AMPLIFIED GRAPH C*-ALGEBRAS II: RECONSTRUCTION

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Abstract. Let E be a countable directed graph that is amplified in the sense that whenever there is an edge from v to w, there are infinitely many edges from v to w. We show that E can be recovered from $C^*(E)$ together with its canonical gauge-action, and also from $L_{\mathbb{K}}(E)$ together with its canonical grading.

1. Introduction

The purpose of this paper is to investigate the gauge-equivariant isomorphism question for C^*-algebras of countable amplified graphs, and the graded isomorphism question for Leavitt path algebras of countable amplified graphs. A directed graph E is called an *amplified* graph if for any two vertices v, w, the set of edges from v to w is either empty or infinite.

The geometric classification (that is, classification by the underlying graph modulo the equivalence relation generated by a list of allowable graph moves) of the C^*-algebras of finite-vertex amplified graph C^*-algebras was completed in [12], and was an important precursor to the eventual geometric classification of all finite graph C^*-algebras [13]. But there has been increasing recent interest in understanding isomorphisms of graph C^*-algebras that preserve additional structure: for example the canonical gauge action of the circle; or the canonical diagonal subalgebra isomorphic to the algebra of continuous functions vanishing at infinity on the infinite path space of the graph; or the smaller coefficient algebra generated by the vertex projections; or some combination of these (see, for example, [5, 6, 7, 8, 9, 10, 19]).

A program of geometric classification for these various notions of isomorphism was initiated by the first two authors in [11]. They discuss xyz-isomorphism of graph C^*-algebras, where x is 1 if we require exact isomorphism, and 0 if we require only stable isomorphism; y is 1 if the isomorphism is required to be gauge-equivariant, and 0 otherwise; and

2010 *Mathematics Subject Classification.* Primary: 46L35.

Key words and phrases. amplified graph, graph C^*-algebra.

This research was supported by Australian Research Council Discovery Project DP200100155, by DFF-Research Project 2 'Automorphisms and Invariants of Operator Algebras', no. 7014-00145B, and by a Simons Foundation Collaboration Grant, #567380.

z is 1 if the isomorphism is required to preserve the diagonal subalgebra and 0 otherwise. They also identified a set of moves on graphs that preserve various kinds of xyz-isomorphism, and conjectured that for all xyz other than x10, the equivalence relation on graphs with finitely many vertices induced by xyz-isomorphism of C^*-algebras is generated by precisely those of their moves that induce xyz-isomorphisms.

This was an important motivation for the present paper. None of the moves in [11] takes an amplified graph to an amplified graph. And although we know of one important instance where one amplified graph can be transformed into another via a sequence of 101-preserving moves passing through non-amplified graphs (see Diagram (3.1) in Remark 3.5), we had given up on envisioning such a sequence consisting only of x1z-preserving moves. Based on the main conjecture of [11], this led us to expect that an amplified graph C^*-algebra together with its gauge action should remember the graph itself.

Our main theorem shows that, indeed, any countable amplified graph E can be reconstructed from either the circle-equivariant K_0-group of its C^*-algbra, or the graded K_0-group of its Leavitt path algebra over any field. That is:

Theorem A. *Let E and F be countable amplified graphs and let $\mathbb{K}$ be a field. Then the following are equivalent:*

1. $E \cong F$;
2. *there is a $\mathbb{Z}[x, x^{-1}]$-module order-isomorphism* $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E)) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(F))$; *and*
3. *there is a $\mathbb{Z}[x, x^{-1}]$-module order-isomorphism* $K_0^{\mathbb{T}}(C^*(E), \gamma)) \cong K_0^{\mathbb{T}}(C^*(F), \gamma))$ *of $\mathbb{T}$-equivariant K_0-groups.*

We spell out a number of consequences of this theorem in Remark 3.9, Theorem 3.4, and Theorem 3.8. The headline is that for amplified graphs, and for any x, z, the graph C^*-algebras $C^*(E)$ and $C^*(F)$ are x1z-isomorphic if and only if E and F are isomorphic. Combined with results of [4, 13], this confirms [11, Conjecture 5.1] for amplified graphs (see Remark 3.5).

Another immediate consequence is that, since ordered graded K_0 is an isomorphism invariant of graded rings, and ordered $\mathbb{T}$-equivariant K_0 is an isomorphism invariant of C^*-algebras carrying circle actions, our theorem confirms a special case of Hazrat's conjecture: ordered graded K_0 is a complete graded-isomorphism invariant for amplified Leavitt path algebras; and we also obtain that ordered $\mathbb{T}$-equivariant K_0 is a complete gauge-isomorphism invariant of amplified graph C^*-algebras.

A third consequence is related to different graded stabilisations of Leavitt path algebras (and different equivariant stabilisations of graph C^*-algebras). Each Leavitt path algebra has a canonical grading, and, as alluded to above, significant work led by Hazrat has been done on determining when graded K-theory completely classifies graded Leavitt path algebras. Historically, in the classification program for C^*-algebras, significant progress has been made by first considering classification up to stable isomorphism; so it is natural to consider the same approach to Hazrat's graded classification question. But almost immediately, there is a difficulty: which grading on $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})$ should we consider? It seems natural enough to use the grading arising from the graded tensor product of the graded algebras $L_{\mathbb{K}}(E)$ and $M_\infty(\mathbb{K})$. But there are many natural gradings on $M_\infty(\mathbb{K})$: given any $\overline{\delta} \in \prod_i \mathbb{Z}$, we obtain a grading of $M_\infty(\mathbb{K})$ in which the m, n matrix unit is homogeneous of degree $\overline{\delta}_m - \overline{\delta}_n$. Different nonzero choices for $\overline{\delta}$ correspond to different ways of stabilising $L_{\mathbb{K}}(E)$ by modifying the graph E (for example by adding heads [23]), while taking $\overline{\delta} = (0, 0, 0, \dots)$ corresponds to stabilising the associated groupoid by taking its cartesian product with the (trivially graded) full equivalence relation $\mathbb{N} \times \mathbb{N}$.

In Section 3.2, we show that for amplified graphs it doesn't matter what value of $\overline{\delta}$ we pick. Specifically, using results of Hazrat, we prove that $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta})) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ regardless of $\overline{\delta}$. Consequently, for any choice of $\overline{\delta}$ we have $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$ if and only if there exists a $\mathbb{Z}[x, x^{-1}]$-module order-isomorphism $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E)) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(F))$. A similar result holds for C^*-algebras with the gradings on Leavitt path algebras replaced by gauge actions on graph C^*-algebras, and the gradings of $M_\infty(\mathbb{K})$ corresponding to different elements $\overline{\delta}$ replaced by the circle actions on $\mathcal{K}(\ell^2)$ implemented by different strongly continuous unitary representations of the circle on ℓ^2.

We prove our main theorem in Section 2. We use general results to see that the graded K_0-group of $L_{\mathbb{K}}(E)$ and the equivariant K_0-group of $C^*(E)$ are isomorphic as ordered $\mathbb{Z}[x, x^{-1}]$-modules to the K_0-groups of the Leavitt path algebra and the graph C^*-algebra (respectively) of the skew-product graph $E \times_1 \mathbb{Z}$. These are known to coincide, and their lattice of order ideals (with canonical $\mathbb{Z}$-action) is isomorphic to the lattice of hereditary subsets of $(E \times_1 \mathbb{Z})^0$ with the $\mathbb{Z}$-action of translation in the second variable. So the bulk of the work in Section 2 goes into showing how to recover E from this lattice. We then go on in Section 3.2 to establish the consequences of our main theorem for stabilizations. Here the hard work goes into showing that $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta})) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ for any $\overline{\delta} \in \prod_i \mathbb{Z}$ and that $K_0^{\mathbb{T}}(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \mathrm{Ad}_u) \cong K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ for any strongly continuous unitary representation u of $\mathbb{T}$.

