AF-EMBEDDABILITY OF 2-GRAPH ALGEBRAS AND STABLE FINITENESS OF k-GRAPH ALGEBRAS

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ABSTRACT. We characterise stable finiteness of the C^*-algebra of a cofinal k-graph in terms of an algebraic condition involving the coordinate matrices of the graph. This result covers all simple k-graph C^*-algebras. In the special case of cofinal 2-graphs we further prove that AF-embeddability, quasidiagonality and stable finiteness of the 2-graph algebra are all equivalent.

1. INTRODUCTION

Finite-dimensional approximation properties for C^*-algebras play a very important role in their structure theory [7]. In particular, for stably finite C^*-algebras, AF-embeddability and the weaker notion of quasidiagonality are important in the context of recent advances in classification theory for various classes of simple C^*-algebras [19], as well as structure theory of more general stably finite C^*-algebras [6]. In this paper, we determine exactly which k-graphs have stably finite C^*-algebras, and establish the equivalence of AF-embeddability, quasidiagonality and stable finiteness for large classes of 2-graph C^*-algebras, including all simple 2-graph C^*-algebras.

Our motivation, and our key tool, is a theorem of Brown [5, Theorem 0.2], which says that if A is an AF algebra and α is an automorphism of A, then AF-embeddability, quasidiagonality and stable finiteness of the crossed product $A \times_\alpha \mathbb{Z}$ are equivalent and are characterised by a condition on the map in K-theory induced by α. Since every AF-embeddable C^*-algebra is quasidiagonal and every quasidiagonal C^*-algebra is stably finite, the crucial part of Brown's theorem says that $A \times_\alpha \mathbb{Z}$ is AF-embeddable if and only if the image of the homomorphism $1 - K_0(\alpha)$ contains no nontrivial elements of the positive cone of $K_0(A)$: in Brown's words, "$K_0(\alpha)$ compresses no elements in $K_0(A)$." The difficult part of this theorem is the implication that this K-theoretic condition on the automorphism α implies that the crossed product is AF-embeddable.

It is well-known, due to results of Kumjian–Pask–Raeburn [16] that a simple graph C^*-algebra is AF if the graph contains no cycles, and purely infinite otherwise. More generally, Schafhauser [28] has proved that AF-embeddability, quasidiagonality and stable finiteness of $C^*(E)$ are all equivalent to the absence of a cycle with an entrance in the graph E. The hard implication is that the absence of a cycle with an entrance implies AF-embeddability, and Schafhauser proves this by direct construction. But it can also be

recovered from Brown's result using the standard realisation of a graph C^*-algebra, up to stable isomorphism, as a crossed-product of the C^*-algebra of an associated skew-product graph. To do so, one shows that both Brown's K-theoretic condition and the absence of cycles with entrances in E are equivalent to the condition that the vertex matrix A of E satisfies $(1 - A^t)\mathbb{Z}E^0 \cap \mathbb{N}E^0 = \{0\}$ (for details, see Lemma 3.4 and Lemma 4.2.)

Characterising pure infiniteness, stable finiteness or approximate finite dimensionality for k-graph C^*-algebras (even assuming simplicity) has proven much more complicated than for directed graphs (see the partial results in [10, 11, 30]). Each k-graph algebra is a crossed product of an AF algebra by $\mathbb{Z}^k$ [15] rather than $\mathbb{Z}$. So we cannot typically apply Brown's result to understand when the C^*-algebra $C^*(\Lambda)$ of a k-graph Λ is quasidiagonal. Nevertheless, we are led to investigate the relationship between stable finiteness of $C^*(\Lambda)$ and the group $\sum_{i=1}^{k}(1 - A_i^t)\mathbb{Z}\Lambda^0$ where A_i denotes the vertex matrix of the ith coordinate subgraph of Λ.

To describe our conclusions, we first recall two key concepts. A k-graph Λ is *cofinal* if it is possible to reach any vertex of Λ from some point on any infinite path in Λ (see page 9 for details). Since a k-graph Λ is cofinal if and only if every vertex projection in $C^*(\Lambda)$ is full, the C^*-algebras of cofinal k-graphs include all simple k-graph C^*-algebras. A *graph trace* on a k-graph Λ is a function g from the set of vertices of Λ into $[0, \infty)$ that respects the Cuntz–Krieger relation, and is *faithful* if $g(v) \neq 0$ for every vertex v (see page 9 for details). Our main result is the following:

Theorem 1.1. *Let Λ be a row-finite and cofinal k-graph with no sources, and coordinate matrices $A_1, \ldots, A_k$.*

(1) *The following are equivalent.*
  (a) *$C^*(\Lambda)$ is stably finite.*
  (b) *$\left(\sum_{i=1}^{k} \mathrm{im}(1 - A_i^t) \right) \cap \mathbb{N}\Lambda^0 = \{0\}$.*
  (c) *Λ admits a faithful graph trace.*
(2) *If $k = 2$, then the following are equivalent.*
  (a) *$C^*(\Lambda)$ is embeddable in an approximately finite-dimensional C^*-algebra.*
  (b) *$C^*(\Lambda)$ is quasidiagonal.*
  (c) *$C^*(\Lambda)$ is stably finite.*
  (d) *$\left(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t) \right) \cap \mathbb{N}\Lambda^0 = \{0\}$.*

It follows that if cofinal k-graphs Λ and Γ have the same skeleton, then $C^*(\Lambda)$ is stably finite if and only if $C^*(\Gamma)$ is.

We prove part (1) of this theorem in Section 3. Let A be a C^*-algebra with cancellation in K_0. Brown's proof of the implications *AF-embeddability implies quasidiagonality*, *quasidiagonality implies stable finiteness*, and *stable finiteness implies that $K_0(\alpha)$ compresses no elements on $K_0(A)$* goes through, with a suitably modified version of the last condition, to crossed products of A by finitely-generated groups. We apply this to the usual realisation of $C^*(\Lambda)$ up to stable isomorphism as a crossed product of an AF algebra by $\mathbb{Z}^k$ to prove (1a) $\implies$ (1b). Results of [22] show that faithful graph traces on Λ are in bijection with faithful traces on $C^*(\Lambda)$, and we use this to deduce (1c) $\implies$ (1a). To complete the circle of implications, we use the Separating Hyperplane Theorem from convex analysis and the finite-intersection property in $[0, 1]^{\Lambda^0}$ to deduce from the matrix condition (1b) that there is a hereditary subgraph $H\Lambda$ of Λ that admits a faithful graph trace, and then

use stable isomorphism of $C^*(H\Lambda)$ and $C^*(\Lambda)$ to obtain a faithful trace on the latter and hence a faithful graph trace on Λ.

The proof of part (2) of our main theorem occupies Sections 5 and 6. Most of the work is in Section 5, where we deal with the situation where one or the other of the coordinate graphs contains no cycles. Our results in this section do not require that Λ is cofinal. We show that $C^*(\Lambda)$ can be realised up to stable isomorphism as a crossed-product of the graph C^*-algebra of this coordinate graph by an automorphism α. Since the coordinate graph has no cycles, its C^*-algebra is AF, and so Brown's theorem implies that conditions (2a)–(2c) are all equivalent to the condition that $K_0(\alpha)$ compresses no elements of K_0. The bulk of the work in this section is involved in establishing that this K-theoretic condition is equivalent to condition (2d).

We are then left to deal with the situation where Λ is cofinal and has cycles of both colours, which we consider in Section 6. If either of these cycles has an entrance of the same colour, it is straightforward to see that $C^*(\Lambda)$ is not stably finite and that condition (2d) fails. So it remains to show that if Λ contains cycles of both colours, but none of these cycles has an entrance, then condition (2d) holds and $C^*(\Lambda)$ is AF-embeddable. We do this by arguing directly that $C^*(\Lambda)$ is stably isomorphic to $C(\mathbb{T}^2)$ and therefore AF-embeddable; that condition (2d) holds then follows from (1a) $\implies$ (1b) in part (1) of the theorem. In the final section, Section 7, we detail three examples of applications of our results to previously-considered classes of 2-graphs.

2. BACKGROUND

Let A be a C^*-algebra. We say that A is *AF-embeddable* if there exists an injective homomorphism from A into an AF algebra. A projection in A is *infinite* if it is Murray-von Neumann equivalent to a subprojection of itself. A projection which is not infinite is called *finite*. The C^*-algebra A is *infinite* if it contains an infinite projection, and is called *finite* if it admits an approximate unit of projections and all projections in A are finite; if $A \otimes \mathcal{K}$ is finite, then A is *stably finite* [26, page 7]. Finally, A is *quasidiagonal* if there exists a faithful representation $\pi : A \to B(\mathcal{H})$ such that $\pi(A)$ is a set of quasidiagonal operators in the sense of [6, Definition 3.5] (we call π a *quasidiagonal representation*).

We will frequently need to compute in the free abelian group on generators indexed by X for a countable set X. We denote this group by $\mathbb{Z}X$, and regard it as the group of finitely supported integer-valued functions on X; so we denote the generator corresponding to $x \in X$ by δ_x. For $a \in \mathbb{Z}X$, we also write a_x for $a(x)$.

If A and B are stably isomorphic and have approximate identities of projections, then A is stably finite if and only if B is. The next lemma is presumably well known.

Lemma 2.1. *Suppose that C^*-algebras A and B are stably isomorphic. Then A is AF-embeddable (respectively, quasidiagonal, AF) if and only if B is.*

Proof. Since A and B are stably isomorphic, B is isomorphic to a corner of $A \otimes \mathcal{K}$ (and vice-versa), so we just have to check that each of the three properties passes to stabilisations and to corners.

If $A = \overline{\bigcup A_n}$ is AF, then so is $A \otimes \mathcal{K} = \overline{\bigcup A_n \otimes M_n(\mathbb{C})}$; and then if ρ is an AF-embedding of A, then $\rho \otimes 1_{\mathcal{K}}$ is an AF embedding of $A \otimes \mathcal{K}$. Quasidiagonality passes to stabilisations by [12, Corollary 15].

AF-embeddability clearly passes to subalgebras. So does quasidiagonality: if π is quasidiagonal representation of A, it restricts to a quasidiagonal representation of each subalgebra. It is standard that corners of AF algebras are AF [9, Exercise III.2]. $\square$

A *k-graph* is a countable category Λ equipped with a map $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, that carries composition to addition and satisfies the *factorisation property*: if $d(\lambda) = m + n$ then λ has a unique factorisation $\lambda = \mu\nu$ such that $d(\mu) = m$ and $d(\nu) = n$. We write $\Lambda^n := d^{-1}(n)$, and then the factorisation property implies that Λ^0 is precisely the collection of identity morphisms in Λ. Hence the domain and codomain maps determine functions $r, s : \Lambda \to \Lambda^0$ such that $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all $\lambda \in \Lambda$. We call these the range and source maps, and we call elements of Λ^0 *vertices*, and other elements of Λ *paths*. We use the notational convention where, for example, $v\Lambda^n = \{\lambda \in \Lambda^n : r(\lambda) = v\}$. We say that Λ is *row-finite* if $v\Lambda^n$ is finite for all n, v, and that Λ has *no sources* if $v\Lambda^n$ is nonempty for all n, v. For $\mu, \nu \in \Lambda$, we write $\Lambda^{\min}(\mu, \nu)$ for the set $\{(\alpha, \beta) \in s(\mu)\Lambda \times s(\nu)\Lambda : \alpha\mu = \beta\nu \in \Lambda^{d(\mu)\vee d(\nu)}\}$.

