(\mu)^* : \lambda, \mu \in \Lambda, f \in C_c(\widehat{A})\}, \tag{1}$$

and any c-representation (ϕ, π) of (Λ, A) in a C^*-algebra B induces a homomorphism $\phi \times \pi : C^*(\Lambda, A, c) \to B$ such that $(\phi \times \pi) \circ i_\Lambda = \phi$ and $(\phi \times \pi) \circ i_A = \pi$.

Proof. Given a c-representation (ϕ, π) of (Λ, A) in a C^*-algebra B, calculations like those of [15, Lemma 7.2] show that for $\mu, \nu \in \Lambda$, we have

$$\phi(\mu)^*\phi(\nu) = \sum_{\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)} \pi(c(\nu, \beta) - c(\mu, \alpha))\phi(\alpha)\phi(\beta)^*. \tag{2}$$

Since $\pi(C^*(A))$ is central, $\overline{\operatorname{span}}\{\phi(\lambda)\pi(f)\phi(\mu)^* : \lambda, \mu \in \Lambda, f \in C_c(\widehat{A})\}$ is therefore a C^*-subalgebra of B. The $\phi(\lambda)$ are partial isometries by (R2) and (R4). Thus $\|\phi(\lambda)\pi(f)\phi(\mu)^*\| \leq \|f\|$ for each λ. An argument like [20, Proposition 1.21] now proves the lemma. □

Recall that if Λ is a k-graph, A is an abelian group and $b : \Lambda \to A$, then the 2-coboundary $\underline{\delta}^1 b : \Lambda^{*2} \to A$ is given by $\underline{\delta}^1 b(\lambda_1, \lambda_2) = b(\lambda_1) - b(\lambda_1\lambda_2) + b(\lambda_2)$.

Lemma 2.4. *Let Λ be a row-finite k-graph with no sources, and let A be an abelian group. Suppose that $c, c' \in \underline{Z}^2(\Lambda, A)$ and $b : \Lambda \to A$ satisfy $c - c' = \underline{\delta}^1 b$. Then there is an isomorphism $C^*(\Lambda, A, c) \cong C^*(\Lambda, A, c')$ determined by $i^c_\Lambda(\lambda)i^c_A(f) \mapsto i^{c'}_A(b(\lambda))i^{c'}_\Lambda(\lambda)i^{c'}_A(f)$.*

Proof. Define $\pi := i^{c'}_A$ and $\phi(\lambda) := i^{c'}_A(b(\lambda))i^{c'}_\Lambda(\lambda)$. Then (ϕ, π) is a c-representation of (Λ, A) in $C^*(\Lambda, A, c')$. The homomorphism $\phi \times \pi : C^*(\Lambda, A, c) \to C^*(\Lambda, A, c')$ satisfies the desired formula. Symmetry provides an inverse. □

Recall (from [27, Appendix C] for example) that if X is a locally compact Hausdorff space, then a $C_0(X)$-algebra is a C^*-algebra B equipped with a homomorphism $i : C_0(X) \to \mathcal{ZM}(B)$ such that $\overline{\operatorname{span}}\{i(f)b : f \in C_0(X) \text{ and } b \in B\} = B$. For $x \in X$, we write I_x for the ideal $\{i(f)b : b \in B, f(x) = 0\}$, we write $B_x := B/I_x$, and $q_x : B \to B_x$ denotes the quotient map.

Proposition 2.5. *Let Λ be a row-finite k-graph with no sources, let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. Then $C^*(\Lambda, A, c)$ is a $C_0(\widehat{A})$-algebra with respect to $i_A : C^*(A) \to \mathcal{M}(C^*(\Lambda, A, c))$. For each character $\chi \in \widehat{A}$, there is a unique homomorphism $\pi_\chi : C^*(\Lambda, A, c) \to C^*(\Lambda, \chi \circ c)$ such that $\pi_\chi(i_\Lambda(\lambda) i_A(f)) = f(\chi)s_\lambda$. This π_χ descends to an isomorphism $C^*(\Lambda, A, c)_\chi \cong C^*(\Lambda, \chi \circ c)$. The map $\chi \mapsto \|\pi_\chi(b)\|$ is upper semicontinuous for each $b \in C^*(\Lambda, A, c)$, and $b \mapsto (\chi \mapsto \pi_\chi(b))$ is an isomorphism of $C^*(\Lambda, A, c)$ onto the section algebra of an upper-semicontinuous C^*-bundle with fibres $C^*(\Lambda, \chi \circ c)$.*

Proof. Condition (R1) and equation (1) imply that each $C^*(\Lambda, A, c)$ is a $C_0(\widehat{A})$-algebra with respect to i_A. Fix $\chi \in \widehat{A}$. The map $\chi \circ c$ belongs to $\underline{Z}^2(\Lambda, \mathbb{T})$, and $(\lambda \mapsto s_\lambda, f \mapsto f(\chi))$ is a c-representation of (Λ, A) in $C^*(\Lambda, \chi \circ c)$. The universal property of $C^*(\Lambda, A, c)$ yields the desired homomorphism π_χ. We have $I_\chi \subseteq \ker(\pi_\chi)$, and so π_χ descends to a homomorphism $\widetilde{\pi}_\chi : C^*(\Lambda, A, c)_\chi \to C^*(\Lambda, \chi \circ c)$. Since $q_\chi(f) = f(\chi)$ for all $\chi \in \widehat{A}$, the partial isometries $t_\lambda := q_\chi(i_\Lambda(\lambda))$ constitute a Cuntz-Krieger $(\Lambda, \chi \circ c)$-family in $C^*(\Lambda, A, c)_\chi$. Thus the universal property of $C^*(\Lambda, \chi \circ c)$ gives an inverse for $\widetilde{\pi}_\chi$. Proposition C.10(a) of [27] implies that $\chi \mapsto \|q_\chi(b)\| = \|\pi_\chi(b)\|$ is upper semicontinuous. The final statement is proved in the first paragraph of the proof of [27, Theorem C.26]. ☐

We digress to characterise when $\pi \times \phi$ is injective.

Theorem 2.6. *Let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. Suppose that (ϕ, π) is a c-representation of (Λ, A) in a C^*-algebra B. Suppose that there is an action β of $\mathbb{T}^k$ on B such that $\beta_z(\phi(\lambda)\pi(f)) = z^{d(\lambda)}\phi(\lambda)\pi(f)$ for all λ, f. Suppose that $f \mapsto \phi(w)\pi(f)$ is injective on $C^*(A)$ for each $w \in \Lambda^0$. Then $\phi \times \pi : C^*(\Lambda, A, c) \to B$ is injective.*

Proof. We may assume that $B = C^*(\phi, \pi)$, and hence that π maps $C^*(A) = C_0(\widehat{A})$ into $\mathcal{ZM}(B)$. Thus B is a $C_0(\widehat{A})$-algebra. The series $\sum_{w \in \Lambda^0} \phi(w)$ is an approximate identity for B. Hence each fibre B_χ of B is the quotient by the ideal generated by $\{\phi(w)\pi(f) : w \in \Lambda^0, f \in C_0(\widehat{A}), f(\chi) = 0\}$.

Fix $\chi \in \widehat{A}$, and let $q_\chi : B \to B_\chi$ be the quotient map. Since $f \mapsto \phi(w)\pi(f)$ induces a faithful representation of $C^*(A)$ we have $q_\chi(\phi(w)) \neq 0$ for all $w \in \Lambda^0$ and $\chi \in \widehat{A}$. The action β descends to an action $\widetilde{\beta}$ of $\mathbb{T}^k$ on B_χ satisfying $\widetilde{\beta}_z(q_\chi(\phi(\lambda))) = z^{d(\lambda)} q_\chi(\phi(\lambda))$. The elements $\{q_\chi(\phi(\lambda)) : \lambda \in \Lambda\}$ form a Cuntz-Krieger $(\Lambda, \chi \circ c)$-family. Thus the gauge-invariant uniqueness theorem [15, Corollary 7.7] implies that $s_\lambda \mapsto q_\chi(\phi(\lambda))$ determines an isomorphism $C^*(\Lambda, \chi \circ c) \cong B_\chi$. Proposition 2.5 implies that $\pi_\chi(i_\Lambda(\lambda) i_A(f)) \mapsto \chi(f)s_\lambda$ determines an isomorphism $C^*(\Lambda, A, c)_\chi \cong C^*(\Lambda, \chi \circ c)$. Composing these two isomorphisms gives an isomorphism $C^*(\Lambda, A, c)_\chi \cong B_\chi$ which carries each $\pi_\chi(i_\Lambda(\lambda) i_A(f))$ to $q_\chi(\phi(\lambda)\pi(f))$. Thus $\phi \times \pi$ is a fibrewise-isometric homomorphism of $C_0(\widehat{A})$-algebras; Proposition C.10(c) of [27] implies that $\phi \times \pi$ is isometric. ☐

3. Lower semicontinuity

We adapt an argument due to Rieffel (see the proof of [21, Theorem 2.5]) to show that the bundles described in Proposition 2.5 are continuous bundles.