2. Gauge-invariant classification of amplified graph C^*-algebras

Throughout the paper, a countable directed graph E is a quadruple $E = (E^0, E^1, r, s)$ where E^0 is a countable set whose elements are called *vertices*, E^1 is a countable set whose elements are called *edges*, and $r, s\colon E^1 \to E^0$ are functions. We think of the elements of E^0 as points or dots, and each element e of E^1 as an arrow pointing from the vertex $s(e)$ to the vertex $r(e)$. We follow the conventions of, for example [14], where a path is a sequence $e_1 \dots e_n$ of edges in which $s(e_{n+1}) = r(e_n)$. This is not the convention used in Raeburn's monograph [21], but is the convention consistent with all of the Leavitt path algebra literature as well as much of the graph C^*-algebra literature. In keeping with this, for $v, w \in E^0$ and $n \geq 0$, we define

$$vE^1 = s^{-1}(v), \quad E^1 w = r^{-1}(w), \quad \text{and} \quad vE^1 w = s^{-1}(v) \cap r^{-1}(w).$$

We will also write vE^n for the sets of paths of length n that are emitted by v, $E^n w$ for the set of paths of length n received by w, and $vE^n w$ for the set of paths of length n pointing from v to w.

A vertex v is *singular* if vE^1 is either empty or infinite, so v is either a sink or an infinite emitter; and for any edge e, we have $s_e^* s_e = p_{r(e)}$ and $p_{s(e)} \geq s_e s_e^*$ in the graph C^*-algebra $C^*(E)$. We will also consider the Leavitt path algebras, $L_{\mathbb{K}}(E)$ for any field $\mathbb{K}$, the so-called algebraic cousin of graph C^*-algebras. Leavitt path algebras are defined via generators and relations similar to those for graph C^*-algebras (see [1]).

Countable directed graphs E and F are *isomorphic*, denoted $E \cong F$, if there is a bijection $\phi : E^0 \sqcup E^1 \to F^0 \sqcup F^1$ that restricts to bijections $\phi^0\colon E^0 \to F^0$ and $\phi^1\colon E^1 \to F^1$ such that

$$\phi^0(r(e)) = r(\phi^1(e)) \quad \text{and} \quad \phi^0(s(e)) = s(\phi^1(e)).$$

In this paper, we consider amplified graphs. The classification of amplified graph C^*-algebras was the starting point in the classification of unital graph C^*-algebras via moves (see [12] and [13]).

Definition 2.1 (Amplified Graph and Amplified graph algebra)**.** A directed graph E is an *amplified graph* if for all $v, w \in E^0$, the set $vE^1 w = s^{-1}(v) \cap r^{-1}(w)$ is either empty or infinite. An *amplified graph C^*-algebra* is a graph C^*-algebra of an amplified graph and an *amplified Leavitt path algebra* is a Leavitt path algebra of an amplified graph.

Observe that in an amplified graph, every vertex is singular.

Recall that a set $H \subseteq E^0$ is *hereditary* if $s(e) \in H$ implies $r(e) \in H$ for every $e \in E^1$, and is *saturated* if whenever v is a regular vertex such that $r(vE^1) \subseteq H$, we have $v \in H$. Again since every vertex in an amplified graph is singular, every set of vertices is saturated.

Recall from [18] that if E is a directed graph, then the skew-product graph $E \times_1 \mathbb{Z}$ is the graph with vertices $E^0 \times \mathbb{Z}$ and edges $E^1 \times \mathbb{Z}$ with $s(e, n) = (s(e), n)$ and $r(e, n) = (r(e), n + 1)$. If E is an amplified graph, then so is $E \times_1 \mathbb{Z}$.

For a countable amplified graph, E, we write $\mathcal{H}(E \times_1 \mathbb{Z})$ for the lattice (under set inclusion) of hereditary subsets of the vertex-set of the skew-product graph $E \times_1 \mathbb{Z}$. The action of $\mathbb{Z}$ on $E \times_1 \mathbb{Z}$ by given by $n \cdot (e, m) = (e, n+m)$ induces an action lt of $\mathbb{Z}$ on $\mathcal{H}(E \times_1 \mathbb{Z})$. There is also a distinguished element $H_0 \in \mathcal{H}(E \times_1 \mathbb{Z})$ given by $H_0 := \{(v, n) : v \in E^0, n \geq 0\} \subseteq (E \times_1 \mathbb{Z})^0$.

Throughout this section, given $v \in E^0$ and $n \in \mathbb{Z}$, we write $H(v, n)$ for the smallest hereditary subset of $(E \times_1 \mathbb{Z})^0$ containing (v, n). So $H(v, n) = \{(r(\mu), n + |\mu|) : \mu \in vE^*\}$ is the set of vertices that can be reached from (v, n) in $E \times_1 \mathbb{Z}$.

If $(\mathcal{L}, \preceq)$ is a lattice, we say that $L \in \mathcal{L}$ has a unique predecessor if there exists $K \in \mathcal{L}$ such that $K \prec L$, and every K' with $K' \prec L$ satisfies $K' \preceq K$. The next proposition is the engine-room of our main result.

Proposition 2.2. *Let E be a countable amplified graph. Define $\mathcal{H}_{\mathrm{vert}} \subseteq \mathcal{H}(E \times_1 \mathbb{Z})$ to be the subset*

$$\mathcal{H}_{\mathrm{vert}} = \{H \in \mathcal{H}(E \times_1 \mathbb{Z}) : H \textit{ has a unique predecessor}\}.$$

Then $\mathcal{H}_{\mathrm{vert}} = \{H(v, n) : v \in E^0 \text{ and } n \in \mathbb{Z}\}$. Let

$$\overline{E}^0 := \{H \in \mathcal{H}_{\mathrm{vert}} : H \subseteq H_0 \text{ and } H \not\subseteq \mathrm{lt}_1(H_0)\}.$$

Define $\overline{E}^1 := \{(H, n, K) : H, K \in \overline{E}^0, \mathrm{lt}_1(K) \subseteq H, \text{ and } n \in \mathbb{N}\}$. Define $\bar{s}, \bar{r} : \overline{E}^1 \to \overline{E}^0$ by $\bar{s}(H, n, K) = H$ and $\bar{r}(H, n, K) = K$. Then $\overline{E} := (\overline{E}^0, \overline{E}^1, \bar{r}, \bar{s})$ is a countable amplifed directed graph, and there is an isomorphism $E \cong \overline{E}$ that carries each $v \in E^0$ to the hereditary subset of $(E \times_1 \mathbb{Z})^0$ generated by $(v, 0)$.

Proof. The argument of [12, Lemma 5.2] shows that $\mathcal{H}_{\mathrm{vert}} = \{H(v, n) : v \in E^0, n \in \mathbb{Z}\}$.

We clearly have $H(v, n) \subseteq H_0$ if and only if $n \geq 0$, and $H(v, n) \subseteq \mathrm{lt}_1(H_0)$ if and only if $n \geq 1$, so $\overline{E}^0 = \{H(v, 0) : v \in E^0\}$. Since $E \times_1 \mathbb{Z}$ is acyclic, the $H(v, 0)$ are distinct, and we deduce that $\theta^0 : v \mapsto H(v, 0)$ is a bijection from E^0 to $\overline{E}^0$.