If Λ is a k-graph and $j \leq k$, then the jth coordinate graph of Λ, denoted $\Lambda^{\mathbb{N}e_j}$, is the directed graph with vertices Λ^0, edges Λ^{e_j}, and range and source maps inherited from Λ. We write $A_1, \ldots, A_k \in M_{\Lambda^0}(\mathbb{N})$ for the *coordinate matrices* of Λ given by $A_i(v, w) = |v\Lambda^{e_i}w|$. In the specific case when $k = 2$, we call the first coordinate graph the *blue subgraph* and the second coordinate graph the *red subgraph*, and then call a path in the red subgraph a red path and so forth.

A red path $\lambda = \lambda_1\lambda_2 \cdots \lambda_n \in \Lambda$ (where each $\lambda_i \in \Lambda^{e_2}$) is called a *red cycle* if $d(\lambda) \neq 0$, $r(\lambda) = s(\lambda)$ and $s(\lambda_i) \neq s(\lambda_j)$ for $i \neq j$. It is a *red cycle with a red entrance* if λ is a cycle with an entrance in the directed graph $\Lambda^{\mathbb{N}e_2}$.

The C^*-algebra $C^*(\Lambda)$ of a row-finite k-graph Λ with no sources is the universal C^*-algebra generated by elements $\{s_\lambda : \lambda \in \Lambda\}$ satisfying the *Cuntz–Krieger relations*:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections.
(CK2) $s_\lambda s_\mu = s_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$.
(CK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$.
(CK4) $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

It follows from these relations that $s_\mu^* s_\nu = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\mu,\nu)} s_\alpha s_\beta^*$ for all μ, ν (with the convention that the empty sum is equal to zero). We often write p_v rather than s_v for the projection associated to a vertex $v \in \Lambda^0$.

Every higher-rank graph C^*-algebra has a countable approximate identity of projections: enumerate $\Lambda^0 = \{v_1, v_2, \ldots\}$ and then put $e_n := \sum_{i=1}^n p_{v_n}$. It follows from [4] that two k-graph C^*-algebras are stably isomorphic if and only if they are Morita equivalent.

3. STABLE FINITENESS OF k-GRAPH C^*-ALGEBRAS

In this section we prove part (1) of Theorem 1.1 (see Theorem 3.8). We first extend the "easy" implication (3) $\implies$ (4) in Brown's theorem [5, Theorem 0.2] from $\mathbb{Z}$ to a finitely generated discrete group G. The other implications in Proposition 3.1 are well-known; we just record them for ease of reference.

Proposition 3.1. *Let $\alpha : G \to \operatorname{Aut} A$ be an action of a discrete group G on a C^*-algebra A. Let H_α be the subgroup of $K_0(A)$ generated by $\{(\operatorname{id} - K_0(\alpha_g))K_0(A) : g \in G\}$. Consider the following statements:*

(1) $A \times_\alpha G$ *is AF embeddable;*

(2) $A \times_\alpha G$ is quasidiagonal;
(3) $A \times_\alpha G$ is stably finite;
(4) $H_\alpha \cap K_0(A)^+ = \{0\}$.

Then (1) $\implies$ (2) and (2) $\implies$ (3). If $K_0(A)$ has cancellation and G is finitely generated, then (3) $\implies$ (4).

We need two technical lemmas to prove the implication (3) $\implies$ (4) of the proposition. The proof of the first is an adaptation of Brown's proof of the corresponding implication in [5, Theorem 0.2].

Lemma 3.2. Let $k \geq 1$ and $\alpha : G \to \operatorname{Aut} A$ be an action of a discrete group G on a C^*-algebra A. For $l \geq 1$, write α^l for the action of G on $M_l(A)$ by entrywise application of α. Let H_α be the subgroup of $K_0(A)$ generated by

$$\{(\operatorname{id} - K_0(\alpha_g))(a) : g \in G, a \in K_0(A)\}.$$

Suppose that $g_1, \ldots, g_k$ are elements of G, and that p_i, q_i $(1 \leq i \leq k)$ and r are projections in $M_l(A)$ such that $\sum_i (\operatorname{id} - K_0(\alpha_{g_i}))([p_i] - [q_i]) = [r]$ and $r \neq 0$. If $K_0(A)$ has cancellation, then

$$\operatorname{diag}(\alpha^l_{g_1}(p_1), \ldots, \alpha^l_{g_k}(p_k), q_1, \ldots, q_k, 0)$$

is an infinite projection in $M_{2k+1}(M_l(A \times_\alpha G))$.

Proof. Rearranging the expression $\sum_i (\operatorname{id} - K_0(\alpha_{g_i}))([p_i] - [q_i]) = [r]$, we have

$$[\operatorname{diag}(p_1, \ldots p_k, \alpha^l_{g_1}(q_1), \ldots, \alpha^l_{g_k}(q_k), 0)] = [\operatorname{diag}(\alpha^l_{g_1}(p_1), \ldots, \alpha^l_{g_k}(p_k), q_1, \ldots, q_k, r)].$$

Since $K_0(A)$ has cancellation, there is a partial isometry V such that

$$VV^* = \operatorname{diag}(p_1, \ldots, p_k, \alpha^l_{g_1}(q_1), \ldots, \alpha^l_{g_k}(q_k), 0),$$
$$V^*V = \operatorname{diag}(\alpha^l_{g_1}(p_1), \ldots, \alpha^l_{g_k}(p_k), q_1, \ldots, q_k, r).$$

Let $u : g \mapsto u_g$ be the universal unitary representation of G in $\mathcal{M}(A \times_\alpha G)$, and for each $i \leq k$ let $U_{g_i} = \operatorname{diag}(u_{g_i}, \ldots, u_{g_i}) \in M_l(A \times_\alpha G)$ so that each U_{g_i} implements $\alpha^l_{g_i}$. A quick calculation shows that

$$W := \operatorname{diag}(U_{g_1} p_1, \ldots, U_{g_k} p_k, q_1 U^*_{g_1}, \ldots, q_k U^*_{g_k}, 0)$$

satisfies

$$WW^* = \operatorname{diag}(\alpha^l_{g_1}(p_1), \ldots, \alpha^l_{g_k}(p_k), q_1, \ldots, q_k, 0),$$
$$W^*W = \operatorname{diag}(p_1, \ldots p_k, \alpha^l_{g_1}(q_1), \ldots, \alpha^l_{g_k}(q_k), 0) = VV^*,$$
$$(WV)^*(WV) = V^*V = \operatorname{diag}(\alpha^l_{g_1}(p_1), \ldots, \alpha^l_{g_k}(p_k), q_1, \ldots, q_k, r),$$
$$(WV)(WV)^* = WW^* = \operatorname{diag}(\alpha^l_{g_1}(p_1), \ldots, \alpha^l_{g_k}(p_k), q_1, \ldots, q_k, 0).$$

Thus WV is a partial isometry. Since $r \neq 0$, this shows that V^*V is equivalent to its proper subprojection WW^*, and hence is an infinite projection in $M_{2k+1}(M_l(A \times_\alpha G))$. $\square$

The second technical lemma describes the group H_α in terms of generators of G.

Lemma 3.3. Let $\alpha : G \to \operatorname{Aut} B$ be an action of a discrete group G on a C^*-algebra B. Let H_α be the subgroup of $K_0(B)$ generated by $\{(\operatorname{id} - K_0(\alpha_g))K_0(B) : g \in G\}$. If $g_1, \ldots, g_k$ generate G, then $H_\alpha = \sum_{i=1}^k (\operatorname{id} - K_0(\alpha_{g_i}))K_0(B)$.

Proof. It helps to name the right-hand side, so we set $R := \sum_{i=1}^{k}(\mathrm{id} - K_0(\alpha_{g_i}))K_0(B)$. Clearly $R \subseteq H_\alpha$. For the other inclusion we first observe that each

$$(\mathrm{id} - K_0(\alpha_{g_i^{-1}}))K_0(B) = (K_0(\alpha_{g_i}) - 1)K_0(\alpha_{g_i^{-1}})K_0(B) \subseteq R.$$

Fix $g \in G$, and write $g = h_1 \ldots h_n$ where each $h_j \in \{g_1, \ldots, g_k, g_1^{-1}, \ldots, g_k^{-1}\}$. For $a \in K_0(B)$, we have

$$\big(\mathrm{id} - K_0(\alpha_g)\big)a = (\mathrm{id} - K_0(\alpha_{h_n}))a + (\mathrm{id} - K_0(\alpha_{h_{n-1}}))K_0(\alpha_{h_n}(a))$$
$$+ (\mathrm{id} - K_0(\alpha_{h_{n-2}}))K_0(\alpha_{h_{n-1}h_n}(a)) + \cdots + (\mathrm{id} - K_0(\alpha_{h_1}))K_0(\alpha_{h_1^{-1}g}(a)).$$

Each term on the right-hand side belongs to R. Since R is closed under addition, it follows that $\big(\mathrm{id} - K_0(\alpha_g)\big)a \in R$. Using again that R is closed under addition, we deduce that $H_\alpha \subseteq R$. $\qquad\square$

Proof of Proposition 3.1. The implications $(1) \implies (2)$ and $(2) \implies (3)$ are special cases of the general facts that every AF-embeddable C^*-algebra is quasidiagonal [7, Propositions 7.1.9 and 7.1.10], and every quasidiagonal C^*-algebra is stably finite [7, Proposition 7.1.15].

For $(3) \implies (4)$ we adapt the proof of the corresponding assertion in [5, Theorem 0.2]. Assume that G is generated by $\{g_1, g_2, \ldots, g_k\}$ and that $K_0(A)$ is cancellative. We prove the contrapositive. Suppose that $H_\alpha \cap K_0(A)^+ \neq \{0\}$. Thus there exist $l \geq 1$ and a nonzero projection $r \in M_l(A)$ such that $[r] \in H_\alpha$. By Lemma 3.3, there exist projections $p_i, q_i \in M_l(A)$ such that $\sum_i (\mathrm{id} - K_0(\alpha_{g_i}))([p_i] - [q_i]) = [r]$. Now Lemma 3.2 shows that there exists $L \geq 1$ such that $M_L(A \times_\alpha G)$ contains an infinite projection. So $A \times_\alpha G$ is not stably finite. $\qquad\square$

The natural question is under what circumstances some or all of the reverse implications in Proposition 3.1 also hold. We consider this question in the context of k-graphs, so our first order of business is to recall how to realise a k-graph C^*-algebra, up to stable isomorphism, as a crossed-product of an AF algebra, and relate condition (4) of Proposition 3.1 to the coordinate matrices of the k-graph.

Recall from [15, Definition 5.1] that if Λ is a row-finite k-graph with no sources, and $c : \Lambda \to G$ is a functor into a discrete abelian group, then the *skew-product* k-graph $\Lambda \times_c G$ is equal as a set to $\Lambda \times G$, and has structure maps

$$r(\lambda, g) = (r(\lambda), g), \quad s(\lambda, g) = (s(\lambda), g + c(\lambda)),$$

$(\lambda, g)(\mu, g + c(\lambda)) = (\lambda\mu, g)$ and $d(\lambda, g) = d(\lambda)$. There is an action β^c of $\widehat{G}$ on $C^*(\Lambda)$ such that $\beta_\chi^c(s_\lambda) = \chi(c(\lambda))s_\lambda$ for all λ, and there is an isomorphism

$$C^*(\Lambda \times_c G) \cong C^*(\Lambda) \times_{\beta^c} \widehat{G}$$

that carries $s_{(\lambda, g)}$ to the function $\chi \mapsto \chi(g)s_\lambda \in C(\widehat{G}, C^*(\Lambda))$.