We recall some background regarding k-graph groupoids; for more detail, see [12, 15]. Fix a row-finite k-graph Λ with no sources. There is a groupoid $\mathcal{G}_\Lambda$ with unit space

Λ^∞, the space of infinite paths in Λ, whose elements have the form $(\alpha x, d(\alpha)-d(\beta), \beta x)$ where $x \in \Lambda^\infty$ and $\alpha, \beta \in \Lambda$ satisfy $s(\alpha) = s(\beta) = r(x)$. We use lower-case fraktur letters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ for elements of $\mathcal{G}_\Lambda$, and the letters x, y, z for elements of $\mathcal{G}_\Lambda^{(0)} = \Lambda^\infty$. For $\mu, \nu \in \Lambda *_s \Lambda := \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\mu) = s(\nu)\}$, we write $Z(\mu, \nu)$ for the basic compact open subset $\{(\mu x, d(\mu) - d(\nu), \nu x) : x \in s(\mu)\Lambda^\infty\}$ of $\mathcal{G}_\Lambda$. We showed in [15, §6] that there is a countable set $\mathcal{P}$ of pairs $(\mu, \nu) \in \Lambda *_s \Lambda$ such that $(\lambda, s(\lambda)) \in \mathcal{P}$ for all $\lambda \in \Lambda$ and $\mathcal{G}_\Lambda = \bigsqcup\{Z(\mu, \nu) : (\mu, \nu) \in \mathcal{P}\}$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Lemma 6.3 of [15] yields a continuous groupoid cocycle σ_c on $\mathcal{G}_\Lambda$ and an isomorphism $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$ which carries each s_λ to the indicator function $1_{Z(\lambda, s(\lambda))}$. We write $*_{\sigma_c}$ for the convolution product in $C^*(\mathcal{G}_\Lambda, \sigma_c)$.

Regard $\underline{Z}^2(\Lambda, \mathbb{T})$ as a topological subspace of $\mathbb{T}^{\Lambda^{*2}}$. Let $\ell^2(s)$ be the Hilbert $C_0(\mathcal{G}_\Lambda^{(0)})$-module completion of $C_c(\mathcal{G}_\Lambda)$ with respect to the $C_0(\mathcal{G}_\Lambda^{(0)})$-valued inner-product

$$\langle f, g\rangle_{\ell^2(s)}(x) = \textstyle\sum_{s(\mathfrak{a})=x} \overline{f(\mathfrak{a})} g(\mathfrak{a}).$$

Let $R : C^*(\mathcal{G}_\Lambda, \sigma_c) \to C_0(\mathcal{G}_\Lambda^{(0)})$ be the faithful conditional expectation such that $R(f) = f|_{\mathcal{G}_\Lambda^{(0)}}$ for $f \in C_c(\mathcal{G}_\Lambda)$. The map $\langle f, g\rangle_{\sigma_c} := R(f^* *_{\sigma_c} g)$ is a $C_0(\mathcal{G}_\Lambda^{(0)})$-valued inner product on $C_c(\mathcal{G}_\Lambda, \sigma_c)$. The completion $H(\sigma_c)$ of $C_c(\mathcal{G}_\Lambda)$ in the corresponding norm is a right-Hilbert $C_0(\mathcal{G}_\Lambda^{(0)})$-module. Given a right-Hilbert module H, let $\mathcal{L}(H)$ denote the algebra of adjointable operators on H.

Proposition 3.1. *Let Λ be a row-finite k-graph with no sources, and suppose $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Then $\langle f, g\rangle_{\sigma_c} = \langle f, g\rangle_{\ell^2(s)}$ for all $f, g \in C_c(\mathcal{G}_\Lambda)$. There is an injective homomorphism $\pi_c : C^*(\mathcal{G}_\Lambda, \sigma_c) \to \mathcal{L}(\ell^2(s))$ such that $\pi_c(f)g = f *_{\sigma_c} g$ for all $f, g \in C_c(\mathcal{G}_\Lambda)$. Suppose that $c_n \to c$ in $\underline{Z}^2(\Lambda, \mathbb{T})$. Then for all compact open bisections $U, V \subseteq \mathcal{G}_\Lambda$ and all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|(\pi_{c_n}(1_U) - \pi_c(1_U))1_V\| \leq \varepsilon$ for all $n \geq N$. For each compact open bisection $U \subseteq \mathcal{G}_\Lambda$, the map $c \mapsto \pi_c(1_U)$ is strongly continuous from $\underline{Z}^2(\Lambda, \mathbb{T})$ to $\mathcal{L}(\ell^2(s))$.*

Proof. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and $x \in \mathcal{G}_\Lambda^{(0)}$. Then

$$\langle f, g\rangle_{\sigma_c}(x) = \textstyle\sum_{\mathfrak{a}\mathfrak{b}=x} \sigma_c(\mathfrak{a}, \mathfrak{b}) f^*(\mathfrak{a}) g(\mathfrak{b}) = \sum_{s(\mathfrak{b})=x} \sigma_c(\mathfrak{b}^{-1}, \mathfrak{b}) \overline{\sigma_c(\mathfrak{b}^{-1}, \mathfrak{b}) f(\mathfrak{b})} g(\mathfrak{b}).$$

Thus $\langle f, g\rangle_{\sigma_c} = \langle f, g\rangle_{\ell^2(s)}$. Hence $\mathrm{id} : C_c(\mathcal{G}_\Lambda) \to C_c(\mathcal{G}_\Lambda)$ extends to an isometric isomorphism $\phi_c : H(\sigma_c) \to \ell^2(s)$.

Multiplication in $C_c(\mathcal{G}_\Lambda, \sigma_c)$ extends to a left action of $C^*(\mathcal{G}_\Lambda, \sigma_c)$ on $H(\sigma_c)$. Since the map $\phi_c : H(\sigma_c) \to \mathcal{L}(\ell^2(s))$ is isometric for each c, there is a homomorphism $\pi_c : C^*(\mathcal{G}_\Lambda, \sigma_c) \to \mathcal{L}(\ell^2(s))$ as claimed. The $\mathbb{Z}^k$-valued 1-cocycle $\tilde{d} : (x, m, y) \to m$ on $\mathcal{G}_\Lambda$ induces a strongly continuous $\mathbb{T}^k$ action on $C_c(\mathcal{G}_\Lambda)$ (called the gauge action) given by $\gamma_z(f)(\mathfrak{a}) = z^{\tilde{d}(\mathfrak{a})} f(\mathfrak{a})$ for $f \in C_c(\mathcal{G}_\Lambda)$ and $z \in \mathbb{T}^k$. Let ρ denote the $\mathbb{T}^k$ action on $\mathcal{L}(\ell^2(s))$ induced by the gauge action on $C_c(\mathcal{G}_\Lambda)$, then $\rho_z \circ \pi_c = \pi_c \circ \gamma_z$ for all $z \in \mathbb{T}^k$. Since $C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c)$, Corollary 7.7 of [15] implies that π_c is faithful.

Fix compact open bisections U, V of $\mathcal{G}_\Lambda$. For $\mathfrak{a} \in UV$, let $\mathfrak{a}^U \in U$ and $\mathfrak{a}^V \in V$ be the unique elements such that $\mathfrak{a} = \mathfrak{a}^U \mathfrak{a}^V$. Then

$$\begin{aligned}(\pi_c(1_U)1_V)(\mathfrak{a}) &= (1_U *_{\sigma_c} 1_V)(\mathfrak{a}) \\ &= \sum_{\mathfrak{b}\mathfrak{c}=\mathfrak{a}} \sigma_c(\mathfrak{b}, \mathfrak{c}) 1_U(\mathfrak{b}) 1_V(\mathfrak{c}) = \begin{cases} \sigma_c(\mathfrak{a}^U, \mathfrak{a}^V) & \text{if } \mathfrak{a} \in UV \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Recall from [15, Lemma 6.3] that $(\mu_{\mathfrak{a}^U}, \nu_{\mathfrak{a}^U}), (\mu_{\mathfrak{a}^V}, \nu_{\mathfrak{a}^V})$ and $(\mu_{\mathfrak{a}}, \nu_{\mathfrak{a}})$ denote the elements of $\mathcal{P}$ such that $\mathfrak{a}^U \in Z(\mu_{\mathfrak{a}^U}, \nu_{\mathfrak{a}^U})$, $\mathfrak{a}^V \in Z(\mu_{\mathfrak{a}^V}, \nu_{\mathfrak{a}^V})$ and $\mathfrak{a} \in Z(\mu_{\mathfrak{a}}, \nu_{\mathfrak{a}})$. There exist $\alpha, \beta, \gamma \in \Lambda$ and $y \in \Lambda^\infty$ such that

$$\begin{aligned} \mathfrak{a}^U = (\mu_{\mathfrak{a}^U}\alpha y, \widetilde{d}(\mathfrak{a}^U), \nu_{\mathfrak{a}^U}\alpha y), \quad \mathfrak{a}^V = (\mu_{\mathfrak{a}^V}\beta y, \widetilde{d}(\mathfrak{a}^V), \nu_{\mathfrak{a}^V}\beta y), \quad \text{and} \\ \mathfrak{a} = (\mu_{\mathfrak{a}}\gamma y, \widetilde{d}(\mathfrak{a}), \nu_{\mathfrak{a}}\gamma y). \end{aligned} \tag{3}$$