Fix $v, w \in E^0$. We have $\mathrm{lt}_1(H(w, 0)) = H(w, 1)$, and since $(w, 1) \in H(v, 0)$ if and only if $vE^1w \neq \emptyset$, we have $H(w, 1) \subseteq H(v, 0)$ if and only if $vE^1w \neq \emptyset$, in which case vE^1w is infinite because E is amplified. It follows that $|H(v, 0)\overline{E}^1 H(w, 0)| = |vE^1w|$ for all v, w, so we can choose a bijection $\theta^1 : E^1 \to \overline{E}^1$ that restricts to bijections $vE^1w \to \theta^0(v)\overline{E}^1\theta^0(w)$ for all $v, w \in E^0$. The pair (θ^0, θ^1) is then the desired isomorphism $E \cong \overline{E}$. □

In order to use Proposition 2.2 to prove Theorem A, we need to know that if $(\mathcal{H}(E \times_1 \mathbb{Z}), \mathrm{lt}^E)$ is order isomorphic to $(\mathcal{H}(F \times_1 \mathbb{Z}), \mathrm{lt}^F)$ then there is an isomorphism from $(\mathcal{H}(E \times_1 \mathbb{Z}), \mathrm{lt}^E)$ to $(\mathcal{H}(F \times_1 \mathbb{Z}), \mathrm{lt}^F)$ that carries H_0^E to H_0^F. We do this by showing that if E is connected, then we can recognise the sets $\mathrm{lt}_n(H_0)$ amongst all the hereditary subsets of $(E \times_1 \mathbb{Z})^0$ using just the order-structure and the action lt.

Recalling that vE^nw denotes the set of paths of length n from v to w, we have

$$H(w,n) \subseteq H(v,m) \quad \text{if and only if} \quad vE^{n-m}w \neq \emptyset. \tag{2.1}$$

Recall that a graph E is said to be *connected* if the smallest equivalence relation on E^0 containing $\{(s(e), r(e)) : e \in E^1\}$ is all of $E^0 \times E^0$.

Let E be a connected, countable amplified graph. The set $V_0 := \{H(v,0) : v \in E^0\}$ is exactly the set of maximal elements of the collection $\{H \in \mathcal{H}_{\mathrm{vert}} : H \subseteq H_0\}$. The sets H_0 and V_0 have the following properties:

- for each $H \in \mathcal{H}_{\mathrm{vert}}$ there is a unique $n \in \mathbb{Z}$ such that $\mathrm{lt}_n(H) \in V_0$;
- the smallest equivalence relation on V_0 containing $\{(H,K) : \mathrm{lt}_1(K) \subseteq H\}$ is all of $V_0 \times V_0$; and
- if H, K are distinct elements of V_0, and if $n \geq 0$, then $H \not\subseteq \mathrm{lt}_n(K)$.

The next lemma shows that for connected graphs, these properties characterise H_0 up to translation.

Lemma 2.3. *Suppose that E is a connected, countable amplified graph. Take $H \in \mathcal{H}(E \times_1 \mathbb{Z})$, and let V_H be the set of maximal elements of $\{K \in \mathcal{H}_{\mathrm{vert}} : K \subseteq H\}$ with respect to set inclusion. Suppose that*

(1) *for each $K \in \mathcal{H}_{\mathrm{vert}}$ there is a unique $n \in \mathbb{Z}$ such that $\mathrm{lt}_n(K) \in V_H$;*
(2) *the smallest equivalence relation on V_H containing $\{(H,K) : \mathrm{lt}_1(K) \subseteq H\}$ is all of $V_H \times V_H$; and*
(3) *if K, K' are distinct elements of V_H, and if $n \geq 0$, then $K \not\subseteq \mathrm{lt}_n(K')$.*

Then there exists $n \in \mathbb{Z}$ such that $H = \mathrm{lt}_n(H_0)$.

Proof. For each $v \in E^0$, item (1) applied to $K = H(v,0)$ shows that there exists a unique $n_v \in \mathbb{Z}$ such that $H(v, n_v) = \mathrm{lt}_{n_v}(K) \in V_H$. So $V_H = \{H(v, n_v) : v \in E^0\}$. We must show that $n_v = n_w$ for all $v, w \in E^0$. To do this, it suffices to show that for any $u \in E^0$, we have $n_w \geq n_u$ for all $w \in E^0$.

So fix $u \in E^0$. Define

$$L_u := \{v \in E^0 : n_v < n_u\} \quad \text{and} \quad G_u := \{w \in E^0 : n_w \geq n_u\}$$

We prove that if $v \in L_u$ and $w \in G_u$, then

$$\mathrm{lt}_1(H(v,n_v)) \not\subseteq H(w,n_w) \quad \text{and} \quad \mathrm{lt}_1(H(w,n_w)) \not\subseteq H(v,n_v). \tag{2.2}$$

For this, fix $v \in L_u$ and $w \in G_u$; note that in particular $v \neq w$.

To see that $\mathrm{lt}_1(H(v,n_v)) \not\subseteq H(w,n_w)$, suppose otherwise for contradiction. Then $H(v, n_v+1) \subseteq H(w, n_w)$. Hence (2.1) shows that $wE^{n_v+1-n_w}v \neq \emptyset$, which forces $n_v \geq n_w - 1$. Since $v \in L_u$ and $w \in G_u$, we also have $n_v \le n_w - 1$, and we conclude that $n_v + 1 - n_w = 0$. This forces $wE^0v \neq \emptyset$, contradicting that $v \neq w$.

To see that $\mathrm{lt}_1(H(w, n_w)) \not\subseteq H(v, n_v)$, we first claim that there is no $e \in E^1$ satisfying $s(e) \in L_u$ and $r(e) \in G_u$. To see this, fix $x \in L_u$ and $y \in G_u$. Then $n_y > n_x$, and in particular $n_y - 1 - n_x \ge 0$. Hence Item (3) shows that $H(y, n_y) \not\subseteq \mathrm{lt}_{n_y-1-n_x}(H(x, n_x))$. Applying lt_{1-n_y} on both sides shows that $\mathrm{lt}_1(H(y, 0)) \not\subseteq H(x, 0)$, and so $xE^1y = \emptyset$. This proves the claim.

Since $v \in L_u$, applying the claim n_w+1-n_v times shows that for any path $\mu \in vE^{n_w+1-n_v}$, we have $r(\mu) \in L_u$. In particular, $vE^{n_w+1-n_v}w = \emptyset$. Thus (2.1) implies that $\mathrm{lt}_1(H(w, n_w)) \not\subseteq H(v, n_v)$.