For $n \in \mathbb{N}^k$, we write A_n for the element of $M_{\Lambda^0}(\mathbb{N})$ given by $A_n(v, w) = |v\Lambda^n w|$. So $A_n^t(v, w) = |w\Lambda^n v|$. We then have $A_m A_n = A_{m+n}$ by the factorisation property. Thus putting $G_n := \mathbb{Z}\Lambda^0$ for each $n \in \mathbb{Z}^k$, and defining $A_{m,n}^t : G_m \to G_n$ for $m \leq n \in \mathbb{Z}^k$ by $A_{m,n}^t := A_{n-m}^t$, we can form the direct limit $\varinjlim_{m \in \mathbb{Z}^k}(\mathbb{Z}\Lambda^0, A_m^t)$. We write $A_{n,\infty}^t : G_n = \mathbb{Z}\Lambda^0 \to \varinjlim(\mathbb{Z}\Lambda^0, A_m^t)$ for the canonical homomorphisms such that $A_{n,\infty}^t \circ A_{m,n}^t = A_{m,\infty}^t$ for all $m \leq n$. We continue to write A_i for A_{e_i} for $1 \leq i \leq k$; to avoid confusion, we will be careful to ensure that the pronumerals i, j are never used for elements of $\mathbb{N}^k$.

Lemma 3.4. *Let Λ be a row-finite k-graph with no sources. The C^*-algebra $C^*(\Lambda \times_d \mathbb{Z}^k)$ is an AF algebra, and there is an order-isomorphism $K_0(C^*(\Lambda \times_d \mathbb{Z}^k)) \cong \varinjlim_{n \in \mathbb{Z}^k}(\mathbb{Z}\Lambda^0, A_n^t)$ that carries $[p_{(v,n)}]$ to $A_{n,\infty}\delta_v$ for all v, n. There is an action $\alpha : \mathbb{Z}^k \to \operatorname{Aut} C^*(\Lambda \times_d \mathbb{Z}^k)$ such that $\alpha_m(s_{(\lambda,n)}) = s_{(\lambda,m-n)}$ for all $\lambda \in \Lambda$ and $n \leq m \in \mathbb{N}^k$. The C^*-algebra $C^*(\Lambda)$ is stably isomorphic to $C^*(\Lambda \times_d \mathbb{Z}^k) \times_\alpha \mathbb{Z}^k$. As in Proposition 3.1, write H_α for the subgroup of $K_0(C^*(\Lambda \times_d \mathbb{Z}^k))$ generated by $\{(\operatorname{id} - K_0(\alpha_n))[p_{(v,m)}] : n \in \mathbb{Z}^k, (v,m) \in (\Lambda \times_d \mathbb{Z}^k)^0\}$. Then*

$$H_\alpha \cap K_0(C^*(\Lambda \times_d \mathbb{Z}^k))^+ = \{0\} \quad \text{if and only if} \quad \left(\textstyle\sum_{i=1}^k \operatorname{im}(1 - A_i^t)\right) \cap \mathbb{N}\Lambda^0 = \{0\}.$$

Proof. To help keep notation manageable, let $B := C^*(\Lambda \times_d \mathbb{Z}^k)$. For $m \in \mathbb{Z}^k$, let $B_m := \overline{\operatorname{span}}\{s_{(\mu,m-d(\mu))}s^*_{(\nu,m-d(\nu))} : s(\mu) = s(\nu)\}$. Observe that the map $b : (\Lambda \times_d \mathbb{Z}^k)^0 \to \mathbb{Z}^k$ given by $b(\mu, m) = m$ satisfies $d(\mu, m) = b(s(\mu, m)) - b(r(\mu, m))$ for all m; so in the language of [17, Lemma 4.1], we have $\underline{\delta}^0 b = d$. Applying Lemma 4.1 of [17] with A equal to the trivial group $\{0\}$ and c the trivial cocycle, we deduce that for each m there is an isomorphism $\pi_m : B_m \to \bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\Lambda v))$ such that

$$\pi_m(s_{(\mu,m-d(\mu))}s^*_{(\nu,m-d(\nu))}) = \theta_{\mu,\nu} \in \mathcal{K}(\ell^2(\Lambda s(\mu))) \subseteq \bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\Lambda v)).$$

Applying [17, Theorem 4.2], we see further that for $m \leq n \in \mathbb{N}^k$ there is an endomorphism $j_{m,n} : \bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\Lambda v)) \to \bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\Lambda v))$ such that $j_{m,n}(\theta_{\mu,\nu}) = \sum_{\alpha \in s(\mu)\Lambda^{n-m}} \theta_{\mu\alpha,\nu\alpha}$, that $\pi_n \circ j_{m,n} = \pi_m$, and that there is an isomorphism

$$\pi_\infty : B \to \varinjlim \Big(\bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\Lambda v)), j_{m,n} \Big)$$

such that $\pi_\infty(s_{(\mu,m-d(\mu))}s^*_{(\nu,m-d(\nu))}) = j_{m,\infty}(\theta_{\mu,\nu})$ for all μ, ν. Hence $B = \overline{\bigcup_n B_n}$ is AF.

The induced map $(j_{m,n})_*$ on K_0 satisfies

$$(j_{m,n})_*([\theta_{v,v}]) = \sum_{\alpha \in \Lambda^m v} [\theta_{\alpha,\alpha}] = \sum_{w \in \Lambda^0} |v\Lambda^{n-m}v|[\theta_{w,w}].$$

Hence, after identifying $K_0\big(\bigoplus_{v \in \Lambda^0} \mathcal{K}(\ell^2(\Lambda v))\big)$ with $\mathbb{Z}\Lambda^0$ via $[\theta_{v,v}] \mapsto \delta_v$, we see that the map $(j_{m,n})_*$ is implemented by A_{n-m}^t. We have now arrived at the desired description of $K_0(B)$: There is an isomorphism $\rho : K_0(B) \to \varinjlim_{m \in \mathbb{Z}^k}(\mathbb{Z}\Lambda^0, A_m^t)$ such that

$$\rho([s_{(\mu,m-d(\mu))}s^*_{(\mu,m-d(\mu))}]) = \rho([s_{(s(\mu),m)}]) = A_{m,\infty}^t \delta_v$$

for all $v \in \Lambda^0$, $\mu \in \Lambda v$ and $m \in \mathbb{Z}^k$. Moreover, this ρ satisfies

$$\rho(K_0(B)^+) = \bigcup_{m \in \mathbb{N}^k} A_{m,\infty}^t(\mathbb{N}\Lambda^0).$$

Corollary 5.3 of [15] implies that

$$B \cong C^*(\Lambda) \times_\gamma \mathbb{T}^k$$

where γ is the gauge action. This isomorphism carries the inverse of the dual action $\hat{\gamma}$ on $C^*(\Lambda) \times_\gamma \mathbb{T}^k$ to an action α of $\mathbb{Z}^k$ on B such that $\alpha_m(s_{(\mu,n)}) = s_{(\mu,n-m)}$ as claimed. Hence $B \times_\alpha \mathbb{Z}^k \cong C^*(\Lambda) \times_\gamma \mathbb{T}^k \times_{\hat{\gamma}^{-1}} \mathbb{Z}^k$, which is stably isomorphic to $C^*(\Lambda)$ by Takai duality [31]. $\qquad\qquad \square$

To understand H_α, we next describe the action of $\mathbb{Z}^k$ on $K_0(B)$ induced by α in terms of the isomorphism ρ we have just described. The action $\alpha : \mathbb{Z}^k \to \operatorname{Aut} B$ satisfies

$$\rho([\alpha_n(p_{(v,m)})]) = \rho([p_{(v,m-n)}]) = A^t_{m-n,\infty}\delta_v = A^t_{m,\infty}(A^t_n\delta_v).$$

Since $\rho([p_{v,m}]) = A^t_{m,\infty}(\delta_v)$, we deduce that the action $\beta : \mathbb{Z}^k \to \operatorname{Aut} K_0(B)$ induced by α is characterised by $\beta_n \circ A^t_{m,\infty} = A^t_{m,\infty} \circ A^t_n$.

Now Lemma 3.3 gives

$$(3.1) \qquad H_\alpha = \sum_{i=1}^{k}(1 - K_0(\alpha_{e_i}))(K_0(B)),$$

and hence

$$\rho(H_\alpha) = \sum_{i=1}^{k}(1 - \beta_{e_i})\varinjlim(\mathbb{Z}\Lambda^0, j_{m,n}) = \bigcup_{m\in\mathbb{Z}^k} A^t_{m,\infty}\Big(\sum_{i=1}^{k}(1 - A^t_i)\mathbb{Z}\Lambda^0\Big).$$

Now to prove the final statement of the lemma, first suppose that $H_\alpha \cap K_0(B)^+ = \{0\}$. Then

$$\{0\} = \rho(H_\alpha) \cap \rho(K_0(B)^+) \supseteq A^t_{0,\infty}\Big(\Big(\sum_{i=1}^{k}(1 - A^t_i)\mathbb{Z}\Lambda^0\Big) \cap \mathbb{N}\Lambda^0\Big).$$

So it suffices to show that each $A^t_{0,\infty}$ is injective on $\mathbb{N}\Lambda^0$. We have

$$\ker A^t_{0,\infty} = \bigcup_{n\in\mathbb{N}^k} \ker A^t_n.$$

Since Λ has no sources, each A^t_n is a nonnegative integer matrix with no zero columns, giving $\ker A^t_n \cap \mathbb{N}\Lambda^0 = \{0\}$ as required. Hence $\Big(\sum_{i=1}^{k}(1 - A^t_i)\mathbb{Z}\Lambda^0\Big) \cap \mathbb{N}\Lambda^0 = \{0\}$.

Now suppose that $H_\alpha \cap K_0(B)^+ \neq \{0\}$. For each n, let $\widetilde{A}^t_n$ denote the automorphism of $\varinjlim \mathbb{Z}\Lambda^0$ induced by $A^t_n : \mathbb{Z}\Lambda^0 \to \mathbb{Z}\Lambda^0$. Using the isomorphism ρ and equation (3.1), we can choose $a_1, \ldots, a_k \in \varinjlim \mathbb{Z}\Lambda^0$ such that $\sum_{i=1}^{k}(1 - \widetilde{A}^t_{e_i})a_i \in (\varinjlim \mathbb{Z}\Lambda^0)^+ \setminus \{0\}$. For each i, we can choose $n_i \in \mathbb{Z}^k$ and $a_{i,0} \in \mathbb{Z}\Lambda^0$ such that $a_i = A_{n_i,\infty}(a_{i,0})$. Putting $n = \bigvee_i n_i$ and $a_{i,1} := A^t_{n_i,n}(a_{i,0})$ for each i, we then have $a_i = A^t_{n,\infty}a_{i,1}$ for each i. Now

$$\sum_{i=1}^{k}(1 - \widetilde{A}^t_{e_i})a = \sum_{i=1}^{k}(1 - \widetilde{A}^t_{e_i})A^t_{n,\infty}(a_{i,1}) = A^t_{n,\infty}\Big(\sum_{i=1}^{k}(1 - A^t_i)a_{i,1}\Big)$$

belongs to $(\varinjlim \mathbb{Z}\Lambda^0)^+ \setminus \{0\}$. We therefore have $A^t_{n,\infty}\Big(\sum_{i=1}^{k}(1 - A^t_i)a_{i,1}\Big) = A^t_{m,\infty}(c)$ for some $m \in \mathbb{Z}^k$ and some $c \in \mathbb{N}\Lambda^0$. Again, replacing each of m, n with $m \vee n$ and applying the connecting maps, we can assume that $m = n$. So

$$\sum_{i=1}^{k}(1 - A^t_i)a_{i,1} - c \in \ker A^t_{n,\infty} = \bigcup_{p\geq n} \ker A^t_p,$$

so there exists p such that $A^t_p\big(\sum_{i=1}^{k}(1 - A^t_i)a_{i,1} - c\big) = 0$. We have $A^t_pc \in \mathbb{N}\Lambda^0$ because A^t_p is a positive matrix, and $A^t_pc \neq 0$ because Λ has no sources. So the elements $a_{i,2} := A^t_pa_{i,1}$

satisfy

$$\sum_{i=1}^{k}(1 - A_i^t)a_{i,2} = A_p^t\Big(\sum_{i=1}^{k}(1 - A_i^t)a_{i,1}\Big) = A_p^t c \in \Big(\sum_{i=1}^{k}\mathrm{im}(1 - A_i^t)\Big) \cap \mathbb{N}\Lambda^0 \setminus \{0\}. \qquad \square$$

Before we can prove part (1) of our main result Theorem 1.1, we need to investigate how to construct graph traces on k-graphs.