Hence $W_0 := U \cap Z(\mu_{\mathfrak{a}^U}\alpha, \nu_{\mathfrak{a}^U}\alpha)$ is a compact open neighbourhood of $\mathfrak{a}^U$ and $W_1 := V \cap Z(\mu_{\mathfrak{a}^V}\beta, \nu_{\mathfrak{a}^V}\beta)$ is a compact open neighbourhood of $\mathfrak{a}^V$. We have $s(W_0) = r(W_1)$ and $W = W_0W_1$ is a neighbourhood of $\mathfrak{a}$ contained in $UV \cap Z(\mu_{\mathfrak{a}}\gamma, \nu_{\mathfrak{a}}\gamma)$. For $\mathfrak{b} \in W$, we have

$$\mu_{\mathfrak{b}^U} = \mu_{\mathfrak{a}^U},\ \nu_{\mathfrak{b}^U} = \nu_{\mathfrak{a}^U},\ \mu_{\mathfrak{b}^V} = \mu_{\mathfrak{a}^V},\ \nu_{\mathfrak{b}^V} = \nu_{\mathfrak{a}^V}, \mu_{\mathfrak{b}} = \mu_{\mathfrak{a}},\ \text{and } \nu_{\mathfrak{b}} = \nu_{\mathfrak{a}},$$

and equations (3) are satisfied with $\mathfrak{b}$ in place of $\mathfrak{a}$ and the same paths α, β, γ but a different y. In particular, by [15, Lemma 6.3],

$$\sigma_c(\mathfrak{b}^U, \mathfrak{b}^V) = \sigma_c(\mathfrak{a}^U, \mathfrak{a}^V) = c(\mu_{\mathfrak{a}^U}, \alpha)\overline{c(\nu_{\mathfrak{a}^U}, \alpha)}c(\mu_{\mathfrak{b}^V}, \beta)\overline{c(\nu_{\mathfrak{b}^V}, \beta)}c(\nu_{\mathfrak{a}}, \gamma)\overline{c(\mu_{\mathfrak{a}}, \gamma)}.$$

By compactness of U, V and UV, there is a finite collection $\mathcal{W}$ of mutually disjoint compact open bisections such that $UV = \bigcup \mathcal{W}$, and for $W \in \mathcal{W}$ there exist μ_W, ν_W, η_W, ζ_W, σ_W, τ_W, α_W, β_W, and γ_W in Λ such that, for all $\mathfrak{a} \in W$,

$$(\pi_c(1_U)1_V)(\mathfrak{a}) = c(\mu_W, \alpha_W)\overline{c(\nu_W, \alpha_W)}c(\eta_W, \beta_W)\overline{c(\zeta_W, \beta_W)}c(\sigma_W, \gamma_W)\overline{c(\tau_W, \gamma_W)}.$$

For each $W \in \mathcal{W}$ let U_W and V_W be the subsets of U and V such that $W = U_WV_W$ (and $s(U_W) = r(V_W)$). For each W,

$$\pi_c(1_{U_W})1_{V_W} = c(\mu_W, \alpha_W)\overline{c(\nu_W, \alpha_W)}c(\eta_W, \beta_W)\overline{c(\zeta_W, \beta_W)}c(\sigma_W, \gamma_W)\overline{c(\tau_W, \gamma_W)}1_W. \tag{4}$$

Now suppose that $c_n \to c$ in $\underline{Z}^2(\Lambda, \mathbb{T})$ and fix $\varepsilon > 0$. Since multiplication in $\mathbb{T}$ is continuous, (4) implies that there exists N such that $n \geq N$ implies $\|\pi_{c_n}(1_{U_W})1_{V_W} - \pi_c(1_{U_W})1_{V_W}\| < \varepsilon/|\mathcal{W}|$ for each $W \in \mathcal{W}$. Since $\bigcup \mathcal{W}$ is a bisection, given any collection $\{a_W : W \in \mathcal{W}\}$ of scalars, we have

$$\Big\| \sum_{W \in \mathcal{W}} a_W 1_W \Big\|^2 = \sup_{x \in \mathcal{G}_\Lambda^{(0)}} \sum_{W, W' \in \mathcal{W}} \overline{a_W} a_{W'} 1_{W^{-1}W'}(x) = \max_{W \in \mathcal{W}} |a_W|^2.$$

That $\pi_c(1_U)1_V = \sum_{W \in \mathcal{W}} \pi_c(1_{U_W})1_{V_W}$ gives $\|\pi_{c_n}(1_U)1_V - \pi_c(1_U)1_V\| < \varepsilon$ for all $n \geq N$.

Fix $f \in \ell^2(s)$ and $\varepsilon > 0$. There is a finite set $\mathcal{V}$ of compact open bisections and a linear combination $f_0 := \sum_{V \in \mathcal{V}} a_V 1_V$ such that $\|f - f_0\| \leq \varepsilon/3$. The preceding paragraph yields N such that $\|\pi_{c_n}(1_U)1_V - \pi_c(1_U)1_V\| < \frac{\varepsilon}{3|\mathcal{V}|\,|a_V|}$ for $V \in \mathcal{V}$ and $n \geq N$. For $n \geq N$,

$$\begin{aligned} \|\pi_{c_n}(1_U)f - \pi_c(1_U)f\| \leq \|\pi_{c_n}(1_U)f - \pi_{c_n}(1_U)f_0\| + \|\pi_{c_n}(1_U)f_0 - \pi_c(1_U)f_0\| \\ + \|\pi_c(1_U)f_0 - \pi_c(1_U)f\| \leq \varepsilon \end{aligned}$$

since $\|\pi_c(1_U)\| \leq \|1_U\| = 1$. Hence $c \mapsto \pi_c(1_U)$ is strongly continuous. □

For the following corollary, recall that the isomorphism $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$ of [15, Theorems 6.7 and 7.9] carries each $s_\lambda s_\mu^*$ to a finite linear combination, with coefficients in $\mathbb{T}$, of elements of the form $1_{Z(\lambda\alpha, \mu\alpha)}$.

Corollary 3.2. *Resume the hypotheses of Proposition 3.1. If $a : \Lambda *_s \Lambda \to \mathbb{C}$ has finite support, then $S_a : c \mapsto \sum_{\lambda,\mu} a(\lambda, \mu)\pi_c(s_\lambda s_\mu^*)$ is strongly continuous from $\underline{Z}^2(\Lambda, \mathbb{T})$ to $\mathcal{L}(\ell^2(s))$, and $c \mapsto \|S_a(c)\|$ is lower-semicontinuous from $\underline{Z}^2(\Lambda, \mathbb{T})$ to $[0, \infty)$.*

Proof. Fix a finitely supported function a, and suppose that $c_n \to c$ in $\underline{Z}^2(\Lambda, \mathbb{T})$. Identify $C^*(\Lambda, c)$ with $C^*(\mathcal{G}_\Lambda, \sigma_c)$ as in [15, Theorems 6.7 and 7.9]. We may then assume that each $\pi_c(s_\lambda s_\mu^*) = \sum_{\tau \in F} z_\tau 1_{Z(\lambda\tau, \mu\tau)}$ for some finite set F and $z_\tau \in \mathbb{T}$. By relabelling we may assume that $S_a(c) = \sum_{\lambda,\mu} \tilde{a}(\lambda, \mu)\pi_c(1_{Z(\lambda,\mu)})$ where $\tilde{a}$ has finite support. Fix $x \in \ell^2(s)$ of norm 1 and $\varepsilon > 0$. Let $M := \sum_{\lambda,\mu} |\tilde{a}_{\lambda,\mu}|$. Whenever $\tilde{a}_{\lambda,\mu} \neq 0$, Proposition 3.1 yields $N_{\lambda,\mu} \in \mathbb{N}$ such that $\big\|\big(\pi_{c_n}(1_{Z(\lambda,\mu)}) - \pi_c(1_{Z(\lambda,\mu)})\big)x\big\| \le \frac{\varepsilon}{M}$. Let $N := \max_{\lambda,\mu} N_{\lambda,\mu}$. For $n \ge N$, we have $\|(S_a(c_n) - S_a(c))x\| \le \frac{\varepsilon}{M} \sum_{\lambda,\mu} |\tilde{a}_{\lambda,\mu}| \le \varepsilon$. Thus S_a is strongly continuous.

To see that $c \mapsto \|S_a(c)\|$ is lower-semicontinuous, fix $\varepsilon > 0$. We must show that there exists $N \in \mathbb{N}$ such that $\|S_a(c_n)\| \ge \|S_a(c)\| - \varepsilon$ whenever $n \ge N$. For this, fix $x \in \ell^2(s)$ such that $\|x\| = 1$ and $\|S_a(c)x\| \ge \|S_a(c)\| - \varepsilon/2$. By the preceding paragraph there exists $N \in \mathbb{N}$ such that $n \ge N$ implies $\|S_a(c_n)x - S_a(c)x\| \le \varepsilon/2$, and in particular

$$\|S_a(c_n)\| \ge \|S_a(c_n)x\| \ge \|S_a(c)x\| - \varepsilon/2 \ge \|S_a(c)\| - \varepsilon.$$

The final assertion is clear because each $C^*(\Lambda, c) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$. □

Corollary 3.3. *Let Λ be a row-finite k-graph with no sources, let A be an abelian group, and let $c \in \underline{Z}^2(\Lambda, A)$. The upper-semicontinuous C^*-bundle over $\widehat{A}$ with fibres $\pi_\chi(C^*(\Lambda, A, c)) \cong C^*(\Lambda, \chi \circ c)$ described in Proposition 2.5 is continuous.*

Proof. The map $\chi \mapsto \chi \circ c$ from $\widehat{A}$ to $\underline{Z}^2(\Lambda, \mathbb{T})$ is continuous. Let $\{s_\lambda : \lambda \in \Lambda\}$ be the generators of $C^*(\Lambda, \chi \circ c)$. Proposition 3.1 implies that for a finitely supported function $a : \Lambda *_s \Lambda \to \mathbb{C}$, we have $\|\sum a(\lambda, \mu) s_\lambda s_\mu^*\| = \|S_a(\chi \circ c)\|$. It therefore follows from Corollary 3.2 that $\chi \mapsto \big\|\pi_\chi\big(\sum a(\lambda, \mu) i_\Lambda(\lambda) i_\Lambda(\mu)^*\big)\big\|$ is lower semicontinuous. Since Proposition 2.5 implies that it is also upper-semicontinuous, the result follows. □