We have now established (2.2). Set

$$\overline{L}_u = \{H(v, n_v) : v \in L_u\} \quad \text{and} \quad \overline{G}_u = \{H(w, n_w) : w \in G_u\}.$$

Then (2.2) shows that $(\overline{L}_u \times \overline{L}_u) \sqcup (\overline{G}_u \times \overline{G}_u)$ is an equivalence relation on V_H containing $\{(H, K) : \mathrm{lt}_1(K) \subseteq H\}$. Thus item (2) implies that either $\overline{L}_u$ or $\overline{G}_u$ is empty. Since $H(u, n_u) \in \overline{G}_u$, we deduce that $\overline{L}_u = \emptyset$ which implies that $L_u = \emptyset$. Hence $G_u = E^0$, and so $n_w \ge n_u$ for all $w \in E^0$ as required. □

Corollary 2.4. *Suppose that E and F are amplified graphs. If there exists an isomorphism $\rho : (\mathcal{H}(E \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^E) \cong (\mathcal{H}(F \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^F)$, then there exists an isomorphism $\overline{\rho} : (\mathcal{H}(E \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^E) \to (\mathcal{H}(F \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^F)$ such that $\overline{\rho}(H_0^E) = H_0^F$.*

Proof. First suppose that E and F are connected as in Lemma 2.3. Since $H \in \mathcal{H}^E_{\mathrm{vert}}$ if and only if H has a unique predecessor in $\mathcal{H}(E \times_1 \mathbb{Z})$ and similarly for F, the map ρ restricts to an inclusion-preserving bijection $\rho : \mathcal{H}^E_{\mathrm{vert}} \to \mathcal{H}^F_{\mathrm{vert}}$. Since H_0^E and V_0^E satisfy (1)–(3) of Lemma 2.3, so do $\rho(H_0^E)$ and $\{\rho(H) : H \in V_0^E\}$. So Lemma 2.3 shows that $\rho(H_0^E) = \mathrm{lt}_n(H_0^F)$ for some $n \in \mathbb{Z}$, and therefore $\overline{\rho} := \mathrm{lt}_{-n} \circ \rho$ is the desired isomorphism.

Now suppose that E and F are not connected. Let $\mathcal{WC}(E)$ denote the set of equivalence classes for the equivalence relation on E^0 generated by $\{(s(e), r(e)) : e \in E^1\}$; so the elements of $\mathcal{WC}(E)$ are the weakly connected components of E. Similarly, let $\mathcal{WC}(F)$ be the set of weakly connected components of F.

Using that vE^*w is nonemtpy if and only if $\mathrm{lt}_n(H(w, 0)) \subseteq H(v, 0)$ for some $n \in \mathbb{Z}$, we see that $vE^*w \neq \emptyset$ if and only if $\bigcup_n \mathrm{lt}_n(H(w, i)) \subseteq \bigcup_n \mathrm{lt}_n(H(v, j))$ for some (equivalently for all) $i, j \in \mathbb{Z}$. Since the same is true in F, we see that for $v, w \in E^0$, writing $x, y \in F^0$ for the elements such that $\rho(H(v, 0)) \in \mathrm{lt}_{\mathbb{Z}}(H(x, 0))$ and $\rho(H(w, 0)) \in \mathrm{lt}_{\mathbb{Z}}(H(y, 0))$, we have $vE^*w \neq \emptyset$ if and only if $xF^*y \neq \emptyset$. Now an induction shows that there is a bijection $\tilde{\rho} : \mathcal{WC}(E) \to \mathcal{WC}(F)$ such that for each $C \in \mathcal{WC}(E)$, we have $\rho(\{H(v, n) : v \in C, n \in \mathbb{Z}\}) = \{H(w, m) : w \in \tilde{\rho}(C), m \in \mathbb{Z}\}$. For each $C \in \mathcal{WC}(E)$, write E_C for the subgraph

(C, CE^1C, r, s) of E and similarly for F. Then the inclusions $E_C \hookrightarrow E$ induce inclusions $(H(E_C \times_1 \mathbb{Z}), \mathrm{lt}) \hookrightarrow (H(E \times_1 \mathbb{Z}), \mathrm{lt})$ whose ranges are lt-invariant and mutually incomparable with respect to $\subseteq$. Hence ρ induces isomorphisms $\rho_C : (\mathcal{H}(E_C \times_1 \mathbb{Z}), \mathrm{lt}) \cong (\mathcal{H}(F_{\tilde{\rho}(C)} \times_1 \mathbb{Z}), \mathrm{lt})$. The first paragraph then shows that for each $C \in \mathcal{WC}(E)$ there is an isomorphism $\overline{\rho}_C : (\mathcal{H}(E_C \times_1 \mathbb{Z}), \mathrm{lt}) \to (\mathcal{H}(F_{\tilde{\rho}(C)} \times_1 \mathbb{Z}), \mathrm{lt})$ that carries $H_0^{E_C}$ to $H_0^{F_{\tilde{\rho}(C)}}$, and these then assemble into an isomorphsm $\overline{\rho} : (\mathcal{H}(E \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^E) \to (\mathcal{H}(F \times_1 \mathbb{Z}), \subseteq, \mathrm{lt}^F)$ such that $\overline{\rho}(H_0^E) = H_0^F$. □

We are now ready to prove Theorem A.

Proof of Theorem A. That (1) implies (2) and that (1) implies (3) are clear.

By [3, Proposition 5.7] the graded $\mathcal{V}$-monoid $\mathcal{V}^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ is isomorphic to the $\mathcal{V}$-monoid $\mathcal{V}(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$, and that this isomorphism is equivariant for the canonical $\mathbb{Z}[x, x^{-1}]$ actions arising from the grading on $\mathcal{V}^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ and from the action on $\mathcal{V}(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$ induced by translation in the $\mathbb{Z}$-coordinate in $E \times_1 \mathbb{Z}$. Hence $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ is order isomorphic to $K_0(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$ as $\mathbb{Z}[x, x^{-1}]$-modules. Hence condition (2) holds if and only if $K_0(L_{\mathbb{K}}(E \times_1 \mathbb{Z})) \cong K_0(L_{\mathbb{K}}(F \times_1 \mathbb{Z}))$ as ordered $\mathbb{Z}[x, x^{-1}]$-modules.

Likewise [20, Theorem 2.7.9] shows that the equivariant K-theory group $K_0^{\mathbb{T}}(C^*(E))$ is order isomorphic, as a $\mathbb{Z}[x, x^{-1}]$-module, to the K_0-group $K_0(C^*(E) \times_\gamma \mathbb{T})$. The canonical isomorphism $C^*(E) \times_\gamma \mathbb{T} \cong C^*(E \times_1 \mathbb{Z})$ is equivariant for the dual action $\hat{\gamma}$ of $\mathbb{Z}$ on the former and the action of $\mathbb{Z}$ on the latter induced by translation in $E \times_1 \mathbb{Z}$. It therefore induces an isomorphism $K_0(C^*(E) \times_\gamma \mathbb{T}) \cong K_0(C^*(E \times_1 \mathbb{Z}))$ of ordered $\mathbb{Z}[x, x^{-1}]$-modules. So condition (3) holds if and only if $K_0(C^*(E \times_1 \mathbb{Z})) \cong K_0(C^*(F \times_1 \mathbb{Z}))$ as ordered $\mathbb{Z}[x, x^{-1}]$-modules.

By [16, Theorem 3.4 and Corollary 3.5] (see also [2]), for any directed graph E there is an isomorphism $K_0(L_{\mathbb{K}}(E)) \cong K_0(C^*(E))$ that carries the class of the module $L_{\mathbb{K}}(E)v$ to the class of the projection p_v in $C^*(E)$ for each $v \in E^0$. It follows that $K_0(L_{\mathbb{K}}(E \times_1 \mathbb{Z})) \cong K_0(C^*(E \times_1 \mathbb{Z}))$ as ordered $\mathbb{Z}[x, x^{-1}]$-modules. This shows that conditions (2) and (3) are equivalent. So it now suffices to show that (2) implies (1).