For a row-finite k-graph Λ with no sources, a function $g : \Lambda^0 \to [0, \infty)$ is a *graph trace on Λ* if

$$g(v) = \sum_{\lambda \in v\Lambda^n} g(s(\lambda))$$

for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ [22, Definition 3.5]. A graph trace g is *faithful* if $g(v) \neq 0$ for all $v \in \Lambda^0$.

For the proof of the following result, we use the Separating-Hyperplane Theorem. For this we recall some terminology. As in [32, Theorem 1.3], a *hyperplane* in $\mathbb{R}^n$ is a subset of the form $P = \{x \in \mathbb{R}^n : x \cdot \xi = t\}$ for some $t \in R$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. The sets $H_1 = \{x \in \mathbb{R}^n : x \cdot \xi < t\}$ and $H_2 = \{x \in \mathbb{R}^n : x \cdot (-\xi) < -t\}$ are the *open half-spaces determined by P*. A set $A \subseteq \mathbb{R}^n$ is *affine* if $\{\lambda x + (1 - \lambda)y : \lambda \in \mathbb{R}\} \subseteq A$ whenever $x, y \in A$. A convex set $C \subseteq \mathbb{R}^n$ is *relatively open* if it is open in the relative topology on the smallest affine subset of $\mathbb{R}^n$ that contains it. The Separating Hyperplane Theorem [32, Theorem 11.2] says that given a relatively open convex set C and an closed affine set X in $\mathbb{R}^n$ such that $C \cap X = \emptyset$, there is a hyperplane P such that $X \subseteq P$ and C is contained in one of the half-spaces determined by P. Putting all the terminology together, we obtain the consequence of the theorem that we will want to apply:

Theorem 3.5 (Separating-Hyperplane Theorem). *Suppose that $C \subseteq \mathbb{R}^n$ is convex and open, that $X \subseteq \mathbb{R}^n$ is affine, and that $C \cap X = \emptyset$. Then there exist a nonzero vector $\xi \in \mathbb{R}^n$ and a real number t such that $\xi \cdot x = t$ for all $x \in X$ and $t < \xi \cdot p$ for all $p \in C$.*

Recall (see Definition 2.1 and Examples 1.7(ii) of [15]) that an infinite path in a k-graph Λ is a map $x : \{(m, n) \in \mathbb{N}^k : m \leq n\} \to \Lambda$ such that $d(x(m, n)) = n - m$ and $x(m, n)x(n, p) = x(m, p)$ for all $m \leq n \leq p$. We say that Λ is *cofinal* if, for every $v \in \Lambda^0$ and every infinite path x there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n, n) \neq \emptyset$. Proposition A.2 of [18] shows that Λ is cofinal if and only if, for all v, w in Λ^0, there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\lambda) \neq \emptyset$ for all $\lambda \in w\Lambda^n$.

We say $H \subseteq \Lambda^0$ is *hereditary* if $\lambda \in \Lambda$ and $r(\lambda) \in \Lambda$ imply $s(\lambda) \in H$.

Proposition 3.6. *Let Λ be a row-finite k-graph with no sources. Let $v \in \Lambda^0$, and let $H := s(v\Lambda) \subseteq \Lambda^0$. Then H is hereditary. If*

$$\Big(\sum_{i=1}^{k}\mathrm{im}(1 - A_i^t)\Big) \cap \mathbb{N}\Lambda^0 = \{0\}$$

then there is a graph trace g on $H\Lambda$ such that $g(v) = 1$. If Λ is cofinal, then g is a faithful graph trace on $H\Lambda$, and there is a unique graph trace $\overline{g}$ on Λ such that $\overline{g}|_H = g$.

Proof. The set H is clearly hereditary.

Fix $n \in \mathbb{N}^k$, set $V_n := \bigcup_{m \leq n} s(v\Lambda^m)$, and define

$$Y_n := \{g : \Lambda^0 \to [0, 1] \mid g(v) = 1 \text{ and } g(w) = \sum_{\alpha \in w\Lambda^{e_i}} g(s(\alpha))$$

whenever $n_i \geq 1$ and $w \in V_{n-e_i}$}.

We claim that $Y_n \neq \emptyset$.

To see this, let

$$X_n := \mathrm{span}_{\mathbb{R}}\{(1 - A_i^t)\delta_{w_i} : 0 \leq i \leq k, n_i \geq 1 \text{ and } w_i \in V_{n-e_i}\}.$$

Then each X_n is a finite-dimensional subspace, and in particular a closed affine subset, of $\mathbb{R}^{V_n}$.

We show that $X_n \cap (0, \infty)^{V_n} = \emptyset$. For this, we suppose that $u \in X_n \cap (0, \infty)^{V_n}$, and derive a contradiction. Fix $x_i \in \mathbb{R}^{V_{n-e_i}}$ such that $u = \sum_{i=1}^k (1 - A_i^t)x_i$. Since V_n is finite, the set $(0, \infty)^{V_n}$ is open. Since $(x_1, \ldots, x_k) \mapsto \sum_{i=1}^k (1 - A_i^t)x_i$ is continuous, we deduce that there are $y_1, \ldots, y_k \in \mathbb{Q}^{V_n}$ sufficiently close to the x_i to ensure that $\sum_{i=1}^k (1 - A_i^t)y_i \in (0, \infty)^{V_n}$. Fix $N \in \mathbb{N}$ such that each $Ny_i \in \mathbb{Z}^{V_n}$. Since the A_i are integer matrices, we obtain

$$\sum_{i=1}^k (1 - A_i^t)Ny_i = N \sum_{i=1}^k (1 - A_i^t)y_i \in \Big(\sum_{i=1}^k \mathrm{im}(1 - A_i^t)\Big) \cap \mathbb{N}\Lambda^0,$$

contradicting our hypothesis. So $X_n \cap (0, \infty)^{V_n} = \emptyset$.

Applying Theorem 3.5 to the affine set X_n and the open convex set $(0, \infty)^{V_n}$, we obtain a nonzero vector $\xi^n \in \mathbb{R}^{V_n}$ such that

$$\xi^n \cdot x = t \text{ and } t < \xi^n \cdot p \quad \text{for all } x \in X_n \text{ and } p \in (0, \infty)^{V_n}.$$

Since X_n is a subspace, we have $0 \in X_n$, forcing $t = \xi^n \cdot 0 = 0$. We then have $0 < \xi^n \cdot p$ for every $p \in (0, \infty)^{V_n}$; applying this to $p = \delta_w$ for each $w \in V_n$ shows that $\xi_w^n > 0$ for every $w \in V_n$.

We show that

$$(3.2) \qquad \xi_v^n = \sum_{\lambda \in v\Lambda^m} \xi_{s(\lambda)}^n \text{ for } m \leq n.$$

To establish (3.2), we argue by induction on $|m| := m_1 + m_2 + \cdots + m_k$. If $|m| = 0$ then $m = 0$ and (3.2) holds trivially, giving a base case. Now suppose that (3.2) holds for all $m \leq n$ with $|m| = N$. Suppose that $m' \in \mathbb{N}^k$ satisfies $m' \leq n$ and $|m'| = N + 1$. Then $m' = m + e_i$ for some $1 \leq i \leq k$ and $m \leq n$ with $|m| = N$. By the inductive hypothesis,

$$\xi_v^n = \sum_{\lambda \in v\Lambda^m} \xi^n(s(\lambda)).$$

For $\lambda \in v\Lambda^m$, we have $(1 - A_i^t)\delta_{s(\lambda)} \in X_n$. Since $\xi^n \in X_n^\perp$, we obtain

$$(3.3) \qquad 0 = \xi^n \cdot ((1 - A_i^t)\delta_{s(\lambda)}) = \xi_{s(\lambda)}^n - \sum_{\alpha \in s(\lambda)\Lambda^{e_i}} \xi_{s(\alpha)}^n.$$

Now

$$\xi^n(v) = \sum_{\lambda \in v\Lambda^m} \sum_{\alpha \in s(\lambda)\Lambda^{e_i}} \xi_{s(\alpha)}^n = \sum_{\mu \in v\Lambda^{m'}} \xi_{s(\mu)}^n$$

because $(\lambda, \alpha) \mapsto \lambda\alpha$ is a bijection between the terms appearing in the sums. This completes the inductive step, and proves (3.2).

Take $w \in V_n$, choose $\lambda' \in v\Lambda w$ with $d(\lambda') \leq n$, and put $m := d(\lambda')$. We have

$$\xi_v^n = \sum_{\lambda \in v\Lambda^m} \xi_{s(\lambda)}^n \geq \xi_w^n.$$

So $\xi^n \in [0, \xi_v^n]^{V_n}$. Since $\xi^n \neq 0$, we deduce that $\xi^n(v) > 0$. By rescaling, we may therefore assume that $\xi_v^n = 1$, and then $\xi_w^n \in [0, 1]$ for all $w \in V_n$.

Now take any $g \in [0, 1]^{\Lambda^0}$ such that $g(v) = \xi_v^n$ for all $v \in V_n$ (for example, put $g(v) = \xi_v^n$ for $v \in V_n$ and $g(v) = 0$ for $v \notin V_n$). We show that $g \in Y_n$. For this, fix i such that $n_i > 0$ and fix $w \in V_{n-e_i}$. Choose $m \leq n - e_i$ such that $v\Lambda^m w \neq \emptyset$. Then $(1 - A_i^t)\delta_w \in X_n$, and the calculation (3.3) with $s(\lambda)$ replaced by w shows that $g(w) = \xi_w^n = \sum_{\alpha \in w\Lambda^{e_i}} \xi^n(s(\alpha)) = \sum_{\alpha \in w\Lambda^{e_i}} g(s(\alpha))$ as needed. So $g \in Y_n$, giving $Y_n \neq \emptyset$ as claimed.

Hence the sets $Y_{(1,\ldots,1)}, Y_{(2,\ldots,2)}, \ldots$ are nonempty closed subsets of the compact space $[0, 1]^{\Lambda^0}$, and since $n \leq m \in \mathbb{N}^k$ implies $V_n \subseteq V_m$, we have $Y_{(1,\ldots,1)} \supseteq Y_{(2,\ldots,2)} \supseteq \ldots$. By the finite intersection property, $\bigcap_{j=1}^{\infty} Y_{(j,\ldots,j)}$ is nonempty; say

$$g \in \bigcap_{j \in \mathbb{N}} Y_{(j,\ldots,j)}.$$

Then g is a graph-trace on $H\Lambda$ such that $g(v) = 1$ by definition of the Y_n.

Suppose that Λ is cofinal. Then $H\Lambda$ is also cofinal. Fix $w \in H$. Using the characterisation of cofinality from [18, Proposition A.2], there exists $n \in \mathbb{N}^k$ such that $w(H\Lambda)s(\lambda) \neq \emptyset$ for all $\lambda \in v(H\Lambda)^n$. Now

$$1 = g(v) = \sum_{\lambda \in v(H\Lambda)^n} g(s(\lambda))$$

implies there exists $\lambda' \in v\Lambda^n$ such that $g(s(\lambda')) > 0$. Since $(wH\Lambda)s(\lambda) \neq \emptyset$, we can choose $\xi \in w(H\Lambda)s(\lambda)$. Then

$$g(w) = \sum_{\mu \in w(H\Lambda)^{d(\xi)}} g(s(\mu)) > g(s(\xi)) > 0.$$

Thus g is a faithful graph trace on $H\Lambda$.