Example 3.4 (Fields of noncommutative tori). Fix $k \ge 2$. Let T_k be a copy of $\mathbb{N}^k$ regarded as a k-graph with degree functor the identity map on $\mathbb{N}^k$. Let A be the free abelian group generated by $\{(i, j) : 1 \le j < i \le k\}$. There is an A-valued 2-cocycle c on T_k given by

$$c(m, n) := \textstyle\sum_{j<i} m_i n_j \cdot (i, j).$$

Using relations (R1)–(R5), one can check that $C^*(T_k, A, c)$ is universal for unitaries $\{U_m : m \in \mathbb{Z}^k\} \cup \{W_a : a \in A\}$ such that, for $m, n \in T_k$ and $a, b \in A$,

$$W_a W_b = W_{a+b}, \qquad U_m U_n = W_{c(m,n)} U_{m+n}, \qquad \text{and} \qquad U_m W_a = W_a U_m.$$

Hence $U_{e_i} U_{e_j} = W_{(i,j)} U_{e_i + e_j} = W_{(i,j)} U_{e_j} U_{e_i}$ for $j < i$. Let G be the group generated by $\{g_1, \dots, g_k\} \cup \{h_{i,j} : 1 \le j < i \le k\}$ such that $g_i g_j = h_{i,j} g_j g_i$ and $g_\ell h_{i,j} = h_{i,j} g_\ell$ for all i, j, ℓ with $j < i$. Then $C^*(T_k, A, c)$ and $C^*(G)$ have the same universal property, so coincide.

Fix scalars $z_{i,j} \in \mathbb{T}$ for $1 \le j < i \le k$. The noncommutative torus A_z is the universal C^*-algebra generated by unitaries $U_1, \dots, U_k$ subject to $U_i U_j = z_{i,j} U_j U_i$ for $j < i$ (see, for example, [6, 1.4]). There is a unique character χ_z of A satisfying $\chi_z(i, j) = z_{i,j}$ for $j < i$. The twisted k-graph C^*-algebra $C^*(T_k, \chi_z \circ c)$ is the quotient of $C^*(T_k, A, c)$ by the ideal generated by $\{W_{(i,j)} - z_{i,j} 1 : j < i\}$. Hence the preceding paragraph implies that $C^*(T_k, \chi_z \circ c)$ is the noncommutative torus A_z. Thus Corollary 3.3 shows that $C^*(T_k, A, c)$ is the algebra of sections of a continuous bundle over $\prod_{j<i\le k} \mathbb{T}$ whose fibre over z is A_z.

When $k = 2$, the group G is the integer Heisenberg group $H_3(\mathbb{Z})$, and we recover the description of $C^*(H_3(\mathbb{Z}))$ as the sections of a continuous field of rotation algebras [1].

Example 3.5 (Fields of Heegaard-type quantum spheres). Consider the Heegaard-type quantum spheres $C(S^3_{00\theta})$ of [2]. It was shown in [14, Example 7.10] that these C^*-algebras are the twisted C^*-algebras of a finite 2-graph Λ, and that $\underline{H}^2(\Lambda, \mathbb{T}) = \mathbb{T}$, with the cohomology class corresponding to $e^{2\pi\imath\theta}$ represented by $c_\theta : (\mu, \nu) \mapsto e^{2\pi\imath(d(\mu)_2 d(\nu)_1)\theta}$. We then have $C(S^3_{00\theta}) \cong C^*(\Lambda, c_\theta)$.

Let $b \in \underline{Z}^2(\Lambda, \mathbb{Z})$ be the cocycle $b(\mu, \nu) = d(\mu)_2 d(\nu)_1$. Identifying $\mathbb{T}$ with $\widehat{\mathbb{Z}}$ by $z \mapsto (\chi_z : n \mapsto z^n)$, each c_θ is equal to $\chi_{e^{2\pi\imath\theta}} \circ b$. Thus Corollary 3.3 shows that $C^*(\Lambda, A, c_\theta)$ is the section algebra of a continuous field over $\mathbb{T}$ of Heegaard-type quantum spheres.

4. A field of AF algebras

If Λ is a k-graph and b is a function from Λ^0 to A, then the corresponding 1-coboundary $\underline{\delta}^0 b : \Lambda \to A$ is given by $\underline{\delta}^0 b(\lambda) = b(s(\lambda)) - b(r(\lambda))$.

Suppose that the degree map on Λ satisfies $d = \underline{\delta}^0 b$ for some $b : \Lambda^0 \to \mathbb{Z}^k$. By [12, Lemma 5.4] it follows that $C^*(\Lambda)$ is AF; and [15, Theorem 8.4] says that for any $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ the twisted algebra $C^*(\Lambda, c)$ is isomorphic to $C^*(\Lambda)$. In Theorem 4.2 we combine these results with the bundle structure of $C^*(\Lambda, A, c)$ given in Proposition 2.5 to show that $C^*(\Lambda, A, c) \cong C^*(A) \otimes C^*(\Lambda)$ for every $c \in \underline{Z}^2(\Lambda, A)$.

Given a nonempty set X, we write $\mathcal{K}_X$ for the unique nonzero C^*-algebra generated by elements $\{\theta_{x,y} : x, y \in X\}$ satisfying $\theta^*_{x,y} = \theta_{y,x}$ and $\theta_{x,y}\theta_{w,z} = \delta_{y,w}\theta_{x,z}$. We call the $\theta_{x,y}$ *matrix units* for $\mathcal{K}_X$.

Lemma 4.1. *Let Λ be a row-finite k-graph with no sources and suppose that $d = \underline{\delta}^0 b$ for some $b : \Lambda^0 \to \mathbb{Z}^k$. Let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. For each $n \in \mathbb{Z}^k$, the subspace $B_n := \overline{\operatorname{span}}\{i_\Lambda(\lambda) i_A(f) i_\Lambda(\mu)^* : s(\lambda) = s(\mu) \in b^{-1}(n), f \in C^*(A)\}$ of $C^*(\Lambda, A, c)$ is a C^*-subalgebra. The formula $f \otimes \theta_{\lambda,\mu} \mapsto i_\Lambda(\lambda) i_A(f) i_\Lambda(\mu)^*$ determines an isomorphism $\rho_n : C^*(A) \otimes \big(\bigoplus_{b(v)=n} \mathcal{K}_{\Lambda v}\big) \to B_n$.*

Proof. As in the proof of [12, Lemma 5.4] (or [15, Lemma 8.4]), $\{i_\Lambda(\lambda) i_\Lambda(\mu)^* : s(\lambda) = s(\mu) \in b^{-1}(n)\}$ is a set of nonzero matrix units, and so spans a copy of $\bigoplus_{b(v)=n} \mathcal{K}_{\Lambda v}$ in $\mathcal{M}(C^*(\Lambda, A, c))$. Each $i_\Lambda(\lambda) i_\Lambda(\mu)^*$ commutes with $i_A(C^*(A))$, and so B_n is a $C_0(\widehat{A})$-algebra. The universal property of the tensor product gives a (clearly surjective) homomorphism $\rho_n : C^*(A) \otimes \big(\bigoplus_{b(v)=n} \mathcal{K}_{\Lambda v}\big) \to B_n$ such that $\rho_n(f \otimes \theta_{\lambda,\mu}) = i_\Lambda(\lambda) i_A(f) i_\Lambda(\mu)^*$.

If $\chi \in \widehat{A}$ and $f \in C^*(A)$ satisfy $f(\chi) = 1$, then π_χ carries the $i_\Lambda(\lambda) i_A(f) i_\Lambda(\mu)^*$ to nonzero matrix units in $C^*(\Lambda, \chi \circ c)$. Hence evaluation at χ induces an isomorphism $\big(C^*(A) \otimes \big(\bigoplus_{b(v)=n} \mathcal{K}_{\Lambda v}\big)\big)_\chi \cong (B_n)_\chi$. Thus ρ_n is an isomorphism. □

The idea of the following proof is due to Ben Whitehead [26].

Theorem 4.2. *Let Λ be a row-finite k-graph with no sources and suppose that $d = \underline{\delta}^0 b$ for some $b : \Lambda^0 \to \mathbb{Z}^k$. Let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. For $m \le n \in \mathbb{N}^k$, the inclusion $j_{m,n} : B_m \subseteq B_n$ is given by*

$$j_{m,n}(i_\Lambda(\lambda) i_A(f) i_\Lambda(\mu)^*) = \sum_{\nu \in s(\lambda)\Lambda^{n-m}} i_A(c(\lambda, \nu) - c(\mu, \nu)) i_\Lambda(\lambda\nu) i_A(f) i_\Lambda(\mu\nu)^*.$$

Furthermore, $C^*(\Lambda, A, c) = \varinjlim(B_n, j_{m,n}) \cong C^*(A) \otimes C^*(\Lambda)$.

Proof. The formula for the inclusion maps $j_{m,n}$ follows from (R3) and (R5). Each spanning element of $C^*(\Lambda, A, c)$ belongs to $\bigcup_n B_n$, and hence $C^*(\Lambda, A, c) = \varinjlim(B_n, j_{m,n})$. For

$n \in \mathbb{N}^k$, let $\rho_n : C^*(A) \otimes \big(\bigoplus_{b(v)=n} \mathcal{K}_{\Lambda v}\big) \to B_n$ be as in Lemma 4.1. For $m \le n \in \mathbb{N}^k$ let $\iota_{m,n} : \bigoplus_{b(v)=m} \mathcal{K}_{\Lambda v} \to \bigoplus_{b(w)=n} \mathcal{K}_{\Lambda w}$ be the inclusion $\iota_{m,n}(\theta_{\lambda,\mu}) = \sum_{\alpha \in s(\lambda)\Lambda^{n-m}} \theta_{\lambda\alpha,\mu\alpha}$.