So suppose that (2) holds. Since E, and therefore $E \times_1 \mathbb{Z}$, is an amplified graph, it admits no breaking vertices with respect to any saturated hereditary set, and every hereditary subset of $E \times_1 \mathbb{Z}$ is a saturated hereditary subset. So the lattice $\mathcal{H}(E \times_1 \mathbb{Z})$ of hereditary sets is identical to the lattice of admissible pairs in the sense of [22] via the map $H \mapsto (H, \emptyset)$. By [3, Theorem 5.11], there is a lattice isomorphism from $\mathcal{H}(E \times_1 \mathbb{Z})$ to the lattice of order ideals of $K_0(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$ that carries a hereditary set H to the class of the module $L_{\mathbb{K}}(E \times_1 \mathbb{Z})H$. this isomorphism clearly intertwines the action of $\mathbb{Z}$ induced by the module structure on $K_0(L_{\mathbb{K}}(E \times_1 \mathbb{Z}))$ and the action lt^E of $\mathbb{Z}$ on $\mathcal{H}(E \times_1 \mathbb{Z})$

induced by translation. By the same argument applied to F, we see that $\big(\mathcal{H}(E\times_1\mathbb{Z}),\subseteq,\mathrm{lt}^E\big)\cong\big(\mathcal{H}(F\times_1\mathbb{Z}),\subseteq,\mathrm{lt}^F\big)$.

Now Corollary 2.4 implies that $\big(\mathcal{H}(E\times_1\mathbb{Z}),\mathrm{lt}^E,H_0^E\big)\cong\big(\mathcal{H}(F\times_1\mathbb{Z}),\mathrm{lt}^F,H_0^F\big)$. This isomorphism induces an isomorphism $\overline{E}\cong\overline{F}$ of the graphs constructed from these data in Proposition 2.2. Thus two applications of Proposition 2.2 give $E\cong\overline{E}\cong\overline{F}\cong F$, which is (1). $\square$

3. Equivariant K-theory and graded K-theory are stable invariants

In this section, we prove that equivariant K-theory and graded K-theory are stable invariants. We suspect that these are well-known results but we have been unable to find a reference in the literature. For the convenience of the reader, we include their proofs here. We use these results to deduce the consequences of Theorem A for graded stable isomorphisms of amplified Leavitt path algebras, and gauge-equivariant stable isomorphisms of amplified graph C^*-algebras.

3.1. Stability of equivariant K-theory.

Theorem 3.1. *Let G be a compact group and let α be an action of G on a C^*-algebra A. If A has an increasing approximate identity consisting of G-invariant projections, then the natural $R(G)$-module isomorphism from $K_0^G(A,\alpha)$ to $K_0(C^*(G,A,\alpha))$ is an order isomorphism.*

Proof. First suppose A has a unit. Then the theorem follows from the proof of Julg's Theorem [17] (see also [20, Theorem 2.7.9]). The isomorphism is given by the composition of two isomorphisms:

$$K_0^G(A,\alpha)\to K_0(L^1(G,A,\alpha))\quad\text{and}$$
$$K_0(L^1(G,A,\alpha))\to K_0(C^*(G,A,\alpha)).$$

The proof that these maps are isomorphisms shows that the maps are order isomorphisms (see the proof of [20, Lemma 2.4.2 and Theorem 2.6.1]).

Now suppose A has an increasing approximate identity S consisting of G-invariant projections. Fix $p\in S$. Let

$$\lambda_A\colon K_0^G(A,\alpha)\to K_0(C^*(G,A,\alpha)),\ \text{and}$$
$$\lambda_p\colon K_0^G(pAp,\alpha)\to K_0(C^*(G,pAp,\alpha)),\quad p\in S$$

be the natural $R(G)$-isomorphisms given in Julg's Theorem. Note that α does indeed induce an action on pAp since p is G-invariant. Let ι_p be the G-equivariant inclusion of pAp into A and let $\widetilde{\iota_p}$ be the induce $*$-homomorphism from $C^*(G,pAp,\alpha)$ to $C^*(G,A,\alpha)$.

Let $x\in K_0^G(A,\alpha)_+$. By [20, Corollary 2.5.5], there exist $p\in S$ and $x'\in K_0^G(pAp,\alpha)_+$ such that $(\iota_p)_*(x')=x$. Naturality of the maps λ_A and λ_p gives $\lambda_A(x)=(\widetilde{\iota_p})_*\circ\lambda_p(x')$. Consequently, $\lambda_A(x)\in K_0(C^*(G,A,\alpha))_+$ since $(\widetilde{\iota_p})_*\circ\lambda_p(x')\in K_0(C^*(G,A,\alpha))_+$. Fix $y\in$

$K_0(C^*(G,A,\alpha))_+$. For $f \in L^1(G)$ and $a \in A$ we write $f \otimes a : G \to A$ for the function $(f \otimes a)(g) = f(g)a$. Since S is an approximate identity of A and since

$$\{f \otimes a : f \in L^1(G), a \in A\}$$

is dense in $C^*(G,A,\alpha)$, the set $\bigcup_{p \in S} \widetilde{\iota_p}(C^*(G,pAp,\alpha))$ is dense in $C^*(G,A,\alpha)$. Thus, there exists a projection $p \in S$ and there exists $y' \in K_0(C^*(G,pAp,\alpha))_+$ such that $(\widetilde{\iota_p})_*(y') = x$. Since λ_p is an order isomorphism, $\lambda_p^{-1}(y') \in K_0^G(pAp,\alpha)_+$. Then $(\iota_p)_* \circ \lambda_p^{-1}(y') \in K_0^G(A,\alpha)_+$. Naturality of the maps λ_A and λ_p implies that $\lambda_A \circ (\iota_p)_* \circ \lambda_p^{-1}(y') = y$. We have shown that $\lambda_A(K_0^G(A,\alpha)_+) = K_0(C^*(G,A,\alpha))_+$ which implies that λ_A is an order isomorphism. □

Lemma 3.2. *Let G be a compact group and let A be a separable C^*-algebra and let α be an action of G on A. If B is a hereditary subalgebra of A such that*

1. *B has an increasing approximate identity of G-invariant projections,*
2. *A has an increasing approximate identity of G-invariant projections,*
3. *$\overline{ABA} = A$, and*
4. *$\alpha_g(B) \subseteq B$ for all $g \in G$,*

then the inclusion $\iota\colon B \to A$ induces an isomorphism $K_0^G(B) \cong K_0^G(A)$ of ordered $R(G)$-modules.

Proof. Since B is G-invariant, α is also an action on B and the inclusion ι is G-equivariant. Let $\lambda_B\colon K_0^G(B,\alpha) \to K_0(C^*(G,B,\alpha))$ and $\lambda_A\colon K_0^G(A) \to K_0(C^*(G,A,\alpha))$ be the natural $R(G)$-module order isomorphisms given in Theorem 3.1. Naturality of λ_B and λ_A implies that the diagram

$$\begin{array}{ccc} K_0^G(B) & \xrightarrow{\iota_*} & K_0^G(A) \\ \downarrow{\scriptstyle \lambda_B} & & \downarrow{\scriptstyle \lambda_A} \\ K_0(C^*(G,B,\alpha)) & \xrightarrow[\widetilde{\iota}_*]{} & K_0(C^*(G,A,\alpha)) \end{array}$$

is commutative. As in the proof of [20, Proposition 2.9.1], $C^*(G,B,\alpha)$ is a hereditary subalgebra of $C^*(G,A,\alpha)$ such that the closed two-sided ideal of $C^*(G,A,\alpha)$ generated by $C^*(G,B,\alpha)$ is $C^*(G,A,\alpha)$. This $\widetilde{\iota}_*$ is an order isomorphism, and so ι_* is also an order isomorphism. □

The corollary below implies that the equivariant K_0-group is a stable invariant.

Corollary 3.3. *Let G be a compact group, let α be an action of G on a separable C^*-algebra A, and let β be an action of G on $\mathcal{K}(\ell^2)$. If both*

A and $\mathcal{K}(\ell^2)$ admit increasing approximate identities consisting of G-invariant projections, then there is a $R(G)$-module order isomorphism from $K_0^G(A,\alpha)$ to $K_0^G(A\otimes\mathcal{K}(\ell^2),\alpha\otimes\beta)$.