We must show that g extends uniquely to a faithful graph trace on Λ. For $w \in H$, set $n_w = 0 \in \mathbb{N}^k$. For each $w \in \Lambda^0 \setminus H$, use the characterisation [18, Proposition A.2] of cofinality to choose $n_w \in \mathbb{N}^k$ such that $v\Lambda s(\lambda) \neq \emptyset$ for all $\lambda \in w\Lambda^n$. Then $s(\lambda) \in H$ for every $w \in \Lambda^0$ and $\lambda \in w\Lambda^{n_w}$. Define $\overline{g} : \Lambda^0 \to \mathbb{R}$ by $\overline{g}(w) := \sum_{\lambda \in w\Lambda^{n_w}} g(s(\lambda))$ for every $w \in \Lambda^0$. By definition of n_w, we have $\overline{g}(w) = g(w)$ for $w \in H$. Since g is a faithful graph trace, we have $\overline{g}(w) > 0$ for all w. Moreover, $\overline{g}$ agrees with g on H because $n_w = 0$ for $w \in H$. If $\tilde{g}$ is any graph trace on Λ that extends g, then the graph-trace condition forces

$$\tilde{g}(w) = \sum_{\lambda \in w\Lambda^{n_w}} \tilde{g}(s(\lambda)) = \sum_{\lambda \in w\Lambda^{n_w}} g(s(\lambda)) = \overline{g}(w)$$

for every $w \in \Lambda^0$, so if $\overline{g}$ is a graph trace, then it is the unique graph trace extending g as claimed. So we just need to prove that $\overline{g}$ is a graph trace. Fix $w \in \Lambda^0$ and $n \in \mathbb{N}^k$. Let

$$N := n_w \vee \Big(\bigvee_{\lambda \in w\Lambda^n} (n + n_{s(\lambda)}) \Big).$$

Since g is a graph trace, we have

$$\overline{g}(w) = \sum_{\mu \in w\Lambda^{n_w}} g(s(\mu)) = \sum_{\mu \in w\Lambda^{n_w}} \Big(\sum_{\tau \in s(\mu)\Lambda^{N-n_w}} g(s(\tau)) \Big) = \sum_{\eta \in w\Lambda^N} g(s(\eta)).$$

Similarly,

$$\sum_{\lambda \in w\Lambda^n} \overline{g}(s(\lambda)) = \sum_{\lambda \in w\Lambda^n} \Big(\sum_{\rho \in s(\lambda)\Lambda^{n_{s(\lambda)}}} g(s(\rho)) \Big)$$

$$= \sum_{\lambda \in w\Lambda^n} \Big(\sum_{\rho \in s(\lambda)\Lambda^{n_{s(\lambda)}}} \Big(\sum_{\zeta \in s(\rho)\Lambda^{N-(n+n_{s(\lambda)})}} g(s(\eta)) \Big) \Big)$$

$$= \sum_{\eta \in w\Lambda^N} g(s(\eta)).$$

So $\overline{g}(w) = \sum_{\lambda \in w\Lambda^n} \overline{g}(s(\lambda))$, completing the proof. $\qquad \square$

To make use of the preceding result, we first need to understand the relationship between the existence of a faithful graph trace and stable finiteness of the associated C^*-algebra. For this, first recall [2, Definitions II.6.7.1 and II.6.8.1] that a semifinite trace τ on a C^*-algebra A is a map from the set A^+ of positive elements of A to $[0, \infty]$ such that $\tau^{-1}([0, \infty))$ is dense in A^+ and such that for all $a, b \in A^+$ and all $\lambda > 0$, we have $\tau(a+b) = \tau(a)+\tau(b)$, $\tau(a^*a) = \tau(aa^*)$ and $\tau(\lambda a) = \lambda \tau(a)$ (with the convention $0 \cdot \infty = 0$). The semifinite trace τ is *faithful* if $\tau(a^*a) > 0$ for all $a \in A \setminus \{0\}$.

Lemma 3.7. *Let Λ be a cofinal row-finite k-graph with no sources. Suppose that there is a faithful graph trace on Λ. Then $C^*(\Lambda)$ is stably finite and carries a faithful semifinite trace.*

Proof. Let g be a faithful graph trace on Λ. Proposition 3.8 of [22] implies that there is a faithful semifinite trace τ on $C^*(\Lambda)$ such that $\tau(p_v) = g(v)$ for all v. Fix $v \in \Lambda^0$. Rescale τ so that $\tau(p_v) = 1$. Then τ restricts to a faithful tracial state on $p_v C^*(\Lambda) p_v$, and so determines a faithful trace τ_n on each $M_n(p_v C^*(\Lambda) p_v)$ by $\tau_n((a_{i,j})) = \sum_{i=1}^n \tau(a_{i,i})$. So each $M_n(p_v C^*(\Lambda) p_v)$ is finite, whence $p_v C^*(\Lambda) p_v$ is stably finite. Since Λ is cofinal, [25, Proposition 3.5] shows that p_v is full. Thus $C^*(\Lambda)$ is Morita equivalent, and hence stably isomorphic [4], to $p_v C^*(\Lambda) p_v$. So $C^*(\Lambda)$ is stably finite as well. $\qquad \square$

We are now ready to prove our key result characterising stable finiteness of k-graph C^*-algebras associated to cofinal k-graphs.

Theorem 3.8. *Let Λ be a row-finite k-graph with no sources.*

(1) *If $C^*(\Lambda)$ is stably finite, then $\Big(\sum_{i=1}^k \text{im}(1 - A_i^t) \Big) \cap \mathbb{N}\Lambda^0 = \{0\}$.*

(2) *If Λ is cofinal and $\Big(\sum_{i=1}^k \text{im}(1 - A_i^t) \Big) \cap \mathbb{N}\Lambda^0 = \{0\}$, then Λ admits a faithful graph trace, and $C^*(\Lambda)$ is stably finite.*

Proof. (1) Resume the notation of Lemma 3.4, and let

$$\rho : \varinjlim_{n \in \mathbb{Z}^k} (\mathbb{Z}\Lambda^0, A_{m,n}) \to K_0(C^*(\Lambda \times_d \mathbb{Z}^k))$$

be the isomorphism described there. Since $C^*(\Lambda)$ is stably finite, Lemma 3.4 shows that $C^*(\Lambda \times_d \mathbb{Z}^k) \times_\alpha \mathbb{Z}^k$ is also stably finite. Since ρ is an isomorphism, Proposition 3.1 implies that

$$\rho(H_\alpha) \cap \rho(K_0(C^*(\Lambda \times_d \mathbb{Z}^k)^+) = \{0\},$$

and then the final statement of Lemma 3.4 gives $\Big(\sum_{i=1}^k \text{im}(1 - A_i^t) \Big) \cap \mathbb{N}\Lambda^0 = \{0\}$.

(2). Suppose that Λ is cofinal and that $\left(\sum_{i=1}^{k} \mathrm{im}(1-A_i^t)\right) \cap \mathbb{N}\Lambda^0 = \{0\}$. Proposition 3.6 shows that Λ carries a faithful graph trace, so Lemma 3.7 shows that $C^*(\Lambda)$ is stably finite. $\qquad\square$

We can now prove the first part of our main result.

Proof of Theorem 1.1(1). For (1), observe that since Λ is cofinal, Theorem 3.8 proves (1a) $\Longleftrightarrow$ (1b) and (1b) $\Longrightarrow$ (1c); and (1c) $\Longrightarrow$ (1a) follows from Lemma 3.7. $\qquad\square$

4. Results about 1-graphs

To apply Brown's theorem [5, Theorem 0.2] in the context of 2-graphs in the next section, we will need an explicit description of the positive cone $K_0(C^*(E))^+$ in the K_0-group of a graph C^*-algebra $C^*(E)$ when E has no cycles. This is folklore, but we could not locate precisely the statement that we need in the literature.

Let E be a row-finite graph with no sources, and let A denote its vertex matrix. The K-theory of $C^*(E)$ is calculated by applying the Pimsner-Voiculescu sequence to the gauge action γ of $\mathbb{T}$ on $C^*(E)$ (see [23, Chapter 7]). Identifying the graph C^*-algebra $C^*(E)$ with the C^*-algebra of the associated 1-graph, and applying Lemma 3.4 with $k = 1$, we see that there is an automorphism β of the AF algebra $C^*(E \times_d \mathbb{Z})$ such that $C^*(E)$ is stably isomorphic to $C^*(E \times_d \mathbb{Z}) \times_\beta \mathbb{Z}$. Moreover, there is an isomorphism $\rho : K_0(C^*(E \times_d \mathbb{Z})) \to \varinjlim(\mathbb{Z}E^0, A^t)$ which carries $K_0(\beta)$ to the automorphism $\widetilde{A}^t$ induced by $A^t : \mathbb{Z}E^0 \to \mathbb{Z}E^0$. Since $C^*(E \times_d \mathbb{Z})$ is an AF algebra, the Pimsner–Voiculescu sequence collapses, giving $K_0(C^*(E)) \cong \mathrm{coker}(\mathrm{id} - K_0(\beta))$. Hence there is an isomorphism $\phi : K_0(C^*(E)) \to \mathrm{coker}(1 - A^t)$ taking $[p_v]$ to $\delta_v + \mathrm{im}(1 - A^t)$ [23, Theorem 7.16].

Lemma 4.1. *Let E be a row-finite directed graph with no sources and no cycles. Let A denote the vertex matrix of E. The isomorphism $\phi : K_0(C^*(E)) \to \mathrm{coker}(1 - A^t)$ carries $K_0(C^*(E))^+$ onto $\{a + \mathrm{im}(1 - A^t) : a \in \mathbb{N}E^0\} \subseteq \mathrm{coker}(1 - A^t)$.*

Proof. The isomorphism $\phi : K_0(C^*(E)) \to \mathrm{coker}(1 - A^t)$ takes $[p_v]$ to $\delta_v + \mathrm{im}(1 - A^t)$. Thus $\mathbb{N}E^0 + \mathrm{im}(1 - A^t) \subseteq \phi(K_0(C^*(E))^+)$.

For the other containment, fix $a \in K_0(C^*(E))^+$. Let $\{F_n\}$ be an increasing sequence of finite subsets of E^1 such that $E^1 = \cup_{n=1}^\infty F_n$. Raeburn and Szymański show in [24, Proof of Theorem 1.5] that there is an increasing sequence $\{E_{F_n}\}$ of finite subgraphs of the dual graph such that $C^*(E) = \varinjlim C^*(E_{F_n})$. Let $\iota_n : C^*(E_{F_n}) \to C^*(E)$ be the injection. Then there is an n such that $a = K_0(\iota_n)(b)$ for some b; and b is positive in $K_0(C^*(E_{F_n}))$ because a is positive.