Let $\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^k$. Define $\kappa : \Lambda \to A$ recursively by

$$\kappa(\mu) = \begin{cases} 0 & \text{if } d(\mu) \not\geq \mathbf{1}, \\ \kappa(\lambda) + c(\lambda, \alpha) & \text{if } \mu = \lambda\alpha \text{ and } d(\alpha) = \mathbf{1}. \end{cases}$$

Let $u_A : A \to \mathcal{UM}(C^*(A))$ denote the canonical map. For $h \in \mathbb{N}$, an $\varepsilon/3$-argument (see, for example, [17]) shows that $\sum_{b(s(\lambda))=h\cdot\mathbf{1}} u_A(\kappa(\lambda)) \otimes \theta_{\lambda,\lambda}$ converges strictly to a unitary $U_h \in \mathcal{UM}\big(C^*(A) \otimes \big(\bigoplus_{b(v)=h\cdot\mathbf{1}} \mathcal{K}_{\Lambda v}\big)\big)$.

Fix $h \in \mathbb{N}$ and $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu) \in b^{-1}(h \cdot \mathbf{1})$. Then

$$\begin{aligned} &j_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}} \circ \rho_{h\cdot\mathbf{1}}(U_h(f \otimes \theta_{\lambda,\mu})U_h^*) \\ &\qquad = j_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}}\big(i_A(\kappa(\lambda) - \kappa(\mu)) i_\Lambda(\lambda) i_A(f) i_\Lambda(\mu)^*\big) \\ &\qquad = \textstyle\sum_{\alpha \in s(\lambda)\Lambda^{\mathbf{1}}} i_A\big((\kappa(\lambda) + c(\lambda,\alpha)) - (\kappa(\mu) + c(\mu,\alpha))\big) i_\Lambda(\lambda\alpha) i_A(f) i_\Lambda(\mu\alpha)^* \\ &\qquad = \textstyle\sum_{\alpha \in s(\lambda)\Lambda^{\mathbf{1}}} \rho_{(h+1)\cdot\mathbf{1}}\big(u_A\big(\kappa(\lambda\alpha) - \kappa(\mu\alpha)\big) f \otimes \theta_{\lambda\alpha,\mu\alpha}\big) \\ &\qquad = \rho_{(h+1)\cdot\mathbf{1}}\big(U_{h+1}(f \otimes \iota_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}}(\theta_{\lambda,\mu}))U_{h+1}^*\big). \end{aligned}$$

That is, $j_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}}\big(\rho_{h\cdot\mathbf{1}} \circ \mathrm{Ad}(U_h)\big) = \big(\rho_{(h+1)\cdot\mathbf{1}} \circ \mathrm{Ad}(U_{h+1})\big)(\mathrm{id}_{C^*(A)} \otimes \iota_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}})$. Thus

$$\begin{aligned} C^*(\Lambda, A, c) &= \varinjlim(B_n, j_{m,n}) \cong \varinjlim(B_{h\cdot\mathbf{1}}, j_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}}) \\ &\cong \varinjlim\big(C^*(A) \otimes \big(\textstyle\bigoplus_{b(v)=h\cdot\mathbf{1}} \mathcal{K}_{\Lambda v}\big), \mathrm{id}_{C^*(A)} \otimes \iota_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}}\big). \end{aligned}$$

Since $C^*(\Lambda) \cong \varinjlim\big(\bigoplus_{b(v)=h\cdot\mathbf{1}} \mathcal{K}_{\Lambda v}, \iota_{h\cdot\mathbf{1},(h+1)\cdot\mathbf{1}}\big)$, the result follows. □

Corollary 4.3. *Let Λ be a row-finite k-graph with no sources. Suppose that $d = \underline{\delta}^0 b$ for some $b : \Lambda^0 \to \mathbb{Z}^k$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{R})$, and identify $\widehat{\mathbb{R}}$ with $\mathbb{R}$ via $t \mapsto (\omega_t : s \mapsto e^{\imath ts})$. Then $C^*(\Lambda, \mathbb{R}, c)|_{[0,1]} \cong C([0,1]) \otimes C^*(\Lambda)$, and the maps $\pi_t : C^*(\Lambda, \mathbb{R}, c) \to C^*(\Lambda, \omega_t \circ c)$ induce isomorphisms $K_*(C^*(\Lambda, \mathbb{R}, c)|_{[0,1]}) \cong K_*(C^*(\Lambda))$ such that $(\pi_t)_*\big([i_\Lambda(v)]_0\big) = [s_v]_0$ for $v \in \Lambda^0$.*

Proof. The first statement is a special case of Theorem 4.2. The Künneth theorem [4, Theorem 23.1.3] and that $K_0(C([0,1]), [1]) \cong (\mathbb{Z}, 1)$ gives the second statement. □

5. K-theory

In this section we prove our main result, Theorem 5.4: if $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ arises by exponentiation of an $\mathbb{R}$-valued 2-cocycle, then $K_*(C^*(\Lambda, c)) \cong K_*(C^*(\Lambda))$. Our argument is essentially that devised by Elliott to prove [6, Theorem 2.2]. We believe that Gwion Evans' argument [7] could also be adapted to obtain a spectral sequence converging to $K_*(C^*(\Lambda, c))$. The advantage of our approach is that when c is of exponential form we reduce the problem of computing $K_*(C^*(\Lambda, c))$ to that of computing $K_*(C^*(\Lambda))$ whether or not the latter is computable using Evans' spectral sequence.

Given an action δ of a discrete group G on a C^*-algebra D, we write $(\jmath_D, \jmath_G)$ for the universal covariant representation of (D, G, δ). That is, $\jmath_D : D \to D \times_\delta G$ (we also write $\jmath_D : \mathcal{M}(D) \to \mathcal{M}(D \times_\delta G)$ for the extension) and $\jmath_G : G \to \mathcal{UM}(D \times_\delta G)$ are the canonical embeddings. The following is surely well known to K-theory experts and is implicitly contained in the proof of [6, Theorem 2.2].

Theorem 5.1. *Let $\phi : B \to C$ be a homomorphism of C^*-algebras which induces an isomorphism $\phi_* : K_*(B) \to K_*(C)$. Suppose that $\beta : \mathbb{Z}^k \to \mathrm{Aut}(B)$ and $\gamma : \mathbb{Z}^k \to \mathrm{Aut}(C)$ are actions such that $\gamma_n \circ \phi = \phi \circ \beta_n$ for all $n \in \mathbb{Z}^k$. There is a homomorphism $\widetilde{\phi} : B \times_\beta \mathbb{Z}^k \to C \times_\gamma \mathbb{Z}^k$ such that $\widetilde{\phi}(\jmath_B(b)\jmath_{\mathbb{Z}^k}(n)) = \jmath_C(\phi(b))\jmath_{\mathbb{Z}^k}(n)$. The induced map $\widetilde{\phi}_* : K_*(B \times_\beta \mathbb{Z}^k) \to K_*(C \times_\gamma \mathbb{Z}^k)$ is an isomorphism, and $\widetilde{\phi}_* \circ (\jmath_B)_* = (\jmath_C)_* \circ \phi_*$.*

Proof. Universality of the crossed product applied to the covariant representation $(\phi \circ \jmath_B, \jmath_{\mathbb{Z}^k})$ yields a homomorphism $\widetilde{\phi} : B \times_\beta \mathbb{Z}^k \to C \times_\gamma \mathbb{Z}^k$ as described.

To see that $\widetilde{\phi}$ induces an isomorphism in K-theory satisfying $\widetilde{\phi}_* \circ (\jmath_B)_* = (\jmath_C)_* \circ \phi_*$, we proceed by induction on k.

The base case $k = 0$ is trivial. Fix $n \geq 0$, suppose as an inductive hypothesis that the result holds for all $k \leq n$, and consider $k = n + 1$. Let $\beta^1 : \mathbb{Z} \to \mathrm{Aut}(B)$ be the restriction of β to $\{(m, 0, \dots, 0) : m \in \mathbb{Z}\} \subseteq \mathbb{Z}^k$ and let β^n be the $\mathbb{Z}^n$ action on $B \times_{\beta^1} \mathbb{Z}$ induced by the restriction of β to the last n coordinates of $\mathbb{Z}^k$. Universality implies that $B \times_\beta \mathbb{Z}^k$ is canonically isomorphic to $(B \times_{\beta^1} \mathbb{Z}) \times_{\beta^n} \mathbb{Z}^n$. Define $\gamma^1 : \mathbb{Z} \to \mathrm{Aut}(C)$ and $\gamma^n : \mathbb{Z}^n \to \mathrm{Aut}(C \times_{\gamma^1} \mathbb{Z})$ similarly; by hypothesis, ϕ is equivariant for β^1 and γ^1, and so it induces a homomorphism $\widetilde{\phi}^1 : B \times_{\beta^1} \mathbb{Z} \to C \times_{\gamma^1} \mathbb{Z}$. Naturality of the Pimsner-Voiculescu sequence induces a commuting diagram