In particular, if $u : G \to \mathcal{U}(\ell^2)$ is a continuous (in the strong operator topology) unitary representation of G and $\beta_g = \mathrm{Ad}(u_g)$, then there is a $R(G)$-module order isomorphism from $K_0^G(A,\alpha)$ and $K_0^G(A\otimes\mathcal{K}(\ell^2),\alpha\otimes\beta)$

Proof. Let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing approximate identity consisting of G-invariant projections in $\mathcal{K}(\ell^2)$. We may assume $p_1 \neq 0$. Then $A\otimes p_1$ is a G-invariant hereditary subalgebra of $A\otimes\mathcal{K}(\ell^2)$ such that $\overline{(A\otimes\mathcal{K}(\ell^2))(A\otimes p_1)(A\otimes\mathcal{K}(\ell^2))} = A\otimes\mathcal{K}(\ell^2)$. From the assumption on A and $\mathcal{K}(\ell^2)$, both $A\otimes p_1$ and $A\otimes\mathcal{K}(\ell^2)$ have increasing approximate identities consisting of G-invariant projections. Lemma 3.2 implies that there is an $R(G)$-module order isomorphism from $K_0^G(A\otimes p_1,\alpha\otimes\beta)$ to $K_0^G(A\otimes\mathcal{K}(\ell^2),\alpha\otimes\beta)$. The result now follows since the map $a\mapsto a\otimes p_1$ is a G-equivariant $*$-isomorphism from A to $A\otimes p_1$.

For the last part of the theorem, since G is compact, u is a direct sum of finite dimensional representations. Thus, $\mathcal{K}(\ell^2)$ has an increasing approximate identity consisting of G-invariant projections. □

To finish this subsection, we describe the consequences of Theorem A for equivariant stable isomorphism of amplified graph C^*-algebras. For the following theorem, given a strong-operator continuous unitary representation $u : \mathbb{T}\to\mathcal{U}(\ell^2)$ of $\mathbb{T}$ on a Hilbert space H, we will write β^u for the action of $\mathbb{T}$ on $\mathcal{B}(\ell^2)$ given by $\beta^u_z = \mathrm{Ad}(u_z)$.

Theorem 3.4. *Let E and F be countable amplified graphs. Then the following are equivalent:*

1. *$E\cong F$;*
2. *$(C^*(E),\gamma^E)\cong(C^*(F),\gamma^F)$;*
3. *$(C^*(E)\otimes\mathcal{K},\gamma^E\otimes\beta^u)\cong(C^*(F)\otimes\mathcal{K},\gamma^F\otimes\beta^u)$, for every strongly continuous representation $u : \mathbb{T}\to\mathcal{U}(\ell^2)$;*
4. *there exists a strongly continuous unitary representation $u : \mathbb{T}\to\mathcal{U}(\ell^2)$ such that $(C^*(E)\otimes\mathcal{K},\gamma^E\otimes\beta^u)\cong(C^*(F)\otimes\mathcal{K},\gamma^F\otimes\beta^u)$; and*
5. *there exist strongly continuous unitary representations $u,v : \mathbb{T}\to\mathcal{U}(\ell^2)$ such that $(C^*(E)\otimes\mathcal{K},\gamma^E\otimes\beta^u)\cong(C^*(F)\otimes\mathcal{K},\gamma^F\otimes\beta^v)$.*

Proof. If $\phi : E\to F$ is an isomorphism, it induces an isomorphism $C^*(E)\cong C^*(F)$, which is gauge invariant because it carries generators to generators. This gives (1) $\implies$ (2).

If (2) holds, say $\phi : C^*(E)\to C^*(F)$ is a gauge-equivariant isomorphism, then for any u the map $\phi\otimes\mathrm{id}_{\mathcal{K}}$ is an equivariant isomorphism from $(C^*(E)\otimes\mathcal{K},\gamma^E\otimes\beta^u)$ to $(C^*(F)\otimes\mathcal{K},\gamma^F\otimes\beta^u)$, giving (3). Clearly (3) implies (4). And if (4) holds for a given $u : \mathbb{T}\to\mathcal{B}(\ell^2)$, then (5) holds with $u = v$. Finally, if (5) holds, then two applications of Corollary 3.3 show that

$$K_0^{\mathbb{T}}(C^*(E), \gamma^E) \cong K_0^{\mathbb{T}}(C^*(E) \otimes \mathcal{K}(\ell^2), \gamma^E \otimes \beta^u) \\ \cong K_0^{\mathbb{T}}(C^*(F) \otimes \mathcal{K}(\ell^2), \gamma^F \otimes \beta^v) \cong K_0^{\mathbb{T}}(C^*(F), \gamma^F)$$

as ordered $\mathbb{Z}[x, x^{-1}]$-modules, and so Theorem A gives (1). □

Remark 3.5. In this remark, we use the notation, moves, and drawing conventions of [11]; we refer the reader there for details.

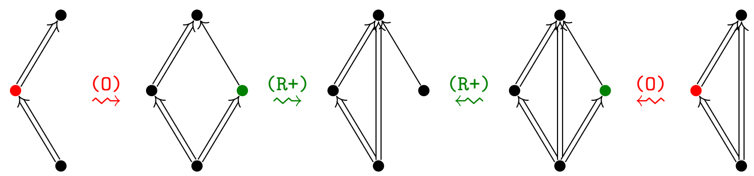
Combined with the results of others, Theorem 3.4 confirms, for the class of amplified graphs, [11, Conjecture 5.1]. The conjecture states that for all xyz other than 010 and 101, the equivalence relation $\overline{\mathtt{xyz}}$ is generated by the moves from {(O), (I-), (I+), (R+), (S), (C+), (P+)} that preserve it.

Theorem 3.4 shows that for amplified graphs,

$$(E, F) \in \overline{\mathtt{010}} \implies E \cong F \implies (E, F) \in \overline{\mathtt{111}}$$

Since we trivially have $\overline{\mathtt{111}} \subseteq \overline{\mathtt{x1z}} \subseteq \overline{\mathtt{010}}$ for all x, z, we deduce that the four equivalence relations $\overline{\mathtt{x1z}}$ are identical and coincide with graph isomorphism. In particular, for amplified graphs, each $\overline{\mathtt{x1z}}$ is trivially contained in the relation generated by the moves that preserve it. For the reverse containment, note that the only moves in the list above that preserve any x1z-equivalences are (O), (I+) and (I-). Of these, neither (I+) nor (I-) can be applied to an amplified graph, and [11, Theorem 3.2] shows that $\langle \mathtt{(O)} \rangle \subseteq \overline{\mathtt{x1z}}$ for all x, z. So we confirm [11, Conjecture 5.1] for amplified graphs for the relations $\overline{\mathtt{x1z}}$.

We now show that a similar result holds for the relations $\overline{\mathtt{x0z}}$. Recall from [12] that if E is an amplified graph then its amplified transitive closure tE is the amplified graph with $tE^0 = E^0$ and $v(tE^1)w \neq \emptyset$ if and only if $vE^*w \setminus \{v\} \neq \emptyset$. Theorem 1.1 of [12] shows that for amplified graphs, if $(E, F) \in \overline{\mathtt{000}}$, then $tE \cong tF$. We claim that this forces $(E, F) \in \overline{\mathtt{101}}$. To see this, first note that by [11, Theorems 3.2 and 3.10], moves (O) and (R+) preserve $\overline{\mathtt{101}}$. So it suffices to show that the graph move t that, given vertices u, v, w such that uE^1v and vE^1w are infinite, adds infinitely many new edges to uE^1w, can be obtained using (O) and (R+). This is achieved as follows:

(3.1)


So as above, for amplified graphs, we see that the four equivalence relations $\overline{\mathtt{x0z}}$ are identical, coincide with isomorphism of amplified transitive closures of the underlying graphs, and are generated by (O) and (R+), and in particular by the moves from [11] that are x0z-invariant.