Since E has no cycles, Lemma 1.3 of [24] implies that E_{F_n} has no cycles. By definition of the E_{F_n} in [24, Definition 1.1],

$$E_{F_n}^0 = F_n \cup \left(r(F_n) \cap s(F_n) \cap r(E^1 \setminus F_n)\right).$$

Inspection of the proof of [24, Lemma 1.2] shows that the injection ι_n carries the vertex projection q_e in $C^*(E_{F_n})$ to $s_e s_e^*$ for $e \in G \subseteq E_{F_n}^0$ and carries q_w to $p_w - \sum_{e \in wG} s_e s_e^*$ for $w \in r(F_n) \cap s(F_n) \cap r(E^1 \setminus F_n)$. Since E is row-finite, we then have

$$\iota_n(q_w) = \sum_{e \in wE^1 \setminus F_n} s_e s_e^*.$$

Since E_{F_n} is a finite graph and has no cycles, there is an isomorphism

$$\bigoplus_{v \in E^0_{F_n}, vE^1_{F_n} = \emptyset} M_{E^*_{F_n} v}(\mathbb{C}) \to C^*(E_{F_n})$$

that carries the matrix unit $\delta_v \theta_{v,v}$ in the direct summand corresponding to a source v in E_{F_n} to the vertex projection $p_v \in C^*(E_{F_n})$ (see, for example, [23, Proposition 1.18]). Since b is positive in $K_0(C^*(E_{F_n}))$ it follows that it is the image of a positive element of

$$K_0\Big(\bigoplus_{v \in E^0_{F_n}, vE^1_{F_n} = \emptyset} M_{E^*_{F_n} v}(\mathbb{C}) \Big) \cong \bigoplus_{v \in E^0_{F_n}, vE^1_{F_n} = \emptyset} \mathbb{Z},$$

and so has the form $\sum_v b_v[q_v]$ for some nonnegative integers b_v. Now we have

$$a = K_0(\iota_n)(b) = \sum_v b_v K_0(\iota_n)([q_v])$$

$$= \sum_{e \in G} b_e[s_e s_e^*] + \sum_{w \in r(F_n) \cap s(F_n) \cap r(E^1 \setminus F_n)} b_w \Big[\sum_{e \in wE^1 \setminus F_n} s_e s_e^* \Big]$$

$$= \sum_{e \in F_n} b_e[p_{s(e)}] + \sum_{w \in r(F_n) \cap s(F_n) \cap r(E^1 \setminus F_n)} \sum_{e \in wE^1 \setminus F_n} b_w[p_{s(e)}]$$

$$= \phi^{-1}\Big(\sum_{e \in F_n} b_e \delta_{s(e)} + \sum_{w \in r(F_n) \cap s(F_n) \cap r(E^1 \setminus F_n)} \sum_{e \in wE^1 \setminus F_n} b_w \delta_{s(e)} + \mathrm{im}(1 - A^t) \Big),$$

as required. $\square$

We next reconcile Theorem 3.8 with Brown's theorem [5, Theorem 0.2] and with recent results of Schafhauser [28]. Schafhauser proves that a graph C^*-algebra $C^*(E)$ is AF embeddable if and only if E contains no cycle with an entrance, and that otherwise it is not stably finite. On the other hand, Brown's theorem combined with Lemma 3.4 shows that $C^*(E)$ is AF-embeddable if and only if the image of $(1 - A^t)$ contains no nontrivial positive elements. Combining the two, we deduce that $\mathrm{im}(1 - A^t)$ contains no nontrivial positive elements if and only if E contains no cycle with an entrance; but it should not be necessary to go through C^*-algebras to prove this combinatorial result. Since we will need one of these implications in the next section, we present a direct proof. We emphasise that most of the implications in the next lemma are due to Schafhauser. Our contribution is the combinatorial proof of (5) $\Longleftrightarrow$ (6).

Lemma 4.2. *Let E be a row-finite directed graph with no sources. The following are equivalent:*

(1) *$C^*(E)$ is AF-embeddable.*
(2) *$C^*(E)$ is quasidiagonal.*
(3) *$C^*(E)$ is stably finite.*
(4) *$C^*(E)$ is finite.*
(5) *No cycle in E has an entrance.*
(6) *The vertex matrix A of E satisfies*

$$\mathrm{im}(1 - A^t) \cap \mathbb{N} E^0 = \{0\}.$$

(7) *The automorphism α of $C^*(E \times_d \mathbb{Z})$ such that $\alpha(s_{(e,n)}) = s_{(e,n+1)}$ satisfies*

$$\mathrm{im}(1 - K_0(\alpha)) \cap K_0(C^*(E \times_d \mathbb{Z}))^+ = \{0\}.$$

Proof. Schafhauser proves that (1)–(5) are equivalent in [28, Theorem 1]. The equivalence (6) $\Longleftrightarrow$ (7) follows from Lemma 3.4 with $k = 1$. So it suffices to establish (5) $\Longleftrightarrow$ (6).

For (5) $\Longleftrightarrow$ (6), first suppose that (5) does not hold, so E has a cycle $\mu = \mu_1 \ldots \mu_n$ with an entrance. Then $r(\mu_i)E^1 \neq \{\mu_i\}$ for some i; say $f \in r(\mu_i)E^1 \setminus \{\mu_i\}$. Let $a := \sum_{i=1}^{n} -\delta_{r(\mu_i)}$. Then

$$(1 - A^t)a = \sum_{i=1}^{n} -\delta_{r(\mu_i)} + \sum_{e \in r(\mu_i)E^1} \delta_{s(e)}$$
$$= \sum_{i=1}^{n} -\delta_{r(\mu_i)} + \sum_{i=1}^{n} \delta_{s(\mu_i)} + \sum_{i=1}^{n} \sum_{e \in r(\mu_i)E^1 \setminus \{\mu_i\}} \delta_{s(e)}$$
$$\geq \delta_{s(f)}.$$

So $(1 - A^t)a \in \big(\mathrm{im}(1 - A^t) \cap \mathbb{N}E^0\big) \setminus \{0\}$.

Conversely, suppose that (6) does not hold. We must show that E contains a cycle with an entrance. Choose $a \in \mathbb{Z}E^0$ such that $(1 - A^t)a \in \mathbb{N}E^0 \setminus \{0\}$, and such that the support of a is minimal in the following sense: if $a' \in \mathbb{Z}E^0$ satisfies $(1 - A^t)a' \in \mathbb{N}E^0 \setminus \{0\}$ and $\mathrm{supp}(a') \subseteq \mathrm{supp}(a)$, then $\mathrm{supp}(a') = \mathrm{supp}(a)$.

We first claim that there is a cycle μ in E such that $a_{r(\mu_i)} \neq 0$ for every $i < |\mu|$. To see this first suppose that $a_v < 0$ for some $v \in E^0$. We have

$$0 \leq ((1 - A^t)a)_v = a_v - \sum_{e \in E^1 v} a_{r(e)},$$

and since a_v is strictly negative, it follows that there exists $e_1 \in E^1 v$ such that $a_{r(e_1)} < 0$. Iterating this argument, we obtain edges e_i such that $s(e_{i+1}) = r(e_i)$ and $a_{r(e_i)} < 0$ for all i. Since $\mathrm{supp}(a)$ is finite, we have $r(e_n) = s(e_m)$ for some $n \geq m$, and then $\mu = e_n \cdots e_m$ is the desired cycle. Now suppose that $a_v \geq 0$ for all v. Since $(1 - A^t)a \neq 0$ we have $a_v > 0$ for some v. Since E has no sources, vE^1 is nonempty; choose $e_1 \in vE^1$. We have

$$0 \leq ((1 - A^t)a)_{s(e_1)} = a_{s(e_1)} - \sum_{f \in E^1 s(e_1)} a_{r(f)} = a_{s(e_1)} - a_v - \sum_{f \in E^1 s(e_1) \setminus \{e_1\}} a_{r(f)}.$$

Since a is nonnegative, we obtain $a_{s(e_1)} - a_v \geq 0$. Since $a_v > 0$, this forces $a_{s(e_1)} > 0$. Again, repeating this argument gives edges e_i with $s(e_i) = r(e_{i+1})$ and $a_{s(e_i)} > 0$ for all i. So as above, since $\mathrm{supp}(a)$ is finite, we have $r(e_n) = s(e_m)$ for some $n \geq m$ and then $\mu = e_n \ldots e_m$ is the desired cycle. This completes the proof of the claim.

Now fix a cycle $\mu = \mu_1 \ldots \mu_n$ with each $a_{r(\mu_i)} \neq 0$. We prove that μ has an entrance. To see this, we suppose to the contrary that $r(\mu_i)E^1 = \{\mu_i\}$ for each i, and derive a contradiction. Since each $r(\mu_i)E^1 = \{\mu_i\}$, the element $b := \sum_{i=1}^{n} \delta_{r(\mu_i)}$ belongs to $\ker(1 - A^t)$, and since each $r(\mu_i)$ belongs to the support of a, the support of b is contained in $\mathrm{supp}(a)$. Hence

$$a' := a - a_{r(\mu)}b$$

satisfies $\mathrm{supp}(a') \subseteq \mathrm{supp}(a) \setminus \{r(\mu)\} \subsetneq \mathrm{supp}(a)$, and

$$(1 - A^t)a' = (1 - A^t)a - a_{r(\mu)}(1 - A^t)b = (1 - A^t)a \in \mathbb{N}E^0 \setminus \{0\},$$

which contradicts minimality of the support of a. Thus μ is a cycle with an entrance. This proves (5) $\Longleftrightarrow$ (6). $\square$

5. 2-graphs with no red cycles

In this section we show how to apply Brown's characterisation of AF-embeddability of crossed products of AF algebras by $\mathbb{Z}$ to characterise when the C^*-algebra of a 2-graph Λ with no red cycles is AF embeddable. Symmetry gives a similar characterisation for 2-graphs with no blue cycles too. The main result in the section is the following; note that, unlike in Theorem 1.1(2), no cofinality hypothesis is required in this result.

Theorem 5.1. *Let Λ be a row-finite 2-graph with no sources and let A_i be the coordinate matrices of Λ with entries $A_i(v, w) = |v\Lambda^{e_i}w|$ for $v, w \in \Lambda^0$. Suppose that Λ contains no red cycles or that Λ contains no blue cycles. Then the following are equivalent:*

(1) *$C^*(\Lambda)$ is AF-embeddable;*
(2) *$C^*(\Lambda)$ is quasidiagonal;*
(3) *$C^*(\Lambda)$ is stably finite;*
(4) *$\big(\operatorname{im}(1 - A_1^t) + \operatorname{im}(1 - A_2^t)\big) \cap \mathbb{N}\Lambda^0 = \{0\}$.*

The bulk of the work involved in proving Theorem 5.1 goes into identifying a skew-product $\Lambda \times_c \mathbb{Z}$ whose C^*-algebra is AF, and then computing the range in $K_0(C^*(\Lambda \times_c \mathbb{Z}))$ of the automorphism of $C^*(\Lambda \times_c \mathbb{Z})$ induced by translation in the $\mathbb{Z}$-coordinate in $\Lambda \times_c \mathbb{Z}$.