$$\begin{array}{ccccccccc}
K_0(B) & & \xrightarrow{1-\beta^1_*} & & K_0(B) & & \xrightarrow{(\jmath_B)_*} & & K_0(B\times_{\beta^1}\mathbb{Z}) \\
\uparrow & \searrow^{\phi_*} & & & \downarrow \phi_* & & & \swarrow^{\widetilde{\phi}^1_*} & \downarrow \\
\uparrow & & K_0(C) & \xrightarrow{1-\gamma^1_*} & K_0(C) & \xrightarrow{(\jmath_C)_*} & K_0(C\times_{\gamma^1}\mathbb{Z}) & & \downarrow \\
\uparrow & & \uparrow & & & & \downarrow & & \downarrow \\
\uparrow & & K_1(C\times_{\gamma^1}\mathbb{Z}) & \xleftarrow{(\jmath_C)_*} & K_1(C) & \xleftarrow{1-\gamma^1_*} & K_1(C) & & \downarrow \\
\uparrow & \nearrow^{\widetilde{\phi}^1_*} & & & \uparrow \phi_* & & & \nwarrow^{\phi_*} & \downarrow \\
K_1(B\times_{\beta^1}\mathbb{Z}) & & \xleftarrow{(\jmath_B)_*} & & K_1(B) & & \xleftarrow{1-\beta^1_*} & & K_1(B)
\end{array}$$

in which the ϕ_* are isomorphisms. The Five Lemma implies that the $\widetilde{\phi}^1_*$ are isomorphisms. The relations $\widetilde{\phi}^1_* \circ (\jmath_B)_* = (\jmath_C)_* \circ \phi_*$ appear as squares in the diagram.

The inductive hypothesis applied to the actions β^n and γ^n and the equivariant homomorphism $\widetilde{\phi}^1$ yields a homomorphism $\widetilde{\phi}^n : (B \times_{\beta^1} \mathbb{Z}) \times_{\beta^n} \mathbb{Z}^n \to (C \times_{\gamma^1} \mathbb{Z}) \times_{\gamma^n} \mathbb{Z}^n$ which induces an isomorphism in K-theory and satisfies $\widetilde{\phi}^n_* \circ (\jmath_{B \times_{\beta^1} \mathbb{Z}})_* = (\jmath_{C \times_{\gamma^1} \mathbb{Z}})_* \circ \widetilde{\phi}^1_*$. Since ϕ intertwines the isomorphism $(B \times_{\beta^1} \mathbb{Z}) \times_{\beta^n} \mathbb{Z}^n \cong B \times_\beta \mathbb{Z}^{n+1}$ and the corresponding isomorphism for C, γ, the squares in the following diagram (in which we have suppressed the subscripts on the inclusion maps $\jmath$) commute.

$$\begin{array}{ccccccc}
K_*(B) & \xrightarrow{\jmath_*} & K_*(B \times_{\beta^1} \mathbb{Z}) & \xrightarrow{\jmath_*} & K_*((B \times_{\beta^1} \mathbb{Z}) \times_{\beta^n} \mathbb{Z}^n) & \xrightarrow{\cong} & K_*(B \times_\beta \mathbb{Z}^k) \\
\downarrow \phi_* & & \downarrow \widetilde{\phi}^1_* & & \downarrow \widetilde{\phi}^n_* & & \downarrow \widetilde{\phi}_* \\
K_*(C) & \xrightarrow{\jmath_*} & K_*(C \times_{\gamma^1} \mathbb{Z}) & \xrightarrow{\jmath_*} & K_*((C \times_{\gamma^1} \mathbb{Z}) \times_{\gamma^n} \mathbb{Z}^n) & \xrightarrow{\cong} & K_*(C \times_\gamma \mathbb{Z}^k)
\end{array}$$

Commutativity of the outside rectangle gives $\widetilde{\phi}_* \circ (\jmath_B)_* = (\jmath_C)_* \circ \phi_*$. □

The Morita equivalence described in the following lemma is an application of Takai duality, and could likely be deduced from arguments like those of [12, Section 5]. However, the details of the isomorphism (5) implementing the Morita equivalence will be important in the proof of our main theorem. Recall from [12] that if Λ is a k-graph, then the skew-product $\Lambda \times_d \mathbb{Z}^k$ of Λ by its degree functor is the set $\Lambda \times \mathbb{Z}^k$ with degree map $d(\lambda, n) = d(\lambda)$, range and source maps $r(\lambda, n) = (r(\lambda), n)$ and $s(\lambda, n) = (s(\lambda), n + d(\lambda))$ and composition $(\lambda, n)(\mu, n + d(\lambda)) = (\lambda\mu, n)$. This skew-product is a k-graph, and is row-finite and has no sources if and only if Λ has the same properties.

Lemma 5.2. *Let Λ be a row-finite k-graph with no sources, and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Define $\phi : \Lambda \times_d \mathbb{Z}^k \to \Lambda$ by $\phi(\lambda, n) = \lambda$. Then $c \circ \phi \in \underline{Z}^2(\Lambda \times_d \mathbb{Z}^k, \mathbb{T})$. There is an action $\mathrm{lt} : \mathbb{Z}^k \to \mathrm{Aut}(C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi))$ determined by $\mathrm{lt}_n(s_{(\lambda,m)}) = s_{(\lambda,m+n)}$. The series*

$$\textstyle\sum_{v\in\Lambda^0} \jmath_{C^*(\Lambda\times_d\mathbb{Z}^k, c\circ\phi)}(s_{(v,0)})$$

converges strictly to a full projection $P_0 \in \mathcal{M}\big(C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k\big)$, and there is an isomorphism

$$C^*(\Lambda, c) \cong P_0\big(C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k\big)P_0, \tag{5}$$

which carries each s_v to $\jmath_{C^*(\Lambda\times_d\mathbb{Z}^k, c\circ\phi)}(s_{(v,0)})$.

Proof. For $n \in \mathbb{N}^k$, $(\lambda, m) \mapsto s_{(\lambda,m+n)}$ determines a Cuntz-Krieger $(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$-family in $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$. Thus the universal property gives a homomorphism $\mathrm{lt}_n : C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \to C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$ satisfying $\mathrm{lt}_n(s_{(\lambda,m)}) = s_{(\lambda,m+n)}$. It is straightforward to check that this determines an action lt of $\mathbb{Z}^k$ on $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$.

To simplify calculations, we identify the generators of $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$ with their images in $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k$. For $m \in \mathbb{Z}^k$ we write u_m for $\jmath_{\mathbb{Z}^k}(m) \in \mathcal{UM}\big(C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k\big)$.

It is standard that the sum defining P_0 converges strictly to a multiplier projection (see, for example, [20, Lemma 2.10]). The series $\sum_{(v,n)\in\Lambda^0\times\mathbb{Z}^k} s_{(v,n)}$ is an approximate identity for $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k$. Hence P_0 is full because

$$s_{(v,n)} = \mathrm{lt}_n(s_{(v,0)}) = u_n s_{(v,0)} u_n^* = u_n P_0 s_{(v,0)} u_n^* \quad \text{for all } (v, n) \in \Lambda^0 \times \mathbb{Z}^k.$$

For $\lambda \in \Lambda$, define $t_\lambda := s_{(\lambda,0)} u_{d(\lambda)}$. Then $t_\lambda t_\lambda^* = s_{(\lambda,0)} s_{(\lambda,0)}^* \le P_0$, and

$$t_\lambda^* t_\lambda = u_{d(\lambda)}^* s_{(s(\lambda),d(\lambda))} u_{d(\lambda)} = s_{(s(\lambda),0)} \le P_0,$$

so the t_λ belong to the corner determined by P_0.

Straightforward calculations using the relation $u_n s_{(\lambda,m)} u_n^* = s_{(\lambda,m+n)}$ show that the t_λ form a Cuntz-Krieger (Λ, c)-family in $P_0(C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k)P_0$. Hence the universal property of $C^*(\Lambda, c)$ gives a homomorphism $\pi_t : C^*(\Lambda, c) \to P_0\big(C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k\big)P_0$ which carries each s_v to $t_v = \jmath_{C^*(\Lambda\times_d\mathbb{Z}^k, c\circ\phi)}(s_{(v,0)})$.

Since the generators of $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$ are nonzero, the t_v are nonzero. The gauge action on $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi)$ induces an action β of $\mathbb{T}^k$ on $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k$ satisfying $\beta_z(t_\lambda) = z^{d(\lambda)} t_\lambda$. The gauge-invariant uniqueness theorem [15, Corollary 7.7] therefore implies that π_t is injective. To check that π_t is surjective, observe that $C^*(\Lambda \times_d \mathbb{Z}^k, c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k$ is densely spanned by elements of the form $s_{(\lambda,m)} s_{(\mu,n)}^* u_p$, where $s(\lambda, m) = s(\mu, n)$

and hence $n - m = d(\lambda) - d(\mu)$. For a nonzero spanning element $s_{(\lambda,m)}s^*_{(\mu,n)}u_p$,

$$P_0 s_{(\lambda,m)}s^*_{(\mu,n)}u_p P_0 = P_0 s_{(\lambda,m)}u_p s^*_{(\mu,n-p)}P_0 = \begin{cases} s_{(\lambda,0)}u_{n-m}s^*_{(\mu,0)} & \text{if } m = n-p = 0 \\ 0 & \text{otherwise.}\end{cases}$$

Since $n - m = d(\lambda) - d(\mu)$ we have $s_{(\lambda,0)}u_{n-m}s^*_{(\mu,0)} = t_\lambda t^*_\mu$. Thus π_t is surjective. □

To prove our main theorem, we need a standard technical lemma, which we include for completeness and to establish notation.