The results of [11] give the reverse containment, so we have confirmed [11, Conjecture 5.1] for amplified graphs for the relations x0z.

3.2. **Stability of graded algebraic K_0.** Next we establish the stable invariance of graded K-theory. Let Γ be an additive abelian group and let A be a Γ-graded ring. For $\overline{\delta} \in \Gamma^n$, we write $M_n(A)(\overline{\delta})$ for the Γ-graded ring $M_n(A)$ with grading given by $(a_{i,j}) \in M_n(A)_\lambda$ if and only if $a_{i,j} \in A_{\lambda+\delta_j-\delta_i}$. Similarly, for $\overline{\delta} \in \prod_n \Gamma$, we write $M_\infty(A)(\overline{\delta})$ for the Γ-graded ring $M_\infty(A)$ with grading given by $(a_{i,j}) \in M_\infty(A)(\overline{\delta})_\lambda$ if and only if $a_{i,j} \in A_{\lambda+\delta_j-\delta_i}$.

Since the tensor product of two graded modules will be key in the proof, we recall the construct given in [15, Section 1.2.6]. Let Γ be an additive abelian group, let A be a Γ-graded ring, let M be a graded right A-module, and let N be a graded left A-module. Then $M \otimes_A N$ is defined to be $M \otimes_{A_0} N$ modulo the subgroup generated by

$$\{ma \otimes n - m \otimes an : m \in M, n \in N, \text{ and } a \in A \text{ are homogeneous}\}$$

with grading induced by the grading on $M \otimes_{A_0} N$ given by

$$(M \otimes_{A_0} N)_\lambda = \left\{ \sum_i m_i \otimes n_i : m_i \in M_{\alpha_i}, n_i \in N_{\beta_i} \text{ with } \alpha_i + \beta_i = \lambda \right\}.$$

Theorem 3.6. *Let Γ be an additive abelian group, let A be a unital Γ-graded ring, and let $\overline{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in \Gamma^n$. Then the inclusion $\iota\colon A \to M_n(A)(\overline{\delta})$ into the $e_{1,1}$ corner induces a $\mathbb{Z}[\Gamma]$-module order isomorphism $K_0^{\mathrm{gr}}(\iota)\colon K_0^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(M_n(A)(\overline{\delta}))$ given by $K_0^{\mathrm{gr}}(\iota)([P]) = [P \otimes_A M_n(A)(\overline{\delta})]$ (the left A-module structure on $M_n(A)(\overline{\delta})$ is given by the inclusion ι).*

Proof. Let $\overline{\alpha} = (0, \delta_2 - \delta_1, \dots, \delta_n - \delta_1)$. By [15, Corollary 2.1.2], there is an equivalence of categories $\phi\colon \text{Pgr-}A \to \text{Pgr-}M_n(A)(\overline{\alpha})$ given by $\phi(P) = P \otimes_A A^n(\overline{\alpha})$. Moreover, ϕ commutes with the suspension map. Since

$$\begin{aligned} M_n(A)(\overline{\alpha})_\lambda &= \begin{pmatrix} A_\lambda & A_{\lambda+\alpha_2-\alpha_1} & \cdots & A_{\lambda+\alpha_n-\alpha_1} \\ A_{\lambda+\alpha_1-\alpha_2} & A_\lambda & \cdots & A_{\lambda+\alpha_n-\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\alpha_1-\alpha_n} & A_{\lambda+\alpha_2-\alpha_n} & \cdots & A_\lambda \end{pmatrix} \\ &= \begin{pmatrix} A_\lambda & A_{\lambda+\delta_2-\delta_1} & \cdots & A_{\lambda+\delta_n-\delta_1} \\ A_{\lambda+\delta_1-\delta_2} & A_\lambda & \cdots & A_{\lambda+\delta_n-\delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\lambda+\delta_1-\delta_n} & A_{\lambda+\delta_2-\delta_n} & \cdots & A_\lambda \end{pmatrix} = M_n(A)(\overline{\delta})_\lambda, \end{aligned}$$

we have $M_n(A)(\overline{\alpha}) = M_n(A)(\overline{\delta})$. Therefore, $\phi(P) = P \otimes_A A^n(\overline{\alpha})$ is an equivalence of categories from Pgr-A to Pgr-$M_n(A)(\overline{\delta})$ and ϕ commutes with the suspension map. Hence, ϕ induces a $\mathbb{Z}[\Gamma]$-module order isomorphism from $K_0^{\mathrm{gr}}(A)$ to $K_0^{\mathrm{gr}}(M_n(A)(\overline{\delta}))$.

We claim that $\phi = K_0^{\mathrm{gr}}(\iota)$. Let M be a graded right A-module. We will show that $M \otimes_A A^n(\overline{\alpha})$ and $M \otimes_A M_n(A)(\overline{\delta})$ are isomorphic as graded modules. Since $1_A \in A_0$ and $M1_A = M$,

$$M \otimes_A M_n(A)(\overline{\delta}) \cong_{\mathrm{gr}} M \otimes_A \iota(1_A)M_n(A)(\overline{\delta}) = M \otimes_A e_{1,1}M_n(A)(\overline{\delta}).$$

By the definitions of the gradings on $e_{1,1}M_n(A)(\overline{\delta})$ and $A^n(\alpha)$, the right $M_n(A)$-module isomorphism

$$e_{1,1}X \mapsto (x_{1,1}, x_{1,2}, \ldots, x_{1,n})$$

is a graded isomorphism. Hence,

$$M \otimes_A M_n(A)(\overline{\delta}) \cong_{\mathrm{gr}} M \otimes_A e_{1,1}M_n(A)(\overline{\delta}) \cong_{\mathrm{gr}} M \otimes_A A^n(\alpha).$$

Thus, $\phi = K_0^{\mathrm{gr}}(\iota)$. Consequently, $K_0^{\mathrm{gr}}(\iota)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism. □

Corollary 3.7. *Let Γ be an additive abelian group and let A be a Γ-graded ring with a sequence of idempotents $\{e_n\}_{n=1}^{\infty} \subseteq A_0$ such that $e_n e_{n+1} = e_n$ for all n, and $\bigcup_n e_n A e_n = A$. For $\overline{\delta} \in \prod_i \Gamma$, the embedding $\iota \colon A \to M_\infty(A)(\overline{\delta})$ into the $e_{1,1}$ corner of $M_\infty(A)(\overline{\delta})$ induces a $\mathbb{Z}[\Gamma]$-module order isomorphism $K_0^{\mathrm{gr}}(\iota) \colon K^{\mathrm{gr}}(A) \to K_0^{\mathrm{gr}}(M_\infty(A)(\overline{\delta}))$.*

In particular, if E is a countable directed graph and $\overline{\delta} \in \prod_i \mathbb{Z}$, then the inclusion of $\iota \colon L_{\mathbb{K}}(E) \to M_\infty(L_{\mathbb{K}}(E))(\overline{\delta})$ of $L_{\mathbb{K}}(E)$ into the $e_{1,1}$ corner of $M_\infty(L_{\mathbb{K}}(E))(\overline{\delta})$ induces a $\mathbb{Z}[x, x^{-1}]$-module order isomorphism from $K_0^{\mathrm{gr}}(L_{\mathbb{K}}(E))$ to $K_0^{\mathrm{gr}}(M_\infty(L_{\mathbb{K}}(E))(\overline{\delta}))$ for any field $\mathbb{K}$.