Proposition 5.2. *Let Λ be a row-finite 2-graph with no sources. Define $c : \Lambda \to \mathbb{N}$ by $c(\lambda) = d(\lambda)_1$ and consider the C^*-algebra $C^*(\Lambda \times_c \mathbb{Z})$ of the skew-product graph $\Lambda \times_c \mathbb{Z}$.*

(1) *For each $n \in \mathbb{Z}$,*
$$B_n := \overline{\operatorname{span}}\{s_{(\mu,p)}s^*_{(\nu,q)} : p + c(\mu) = q + c(\nu) = n\}$$
is a C^*-subalgebra of $C^*(\Lambda \times_c \mathbb{Z})$, and we have $B_n \subseteq B_{n+1}$ for all n.
(2) *$C^*(\Lambda \times_c \mathbb{Z}) = \overline{\bigcup_n B_n}$.*
(3) *For each $n \in \mathbb{Z}$, set $P_n := \sum_{v \in \Lambda^0} p_{(v,n)} \in M(C^*(\Lambda \times_c \mathbb{Z}))$. Then $P_n B_n P_n$ is a full corner in B_n and is canonically isomorphic to the C^*-algebra $C^*(\Lambda^{\mathbb{N}e_2})$ of the directed graph $\Lambda^{\mathbb{N}e_2} := (\Lambda^0, \Lambda^{e_2}, r, s)$.*
(4) *Suppose that Λ has no red cycles. Then $C^*(\Lambda \times_c \mathbb{Z})$ is an AF algebra.*

Proof. (1) Each B_n is self-adjoint. To see that B_n is closed under multiplication, let $a := s_{(\mu,p)}s^*_{(\nu,q)}$ and $b := s_{(\alpha,q)}s^*_{(\beta,m)} \in B_n$ with $r(\alpha) = r(\nu)$; it suffices to see that $ab \in B_n$. Here $c(\nu) = d(\nu)_1 = d(\alpha)_1 = c(\alpha)$, and if $(\sigma, \tau) \in \Lambda^{\min}(\nu, \alpha)$, then $c(\sigma) = d(\sigma)_1 = 0 = d(\tau)_1 = c(\tau)$. We have
$$\Lambda^{\min}((\nu, q), (\alpha, q)) = \{\big((\sigma, c(\nu) + q), (\tau, c(\alpha) + q)\big) : (\sigma, \tau) \in \Lambda^{\min}(\nu, \alpha)\}$$
and hence
$$s_{(\mu,p)}s^*_{(\nu,q)}s_{(\alpha,q)}s^*_{(\beta,m)} = s_{(\mu,p)}\Big(\sum_{(\sigma,\tau)\in\Lambda^{\min}(\nu,\alpha)} s_{(\sigma,q+c(\nu))}s^*_{(\tau,q+c(\alpha))}\Big)s^*_{(\beta,m)}$$
$$= s_{(\mu,p)}\Big(\sum_{(\sigma,\tau)\in\Lambda^{\min}(\nu,\alpha)} s_{(\sigma,p+c(\mu))}s^*_{(\tau,n+c(\beta))}\Big)s^*_{(\beta,m)}$$
$$= \sum_{(\sigma,\tau)\in\Lambda^{\min}(\nu,\alpha)} s_{(\mu\sigma,p)}s^*_{(\beta\tau,m)}.$$

For each $(\sigma, \tau) \in \Lambda^{\min}(\nu, \alpha)$, we have $c(\mu\sigma) + p = c(\mu) + p = n = c(\beta) + m = c(\beta\tau) + m$, and hence $ab \in B_n$ as required. It follows that B_n is a C^*-subalgebra.

To see that $B_n \subseteq B_{n+1}$, let $s_{(\mu,p)}s^*_{(\nu,q)}$ in B_n. Then

$$s_{(\mu,p)}s^*_{(\nu,q)} = \sum_{\alpha \in s(\mu)\Lambda^{e_1}} s_{(\mu,p)}s_{(\alpha,n)}s^*_{(\alpha,n)}s^*_{(\nu,q)} = \sum_{\alpha \in s(\mu)\Lambda^{e_1}} s_{(\mu\alpha,p)}s^*_{(\nu\alpha,q)}$$

with $c(\mu\alpha) + p = c(\mu) + p + c(\alpha) = n + c(\alpha) = n + 1 = c(\nu\alpha) + q$. Thus $s_{(\mu,p)}s^*_{(\nu,q)} \in B_{n+1}$ and it follows that $B_n \subseteq B_{n+1}$.

(2) If $s_{(\mu,p)}s^*_{(\nu,q)}$ is a nonzero spanning element of $C^*(\Lambda \times_c \mathbb{Z})$, then $s(\mu,p) = s(\nu,q)$ implies $c(\mu) + p = c(\nu) + q$, and hence $s_{(\mu,p)}s^*_{(\nu,q)} \in B_{c(\mu)+p}$. Thus $C^*(\Lambda \times_c \mathbb{Z}) = \overline{\bigcup B_n}$.

(3) Each spanning element in B_n can be written $s_{(\mu,p)}s^*_{(\nu,q)} = s_{(\mu,p)}p_{(s(\mu),n)}s^*_{(\nu,q)} = s_{(\mu,p)}P_n s^*_{(\nu,q)}$. Thus $B_n = B_n P_n B_n$, and $P_n B_n P_n$ is a full corner in B_n.

Another calculation of the sort we have been doing shows that

$$P_n B_n P_n = \overline{\text{span}}\{s_{(\mu,n)}s^*_{(\nu,n)} : c(\mu) = c(\nu) = 0\} = \overline{\text{span}}\{s_{(\mu,n)}s^*_{(\nu,n)} : \mu, \nu \in \Lambda^{\mathbb{N}e_2}\}.$$

The collection $\{p_{(v,n)}, s_{(e,n)} : v \in \Lambda^0, e \in \Lambda^{e_2}\}$ forms a Cuntz–Krieger family for the directed graph $\Lambda^{\mathbb{N}e_2}$. It follows that there is a homomorphism $s_\mu \mapsto s_{(\mu,n)}$ from $C^*(\Lambda^{\mathbb{N}e_2})$ to $P_n B_n P_n$, which is surjective because the $\{p_{(v,n)}, s_{(e,n)}\}$ generate $P_n B_n P_n$. Since each $p_{(v,n)} \neq 0$, the gauge-invariant uniqueness theorem [1, Theorem 2.1] for directed graphs implies that $s_\mu \mapsto s_{(\mu,n)}$ is injective.

(4) Since there are no red cycles in Λ, the directed graph $\Lambda^{\mathbb{N}e_2}$ has no cycles, and hence $C^*(\Lambda^{\mathbb{N}e_2})$ is an AF algebra by [16, Theorem 2.4]. By (3), $P_n B_n P_n$ and $C^*(\Lambda^{\mathbb{N}e_2})$ are isomorphic, and hence $P_n B_n P_n$ is an AF algebra. Since $P_n B_n P_n$ is a full corner of B_n, the two are stably isomorphic [4], so Lemma 2.1 implies that B_n is AF. Now $C^*(\Lambda \times_c \mathbb{Z})$ is a direct limit of AF algebras and hence AF. $\square$

Let Λ be a row-finite 2-graph with no sources. Define $c : \Lambda \to \mathbb{N}$ by $c(\lambda) = d(\lambda)_1$. As in [15, Remark 5.6], there is an automorphism α of $C^*(\Lambda \times_c \mathbb{Z})$ such that

$$(5.1) \qquad \alpha(s_{(\lambda,n)}) = s_{(\lambda,n-1)} \quad \text{for } \lambda \in \Lambda \text{ and } n \in \mathbb{Z}.$$

The second statement of [15, Theorem 5.7] implies that the crossed product $C^*(\Lambda \times_c \mathbb{Z}) \times_\alpha \mathbb{Z}$ is stably isomorphic to $C^*(\Lambda)$.

Lemma 5.3. *Let Λ be a row-finite 2-graph with no sources, and suppose that Λ contains no red cycles. Define $c : \Lambda \to \mathbb{N}$ by $c(\lambda) = d(\lambda)_1$.*

(1) *Let A_1 and A_2 denote the coordinate matrices of Λ. Then A_1^t induces a homomorphism*

$$\widetilde{A_1^t} : \text{coker}(1 - A_2^t) \to \text{coker}(1 - A_2^t),$$

and there is an isomorphism

$$\rho : K_0(C^*(\Lambda \times_c \mathbb{Z})) \to \varinjlim(\text{coker}(1 - A_2^t), \widetilde{A_1^t}).$$

Let $(\widetilde{A_1^t})_{n,\infty}$ be the canonical inclusion of the nth approximating copy of $\text{coker}(1 - A_2^t)$ in $\varinjlim(\text{coker}(1 - A_2^t), \widetilde{A_1^t})$. For $v \in \Lambda^0$ and $n \in \mathbb{N}$ we have

$$\rho([p_{(v,n)}]) = (\widetilde{A_1^t})_{n,\infty}(\delta_v + \text{im}(1 - A_2^t)),$$

and

$$(5.2) \qquad \rho(K_0(C^*(\Lambda \times_c \mathbb{Z}))^+) = \bigcup_n (\widetilde{A_1^t})_{n,\infty}(\mathbb{N}\Lambda^0 + \text{im}(1 - A_2^t)).$$

(2) *Let α be the automorphism of $C^*(\Lambda \times_c \mathbb{Z})$ of (5.1). Then the following diagram commutes for each n:*

$$
\begin{array}{ccc}
\operatorname{coker}(1 - A_2^t) \xrightarrow{\ \widetilde{A_1^t}\ } \operatorname{coker}(1 - A_2^t) \xrightarrow{\ (\widetilde{A_1^t})_{n+1,\infty}\ } \varinjlim(\operatorname{coker}((1 - A_2)^t), \widetilde{A_1^t}) \\
\left\downarrow{\widetilde{A_1^t}}\qquad\qquad \left\downarrow{\widetilde{A_1^t}}\qquad\qquad\qquad\qquad \left\downarrow{\rho \circ K_0(\alpha) \circ \rho^{-1}} \\
\operatorname{coker}(1 - A_2^t) \xrightarrow{\ \widetilde{A_1^t}\ } \operatorname{coker}(1 - A_2^t) \xrightarrow{\ (\widetilde{A_1^t})_{n+1,\infty}\ } \varinjlim(\operatorname{coker}((1 - A_2)^t), \widetilde{A_1^t})
\end{array}
$$

$$(\widetilde{A_1^t})_{n,\infty}$$

(3) *Let H_α be the subgroup of $K_0(C^*(\Lambda \times_c \mathbb{Z}))$ generated by*
$$\{(\operatorname{id} - K_0(\alpha_g))K_0(C^*(\Lambda \times_c \mathbb{Z})) : g \in \mathbb{Z}\}.$$

Then

(5.3)
$$
\begin{aligned}
\rho\big(H_\alpha &\cap K_0(C^*(\Lambda \times_c \mathbb{Z}))^+\big) \\
&= \bigcup_n (\widetilde{A_1^t})_{n,\infty}\Big(\big(\operatorname{im}(1 - A_1^t) + \operatorname{im}(1 - A_2^t) \big) \cap \big(\mathbb{N}\Lambda^0 + \operatorname{im}(1 - A_2^t) \big) \Big).
\end{aligned}
$$

Proof. (1) We need to set up some notation. As in Proposition 5.2, let
$$B_n := \overline{\operatorname{span}}\{s_{(\mu,p)}s^*_{(\nu,q)} : p + c(\mu) = q + c(\nu) = n\} \subseteq C^*(\Lambda \times_c \mathbb{Z})$$
and $P_n := \sum_{v \in \Lambda^0} p_{(v,n)} \in M(C^*(\Lambda \times_c \mathbb{Z}))$. By Proposition 5.2, $B_n \subset B_{n+1}$ via the Cuntz–Krieger relation, $C^*(\Lambda \times_c \mathbb{Z}) = \overline{\bigcup B_n}$, and $P_n B_n P_n$ is a full corner in B_n. The inclusion
$$i_n : P_n B_n P_n \to B_n$$
induces an isomorphism $K_0(i_n) : K_0(P_n B_n P_n) \to K_0(B_n)$ by [20, Proposition 1.2]. Also by Proposition 5.2, there is an isomorphism
$$\phi_n : P_n B_n P_n \to C^*(\Lambda^{\mathbb{N}e_2})$$
such that $\phi_n(s_{(\mu,n)}) = s_\mu$. Let
$$\phi : K_0(C^*(\Lambda^{\mathbb{N}e_2})) \to \operatorname{coker}(1 - A_2^t)$$
be the isomorphism of Lemma 4.1, which carries $[p_v]$ to $\delta_v + \operatorname{im}(1 - A_2^t)$.