Lemma 5.3. *Let X be a compact Hausdorff space, and suppose that B is a $C(X)$-algebra with respect to a homomorphism $p : C(X) \to \mathcal{ZM}(B)$. Let α be an action of a discrete group G on B whose extension to $\mathcal{M}(B)$ fixes $p(C(X))$. Then $\jmath_B \circ p$ maps $C(X)$ into $\mathcal{ZM}(B \times_\alpha G)$ and $B \times_\alpha G$ is also a $C(X)$-algebra. For each $x \in X$, α induces an action α^x of G on B_x and the quotient map $\pi^x : B \to B_x$ is equivariant for α and α^x. The induced homomorphism $\widetilde{\pi}^x : B \times_\alpha G \to B_x \times_{\alpha^x} G$ satisfies*

$$\widetilde{\pi}^x \circ \jmath_B = \jmath_{B_x} \circ \pi^x \quad \text{and} \quad \widetilde{\pi}^x \circ \jmath_G = \jmath^x_G, \tag{6}$$

where $(\jmath_{B_x}, \jmath^x_G)$ is the universal covariant representation of (B_x, G, α^x). Moreover, $\widetilde{\pi}^x$ descends to an isomorphism $(B \times_\alpha G)_x \cong B_x \times_{\alpha^x} G$.

Proof. The range of $\jmath_B \circ p$ is central because α fixes $p(C(X))$. Fix $x \in X$. The quotient map π^x is equivariant for α and α^x by definition of the latter. Hence $(\jmath_{B_x} \circ \pi^x, \jmath^x_G)$ is a covariant representation of (B, α, G), and so induces a homomorphism $\widetilde{\pi}^x : B \times_\alpha G \to B_x \times_{\alpha^x} G$ satisfying (6). This $\widetilde{\pi}^x$ descends to a homomorphism $(B \times_\alpha G)_x \to B_x \times_{\alpha^x} G$ because its kernel contains $\{f \in C(X) : f(x) = 0\}$. Let $\theta^x : B \times_\alpha G \to (B \times_\alpha G)_x$ be the quotient map, and continue to write θ^x for the extension to multiplier algebras. Then $\theta^x \circ \jmath_B$ descends to a homomorphism $\widetilde{\theta}^x : B_x \to (B \times_\alpha G)_x$, and the pair $(\widetilde{\theta}^x, \theta^x \circ \jmath_G)$ is a covariant representation of (B_x, α^x, G) in $(B \times_\alpha G)_x$ whose integrated form is inverse to $\widetilde{\pi}^x$. □

Theorem 5.4. *Let Λ be a row-finite k-graph with no sources. Let $c \in \underline{Z}^2(\Lambda, \mathbb{R})$. For $t \in \mathbb{R}$ define $\omega_t \in \widehat{\mathbb{R}}$ by $\omega_t(s) = e^{\imath ts}$; then $C^*(\Lambda, \omega_t \circ c)$ is unital if and only if Λ^0 is finite. There is an isomorphism $K_*(C^*(\Lambda, \omega_t \circ c)) \cong K_*(C^*(\Lambda))$ which carries $[s_v]_0$ to $[s_v]_0$ for each $v \in \Lambda^0$ and carries $[1_{C^*(\Lambda,\omega_t \circ c)}]_0$ to $[1_{C^*(\Lambda)}]_0$ if Λ^0 is finite.*

Proof. Let $t \in \mathbb{R}$. If Λ^0 is finite then $C^*(\Lambda, \omega_t \circ c)$ has unit $\sum_{v \in \Lambda^0} s_v$. If $C^*(\Lambda, \omega_t \circ c)$ is unital then arguing as in [13, Proposition 1.3] it follows that Λ^0 is finite.

By rescaling c if necessary, it suffices to prove the second assertion for $t = 1$. We have $C^*(\Lambda) = C^*(\Lambda, \omega_0 \circ c)$. Corollary 4.3 implies that $C^*(\Lambda, \mathbb{R}, c)|_{[0,1]}$ is a $C([0,1])$-algebra with fibres $C^*(\Lambda, \mathbb{R}, c)_t \cong C^*(\Lambda, \omega_t \circ c)$.

Let $\phi : \Lambda \times_d \mathbb{Z}^k \to \Lambda$ be the functor $\phi(\lambda, n) = \lambda$. Then $c \circ \phi \in \underline{Z}^2(\Lambda \times_d \mathbb{Z}^k, \mathbb{R})$. Lemma 5.2 applied to $\omega_1 \circ c \in \underline{Z}^2(\Lambda, \mathbb{T})$ yields an isomorphism $K_*(C^*(\Lambda, \omega_1 \circ c)) \cong K_*\big(C^*(\Lambda \times_d \mathbb{Z}^k, \omega_1 \circ c \circ \phi) \times_{\mathrm{lt}} \mathbb{Z}^k\big)$ which carries each $[s_v]_0$ to $[\jmath_{C^*(\Lambda \times_d \mathbb{Z}^k, \omega_t \circ c \circ \phi)}(s_{(v,0)})]_0$.

The map $b(v, m) = m$ from $(\Lambda \times_d \mathbb{Z}^k)^0$ to $\mathbb{Z}^k$ satisfies $d = \underline{\delta}^0 b$. Thus Corollary 4.3 shows that the evaluation maps $C^*(\Lambda, \mathbb{R}, c)|_{[0,1]} \to C^*(\Lambda, \mathbb{R}, c)_t \cong C^*(\Lambda \times_d \mathbb{Z}^k, \omega_t \circ c \circ \phi)$ induce isomorphisms in K-theory which carry each $[\jmath_\Lambda(v)]_0$ to $[s_v]_0$. The universal property of $C^*(\Lambda, \mathbb{R}, c)$ implies that it carries an action lt of $\mathbb{Z}^k$ characterised by $\mathrm{lt} \circ \jmath_{\mathbb{R}} = \jmath_{\mathbb{R}}$ and $\mathrm{lt}(\jmath_{\Lambda \times_d \mathbb{Z}^k}((\lambda, m))) = \jmath_{\Lambda \times_d \mathbb{Z}^k}((\lambda, m + d(\lambda)))$. In particular, lt fixes the central copy of $C_0(\mathbb{R})$. Lemma 5.3 now implies that the evaluation maps $C^*(\Lambda, \mathbb{R}, c) \to C^*(\Lambda, \mathbb{R}, c)_t$

satisfy the hypotheses of Theorem 5.1. Hence Theorem 5.1 applied to each of π_1 and π_0 gives isomorphisms

$$K_*(C^*(\Lambda, \omega_1 \circ c)) \cong K_*(C^*(\Lambda, \mathbb{R}, c)|_{[0,1]}) \cong K_*(C^*(\Lambda, \omega_0 \circ c)) \cong K_*(C^*(\Lambda))$$

whose composition carries each $[s_v]_0$ to $[s_v]_0$. The final assertion follows from the first paragraph. □

Example 5.5 (K-theory of noncommutative tori). Resume the notation of Example 3.4, so that A is the free abelian group generated by $\{(i, j) : 1 \le j < i \le k\}$ and $c \in \underline{Z}^2(T_k, A)$ is given by $c(m, n) = \sum_{j<i} m_i n_j \cdot (i, j)$. Fix $z : \{(i, j) : j < i \le k\} \to \mathbb{T}$, and choose elements $r_{i,j} \in \mathbb{R}$ such that $e^{\imath r_{i,j}} = z_{i,j}$. There is a homomorphism $r : A \to \mathbb{R}$ determined by $(i, j) \mapsto r_{i,j}$, and $b := r \circ c$ is then an $\mathbb{R}$-valued 2-cocycle on T_k. For $j < i$, we have $z_{i,j} = e^{\imath r_{i,j}}$. The resulting character $\chi_z \circ c$ of A was used in Example 3.4 to prove that the noncommutative torus for z coincides with $C^*(T_k, \chi_z \circ c)$. We now have $\chi_z \circ c = \omega_1 \circ b$ where ω_1 is the character $\omega_1(s) = e^{\imath s}$ of $\mathbb{R}$. Hence Theorem 5.4 implies that $K_*(C^*(T_k, \chi_z \circ c)) \cong K_*(C^*(T_k)) = K_*(C(\mathbb{T}^k))$. Thus we recover Elliott's calculation of the K-groups of the noncommutative tori, which inspired and informed our results here.

Example 5.6 (K-theory of Heegaard-type quantum spheres). Let Λ be the 2-graph of Example 3.5, and resume the notation used in that example. Let $\mathcal{T}$ denote the Toeplitz algebra with generating isometry S. Recall from [10, Remark 3.3] that $C^*(\Lambda) \cong (\mathcal{T} \otimes \mathcal{T})/(\mathcal{K} \otimes \mathcal{K})$, giving the following six-term exact sequence.

$$\begin{array}{ccccc} K_0(\mathcal{K}\otimes\mathcal{K}) & \longrightarrow & K_0(\mathcal{T}\otimes\mathcal{T}) & \longrightarrow & K_0(C^*(\Lambda)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\Lambda)) & \longleftarrow & K_1(\mathcal{T}\otimes\mathcal{T}) & \longleftarrow & K_1(\mathcal{K}\otimes\mathcal{K}) \end{array}$$

Recall that $(K_0(\mathcal{T}), K_1(\mathcal{T}), [1]_0) \cong (\mathbb{Z}, \{0\}, 1)$ (see [4, 9.4.2]). The inclusion $\iota : \mathcal{K} \to \mathcal{T}$ induces the zero map on K_0 (because ι carries a minimal projection in $\mathcal{K}$ to $S^*S - SS^*$). Naturality of the Künneth formula (see [4, 23.1.3]) therefore implies that the map $\mathbb{Z} \cong K_0(\mathcal{K} \otimes \mathcal{K}) \to K_0(\mathcal{T} \otimes \mathcal{T}) \cong \mathbb{Z}$ is the zero map. The Künneth formula also implies that $K_1(\mathcal{T} \otimes \mathcal{T}) = \{0\} = K_1(\mathcal{K} \otimes \mathcal{K})$. So the six-term sequence above becomes

$$0 \longrightarrow K_1(C^*(\Lambda)) \longrightarrow \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \longrightarrow K_0(C^*(\Lambda)) \longrightarrow 0.$$

Hence $K_j(C^*(\Lambda)) \cong \mathbb{Z}$ for $j = 0, 1$. Theorem 5.4 then implies that $K_j(C(S^3_{00\theta})) = \mathbb{Z}$ for $j = 0, 1$ and $\theta \in [0, 2\pi)$. Thus our methods recover [2, Theorem 4.1].