Proof. Let $\iota_n \colon e_n A e_n \to M_\infty(e_n A e_n)(\overline{\delta})$ be the inclusion of $e_n A e_n$ into the $e_{1,1}$ corner of $M_\infty(e_n A e_n)(\overline{\delta})$. Observe that $A = \varinjlim e_n A e_n$, that $M_\infty(A) = \varinjlim M_\infty(e_n A e_n)$, and that the diagram

$$\begin{array}{ccc} e_n A e_n & \xrightarrow{\subseteq} & A \\ \big\downarrow{\scriptstyle \iota_n} & & \big\downarrow{\scriptstyle \iota} \\ M_\infty(e_n A e_n)(\overline{\delta}) & \xrightarrow{\subseteq} & M_\infty(A)(\overline{\delta}) \end{array}$$

commutes. Therefore, if each $K_0^{\mathrm{gr}}(\iota_n)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism, then $K_0^{\mathrm{gr}}(\iota)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism since the graded K_0-group respects direct limits ([15, Theorem 3.2.4]). Hence, without loss of generality, we may assume that A is a unital Γ-graded ring.

Let $\overline{\delta}_n = (\delta_1, \delta_2, \ldots, \delta_n)$. Let $j_n \colon A \to M_n(A)(\overline{\delta}_n)$ be the inclusion of A into the $e_{1,1}$ corner of $M_n(A)(\overline{\delta}_n)$, and let $\iota_n : M_n(A)(\overline{\delta}_n) \to M_\infty(A)(\overline{\delta})$ be the inclusion map. Then $\varinjlim M_n(A)(\overline{\delta}_n) = M_\infty(A)(\overline{\delta})$ and the diagram

$$\begin{array}{ccc} A & & \\ \big\downarrow{\scriptstyle j_n} & \searrow{\scriptstyle \iota} & \\ M_n(A)(\overline{\delta}_n) & \xrightarrow{\iota_n} & M_\infty(A)(\overline{\delta}) \end{array}$$

commutes. By Theorem 3.6, $K_0^{\mathrm{gr}}(j_n)$ is a $\mathbb{Z}[\Gamma]$-module order isomorphism. Since the graded-K_0 functor respects direct limits, $K_0^{\mathrm{gr}}(\iota)$ is $\mathbb{Z}[\Gamma]$-module order isomorphism.

For the last part of the corollary, let $\{X_n\}$ be a sequence of finite subsets of E^0 such that $X_n \subseteq X_{n+1}$ and $\bigcup_n X_n = E^0$. Then $e_n := \sum_{v \in X_n} v$ defines idempotents of degree zero such that $\bigcup_n e_n L_{\mathbb{K}}(E) e_n = L_{\mathbb{K}}(E)$. □

As in the preceding subsection, we finish by describing the consequences of Theorem A for graded stable isomorphism of amplified Leavitt path algebras.

Theorem 3.8. *Let E and F be countable amplified graphs and let $\mathbb{K}$ be a field. Then the following are equivalent:*

(1) $E \cong F$
(2) $L_{\mathbb{K}}(E) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F)$*;*
(3) $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$ *for every* $\overline{\delta} \in \prod_i \mathbb{Z}$*;*
(4) $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$ *for some* $\overline{\delta} \in \prod_i \mathbb{Z}$*; and*
(5) $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\varepsilon})$ *for some* $\overline{\delta}, \overline{\varepsilon} \in \prod_i \mathbb{Z}$.

Proof. The argument is very similar to that of Theorem 3.4, so we summarise. Any isomorphism of graphs induces a graded isomorphism of their Leavitt path algebras, and any graded isomorphism $\phi : L_{\mathbb{K}}(E) \cong L_{\mathbb{K}}(F)$ amplifies to a graded isomorphism $\phi \otimes \mathrm{id} : L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{\delta})$, giving (1) $\implies$ (2) $\implies$ (3). The implications (3) $\implies$ (4) $\implies$ (5) are trivial. The second statement of Corollary 3.7 shows that if (5) holds then $K^{\mathrm{gr}}(L_{\mathbb{K}}(E)) \cong K^{\mathrm{gr}}(L_{\mathbb{K}}(F))$ as ordered $\mathbb{Z}[x, x^{-1}]$-modules, and then Theorem A gives (1). □

Remark 3.9. Since statement (1) of Theorem 3.8 does not depend on the field $\mathbb{K}$, we deduce that each of the other four statements holds for some field $\mathbb{K}$ if and only if holds for every field $\mathbb{K}$. In particular the graded-isomorphism problem for amplified Leavitt path algebras is field independent, so it suffices, for example, to consider the field $\mathbb{F}_2$.

Remark 3.10. Let E and F be amplified graphs. Theorem 3.4 shows that the existence of an isomorphism $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^u)$ for every u is equivalent to the existence of such an isomorphism for some u, and indeed to the existence of an isomorphism $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^v)$ for some u, v. All of these conditions are formally weaker than the existence of isomoprhisms $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \beta^v)$ for every pair of strongly continuous representations $u, v : \mathbb{T} \to \mathcal{U}(\ell^2)$, and this in turn is clearly equivalent to the existence of an isomorphism $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \mathrm{id})$ for every u. So it is natural to ask for which amplified graphs E, F and which strongly continuous representations $u : \mathbb{T} \to \mathcal{U}(\ell^2)$ we have $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \mathrm{id})$.

This is an intriguing question to which we do not know a complete answer, but we can certainly show that the condition that $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \cong (C^*(F) \otimes \mathcal{K}, \gamma^F \otimes \mathrm{id})$ for every u is in general strictly stronger than the equivalent conditions of Theorem 3.4. Specifically, let $E = F$ be the directed graph with $E^0 = \{v, w\}$ and $E^1 = \{e_n : n \in \mathbb{N}\}$ with $s(e_n) = v$ and $r(e_n) = w$ for all $\mathbb{N}$. Then the only nonzero spectral subspaces for the gauge action on $C^*(E)$ are those corresponding to $-1, 0, -1$, and so the same is true for the spectral subspaces of $C^*(E) \otimes \mathcal{K}$ with respect to $\gamma^E \otimes \mathrm{id}$. On the other hand, if $u : \mathbb{T} \to B(\ell^2(\mathbb{Z}))$ is given by $u_z e_n = z^n e_n$, then each spectral subspace of $C^*(E) \otimes \mathcal{K}$ for $\gamma^E \otimes \beta^u$ is nonempty, so $(C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \beta^u) \ncong (C^*(E) \otimes \mathcal{K}, \gamma^E \otimes \mathrm{id})$. We do not, however, know of an example in which $C^*(E)$ is simple.

A similar question can be posed for amplified Leavitt path algebras: for which amplified graphs E, F and elements $\overline{\delta} \in \prod_i \mathbb{Z}$ do we have $L_{\mathbb{K}}(E) \otimes M_\infty(\mathbb{K})(\overline{\delta}) \cong^{\mathrm{gr}} L_{\mathbb{K}}(F) \otimes M_\infty(\mathbb{K})(\overline{0})$? The same example shows that the existence of such an isomorphism for every $\overline{\delta}$ is in general strictly stronger than the equivalent conditions of Theorem 3.8.

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