Kirchberg algebras. Recall that a Kirchberg algebra is a purely infinite, simple, nuclear C^*-algebra (for more details see [23]). We invoke the Kirchberg-Phillips theorem, which classifies Kirchberg algebras, to see that in many cases the isomorphism class of $C^*(\Lambda, c)$ is independent of c. Recall from [8] that a *generalised cycle with an entrance* in a k-graph Λ is a pair $(\mu, \nu) \in \Lambda \times \Lambda$ such that:

(1) $\mu \ne \nu$, $s(\mu) = s(\nu)$, $r(\mu) = r(\nu)$, and $\mathrm{MCE}(\mu\tau, \nu) \ne \emptyset$ for every $\tau \in s(\mu)\Lambda$; and
(2) there exists $\sigma \in s(\nu)\Lambda$ such that $\mathrm{MCE}(\mu, \nu\sigma) = \emptyset$.

The argument of [8, Lemma 3.7] shows that if (μ, ν) is a generalised cycle with an entrance then $s_{r(\nu)} \ge s_\nu s_\nu^* > s_\mu s_\mu^* \sim s_\nu s_\nu^*$ and so $s_{r(\nu)}$ is infinite. A vertex v in Λ can be *reached* *from a generalised cycle with an entrance* if there is a generalised cycle (μ, ν) with an entrance such that $v\Lambda r(\mu) \neq \emptyset$.

For $v, w \in \Lambda^0$ we denote by $v\Lambda w$ the set $\{\lambda \in \Lambda : s(\lambda) = v, r(\lambda) = w\}$. Recall from [16, Proposition 3.6 and Remark A.3] that a k-graph Λ is *aperiodic* if, whenever $\mu \neq \nu$, there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$, and is *cofinal* if, for all $v, w \in \Lambda^0$, there exists $m \in \mathbb{N}^k$ such that $v\Lambda s(\mu) \neq \emptyset$ for all $\mu \in w\Lambda^m$.

Proposition 5.7. *Let Λ be a row-finite k-graph with no sources. Suppose that every vertex in Λ can be reached from a generalised cycle with an entrance and that Λ is aperiodic. For every $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, every hereditary subalgebra of $C^*(\Lambda, c)$ contains an infinite projection. In particular, if Λ is cofinal, then each $C^*(\Lambda, c)$ is simple and purely infinite.*

Proof. We first show that each s_v is an infinite projection. To see this, fix $v \in \Lambda^0$. Then there exists $\lambda \in v\Lambda$ and a generalised cycle (μ, ν) with an entrance such that $s(\lambda) = r(\mu)$. As discussed above, $s_\lambda^* s_\lambda = s_{s(\lambda)}$ is infinite, and so $s_v \geq s_\lambda s_\lambda^* \sim s_\lambda^* s_\lambda$ is also infinite.

The proof of [3, Proposition 5.3] now shows that every hereditary subalgebra of $C^*(\Lambda, c)$ contains an infinite projection: the argument generalises to k-graph algebras (see, for example [24]), and applies without modification to twisted k-graph C^*-algebras. If Λ is also cofinal, then [15, Corollary 8.2] shows that $C^*(\Lambda, c)$ is also simple. □

Recall that for $t \in \mathbb{R}$, we write ω_t for the character $s \mapsto e^{\imath ts}$ of $\mathbb{R}$.

Corollary 5.8. *Let Λ be an aperiodic, cofinal, row-finite k-graph with no sources. Suppose that $c \in \underline{Z}^2(\Lambda, \mathbb{R})$. If every vertex in Λ can be reached from a generalised cycle with an entrance, then $C^*(\Lambda, \omega_t \circ c)$, $t \in \mathbb{R}$ are mutually isomorphic Kirchberg algebras.*

Proof. Fix $t \in \mathbb{R}$. Proposition 5.7 implies that $C^*(\Lambda)$ and $C^*(\Lambda, \omega_t \circ c)$ are simple and purely infinite. Corollary 8.7 of [15] implies that they are nuclear and UCT-class. Theorem 5.4 implies that $C^*(\Lambda)$ and $C^*(\Lambda, \omega_t \circ c)$ are either both unital or both nonunital, that $K_*(C^*(\Lambda)) \cong K_*(C^*(\Lambda, \omega_t \circ c))$, and that this isomorphism carries $[1_{C^*(\Lambda)}]_0$ to $[1_{C^*(\Lambda, \omega_t \circ c)}]_0$ when both C^*-algebras are unital. The Kirchberg-Phillips theorem [19, Theorem 4.2.4] (see also [11, Corollary C]) then gives $C^*(\Lambda, \omega_t \circ c) \cong C^*(\Lambda)$. □

References

[1] J. Anderson and W. Paschke, *The rotation algebra*, Houston J. Math. **15** (1989), 1–26.

[2] P.F. Baum, P.M. Hajac, R. Matthes and W. Szymański, *The K-theory of Heegaard-type quantum 3-spheres*, K-Theory **35** (2005), 159–186.

[3] T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324.

[4] B. Blackadar, K-theory for operator algebras, Springer-Verlag, New York, 1986, viii+338.

[5] K.R. Davidson and D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math. **61** (2009), 1239–1261.

[6] G.A. Elliott,*On the K-theory of the C^*-algebra generated by a projective representation of a torsion-free discrete abelian group*, Monogr. Stud. Math., 17, Operator algebras and group representations, Vol. I (Neptun, 1980), 157–184, Pitman, Boston, MA, 1984.

[7] D.G. Evans, *On the K-theory of higher-rank graph C^*-algebras*, New York J. Math. **14** (2008), 1–31.

[8] D.G. Evans and A. Sims, *When is the Cuntz-Krieger algebra of a higher-rank graph approximately finite-dimensional?*, J. Funct. Anal., **263** (2012), 183–215.

[9] C. Farthing, P.S. Muhly and T. Yeend, *Higher-rank graph C^*-algebras: an inverse semigroup and groupoid approach*, Semigroup Forum **71** (2005), 159–187.

[10] P.M. Hajac, R. Matthes and W. Szymanski, *A locally trivial quantum Hopf fibration*, Algebr. Represent. Theory **9** (2006), 121–146.

[11] E. Kirchberg, *The classification of purely infinite C^*-algebras using Kasparov's theory*, Preprint, 1994.
[12] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.
[13] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.
[14] A. Kumjian, D. Pask and A. Sims, *Homology for higher-rank graphs and twisted C^*–algebras*, J. Funct. Anal., **263** (2012), 1539–1574.
[15] A. Kumjian, D. Pask and A. Sims, *On twisted higher-rank graph C^*-algebras*, preprint 2011 (arXiv:1112.6233v1 [math.OA]).
[16] P. Lewin and A. Sims, *Aperiodicity and cofinality for finitely aligned higher-rank graphs*, Math. Proc. Cambridge Philos. Soc. **149** (2010), 333–350.
[17] B.K. Maloney, D. Pask and I. Raeburn, *Skew-products of higher-rank graphs and crossed-products by semigroups*, preprint 2011 (arXiv:1203.4860 [math.OA]).
[18] D. Pask, J. Quigg and I. Raeburn, *Fundamental groupoids of k-graphs*, New York J. Math. **10** (2004), 195–207.
[19] N.C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114.
[20] I. Raeburn, Graph algebras, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.
[21] M.A. Rieffel, *Continuous fields of C^*-algebras coming from group cocycles and actions*, Math. Ann. **283** (1989), 631–643.
[22] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. reine angew. Math. **513** (1999), 115–144.
[23] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, 1–145, Encyclopaedia Math. Sci., 126, Springer, Berlin, 2002.
[24] A. Sims, *Gauge-invariant ideals in the C^*-algebras of finitely aligned higher-rank graphs*, Canad. J. Math. **58** (2006), 1268–1290.
[25] A. Skalski and J. Zacharias, *Entropy of shifts on higher-rank graph C^*-algebras*, Houston J. Math. **34** (2008), 269–282.
[26] B. Whitehead, *Twisted relative Cuntz-Krieger algebras of finitely aligned higher-rank graphs*, Honours thesis, University of Wollongong 2012.
[27] D.P. Williams, Crossed products of C^*-algebras, American Mathematical Society, Providence, RI, 2007, xvi+528.

Alex Kumjian, Department of Mathematics (084), University of Nevada, Reno NV 89557-0084, USA
E-mail address: alex@unr.edu

David Pask, Aidan Sims, School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, AUSTRALIA
E-mail address: dpask, asims@uow.edu.au