Since Λ is a 2-graph, the matrices A_1 and A_2 commute, and so do A_1^t and A_2^t. So $A_1^t(1 - A_2^t)\mathbb{Z}\Lambda^0 = (1 - A_2^t)A_1^t\mathbb{Z}\Lambda^0 \subseteq (1 - A_2^t)\mathbb{Z}\Lambda^0$, and it follows that A_1^t induces a homomorphism
$$\widetilde{A_1^t} : \operatorname{coker}(1 - A_2^t) \to \operatorname{coker}(1 - A_2^t).$$
We now consider the composition
$$\phi \circ K_0(\phi_{n+1}) \circ K_0(i_{n+1})^{-1} \circ K_0(i_n) : K_0(P_n B_n P_n) \to \operatorname{coker}(1 - A_2^t).$$
Tracing a generating element $[p_{(v,n)}] \in K_0(P_n B_n P_n)^+$ through the composition we have:
$$
\begin{aligned}
\phi \circ K_0(\phi_{n+1}) &\circ K_0(i_{n+1})^{-1} \circ K_0(i_n)([p_{(v,n)}]) \\
&= \phi \circ K_0(\phi_{n+1}) \circ K_0(i_{n+1})^{-1}([p_{(v,n)}]) \\
&= \phi \circ K_0(\phi_{n+1}) \circ K_0(i_{n+1})^{-1}\Big(\sum_{e \in v\Lambda^{e_1}} [s_{(e,n)}s^*_{(e,n)}] \Big)
\end{aligned}
$$

$$= \phi \circ K_0(\phi_{n+1}) \circ K_0(i_{n+1})^{-1} \Big(\sum_{e \in v\Lambda^{e_1}} [p_{(s(e),n+1)}] \Big)$$

$$= \phi \circ K_0(\phi_{n+1}) \Big(\sum_{e \in v\Lambda^{e_1}} [p_{(s(e),n+1)}] \Big) \quad \text{because } P_{n+1} \geq p_{(s(e),n+1)}$$

$$= \phi \Big(\sum_{e \in v\Lambda^{e_1}} [p_{s(e)}] \Big) = \sum_{e \in v\Lambda^{e_1}} \delta_{s(e)} + \text{im}(1 - A_2^t).$$

On the other hand, tracing $[p_{(v,n)}]$ through the composition

$$\widetilde{A_1^t} \circ \phi \circ K_0(\phi_n) : K_0(P_n B_n P_n) \to \text{coker}(1 - A_2^t)$$

we get

$$\widetilde{A_1^t} \circ \phi \circ K_0(\phi_n)([p_{(v,n)}]) = \widetilde{A_1^t} \circ \phi([p_v]) = \widetilde{A_1^t}(\delta_v + \text{im}(1 - A_2^t))$$

$$= A_1^t(\delta_v) + \text{im}(1 - A_2^t) = \sum_{e \in v\Lambda^{e_1}} \delta_{s(e)} + \text{im}(1 - A_2^t)$$

Thus

$$\phi \circ K_0(\phi_{n+1}) \circ K_0(i_{n+1})^{-1} \circ K_0(i_n) = \widetilde{A_1^t} \circ \phi \circ K_0(\phi_n)$$

and, by composing with $K_0(i_n)^{-1}$, the next square commutes:

$$
\begin{array}{ccc}
K_0(B_n) & \xrightarrow{\ \text{id}\ } & K_0(B_{n+1}) \\
{\scriptstyle \phi \circ K_0(\phi_n) \circ K_0(i_n)^{-1}} \Big\downarrow & & \Big\downarrow {\scriptstyle \phi \circ K_0(\phi_{n+1}) \circ K_0(i_{n+1})^{-1}} \\
\text{coker}(1 - A_2^t) & \xrightarrow[\ \widetilde{A_1^t}\ ]{} & \text{coker}(1 - A_2^t)
\end{array}
$$

By the universal property of the direct limit $\overline{\bigcup K_0(B_n)}$, there is a unique homomorphism $\rho : \overline{\bigcup K_0(B_n)} \to \varinjlim(\text{coker}(1 - A_2^t), \widetilde{A_1^t})$ such that $\rho|_{K_0(B_n)} = (\widetilde{A_1^t})_{n,\infty} \circ \phi \circ K_0(\phi_n) \circ K_0(i_n)^{-1}$. Reversing the roles of $\overline{\bigcup K_0(B_n)}$ and $\varinjlim(\text{coker}(1 - A_2^t), \widetilde{A_1^t})$ implies that ρ is an isomorphism.

We have $\rho([p_{(v,n)}]) = (\widetilde{A_1^t})_n^\infty(\delta_v + \text{im}(1 - A_2^t))$. So

$$\rho(K_0(\varinjlim B_n)^+) = \bigcup \rho(K_0(B_n))^+$$

$$= \bigcup (\widetilde{A_1^t})_{n,\infty} \circ \phi \circ K_0(\phi_n) \circ K_0(i_n)^{-1}(K_0(B_n)^+)$$

$$= \bigcup (\widetilde{A_1^t})_{n,\infty}(\mathbb{N}\Lambda^0 + \text{im}(1 - A_2^t))$$

by Lemma 4.1.

(2) The only arrow in question is the right-most down arrow. By the universal property of $\varinjlim(\text{coker}((1 - A_2)^t), \widetilde{A_1^t})$, there exists a unique homomorphism

$$\Upsilon : \varinjlim(\text{coker}((1 - A_2)^t), \widetilde{A_1^t}) \to \varinjlim(\text{coker}((1 - A_2)^t), \widetilde{A_1^t})$$

such that

$$\Upsilon \circ (\widetilde{A_1^t})_{n,\infty} = (\widetilde{A_1^t})_{n,\infty} \circ \widetilde{A_1^t} \quad \text{for all } n.$$

We claim that $\Upsilon = \rho \circ K_0(\alpha) \circ \rho^{-1}$. We have

$$\alpha(p_{(v,n)}) = \alpha\Big(\sum_{e \in v\Lambda^{e_1}} s_{(e,n)}s^*_{(e,n)}\Big) = \sum_{e \in v\Lambda^{e_1}} s_{(e,n-1)}s^*_{(e,n-1)}$$

and hence

$$K_0(\alpha)([p_{(v,n)}]) = \sum_{e \in v\Lambda^{e_1}} [p_{(s(e,n-1)}] = \sum_{e \in v\Lambda^{e_1}} [p_{(s(e),n)}] = \sum_{w \in \Lambda^0} A_1^t(w,v)[p_{(w,n)}].$$

Thus

$$\rho \circ K_0(\alpha) \circ \rho^{-1} \circ \widetilde{(A_1^t)}_{n,\infty}(\delta_v + \mathrm{im}(1 - A_2^t)) = \rho \circ K_0(\alpha)([p_{(v,n)}])$$

$$= \sum_{w \in \Lambda^0} A_i^t(w,v)\rho([p_{(w,n)}])$$

$$= \sum_{w \in \Lambda^0} A_i^t(w,v)\widetilde{(A_i^t)}_{n,\infty}(\delta_w + \mathrm{im}(1 - A_2^t))$$

$$= \widetilde{(A_i^t)}_{n,\infty} \circ \widetilde{A_i^t}(\delta_v + \mathrm{im}(1 - A_2^t)).$$

Since the $\delta_v + \mathrm{im}(1 - A_2^t)$ generate the positive cone of $\mathrm{coker}(1 - A_2^t)$, this shows that $\Upsilon = \rho \circ K_0(\alpha) \circ \rho^{-1}$ as claimed.

(3) From (5.2) we have

$$\rho(K_0(C^*(\Lambda \times_c \mathbb{Z}))^+) = \bigcup_n \widetilde{(A_1^t)}_{n,\infty}(\mathbb{N}\Lambda^0 + \mathrm{im}(1 - A_2^t)).$$

Since $\mathbb{Z}$ is generated by 1, Lemma 3.3 gives $H_\alpha = \mathrm{im}(\mathrm{id} - K_0(\alpha))$. We have

$$\widetilde{(A_1^t)}_{n,\infty}\big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) = \widetilde{(A_1^t)}_{n,\infty}\big((1 - \widetilde{A_1^t})\,\mathrm{coker}(1 - A_2^t)\big)$$

$$= \rho \circ (\mathrm{id} - K_0(\alpha)) \circ \rho^{-1} \circ \widetilde{(A_1^t)}_{n,\infty}(\mathrm{coker}(1 - A_2^t))$$

$$= \rho \circ (\mathrm{id} - K_0(\alpha))(K_0(B_n)).$$

Thus

$$\rho(H_\alpha) = \rho(\mathrm{im}(1 - K_0(\alpha)))$$

$$= \bigcup_n \rho \circ (\mathrm{id} - K_0(\alpha))(K_0(B_n)) = \bigcup_n \widetilde{(A_1^t)}_n^\infty\big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big),$$

and the result follows. $\square$

Lemma 5.4. *Let Λ be a row-finite 2-graph with no sources and let A_i be the coordinate matrices of Λ. Define $c : \Lambda \to \mathbb{N}$ by $c(\lambda) = d(\lambda)_1$. Let α be the automorphism of $B := C^*(\Lambda \times_c \mathbb{Z})$ of (5.1) and let H_α be the subgroup of $K_0(B)$ generated by*

$$\{(\mathrm{id} - K_0(\alpha_g))(b) : g \in \mathbb{Z}, b \in K_0(B)\}.$$

Suppose that Λ has no red cycles. Then the following are equivalent:

(1) $H_\alpha \cap K_0(B)^+ = \{0\}$;
(2) $\big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) \cap \big(\mathbb{N}\Lambda^0 + \mathrm{im}(1 - A_2^t)\big) = \{0_{\mathrm{coker}(1-A_2^t)}\}$;
(3) $\big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) \cap \mathbb{N}\Lambda^0 = \{0\}$.

Proof. We prove first that (1) and (2) are equivalent. Let ρ be the isomorphism of Lemma 5.3. By (5.3),

$$\rho\big(H_\alpha \cap K_0(B)\big) = \bigcup_n \widetilde{(A_1^t)}_{n,\infty}\Big(\big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) \cap \big(\mathbb{N}\Lambda^0 + \mathrm{im}(1 - A_2^t)\big)\Big).$$

So (2) immediately implies $\rho\big(H_\alpha \cap K_0(B)^+\big) = \{0\}$, and then (1) follows.

Now assume (1). Aiming for a contradiction, we suppose that (2) fails. Then there exist $x, y \in \mathbb{Z}\Lambda^0$ and $0 \neq c \in \mathbb{N}\Lambda^0$ such that $(1 - A_1^t)x + (1 - A_2^t)y = c$.

Since $\rho(H_\alpha \cap K_0(B)^+) = \{0\}$, for every $n \geq 0$ we have $(1 - A_1^t)x + \mathrm{im}(1 - A_2^t) \in \ker(\widetilde{A_1^t})_{n,\infty}$. By [27, Proposition 6.2.5],

$$\ker(\widetilde{A_1^t})_{n,\infty} = \bigcup_{m \geq n} \ker(\widetilde{A_1^t}^m).$$

So for every n there exists $m \geq n$ such that $(1 - A_1^t)x + \mathrm{im}(1 - A_2^t) \in \ker \widetilde{A_1^t}^m$, that is $(A_1^t)^m(1 - A_1^t)x \in \mathrm{im}(1 - A_2^t)$.

In particular, there exists $m \geq 0$ such that $(A_1^t)^m(1 - A_1^t)x = (1 - A_2^t)z$ for some $z \in \mathbb{Z}\Lambda^0$. Now

$$(A_1^t)^m c = (1 - A_2^t)\big(z + (A_1^t)^m y\big) \in \mathbb{N}\Lambda^0.$$

Since Λ has no sources, neither does $\Lambda^{\mathbb{N}e_1} = (\Lambda^0, \Lambda^{e_1}, r, s)$. So $c > 0$ implies $(A_1^t)^m c > 0$. But now $0 \neq (1 - A_2^t)\big(z + (A_1^t)^m y\big) \in \mathbb{N}\Lambda^0$. This contradicts $(5) \implies (6)$ in Lemma 4.2 applied to the directed graph $\Lambda^{\mathbb{N}e_2}$, which has no cycles because Λ has no red cycles. Thus (1) implies (2).

Next, assume (2). Fix $c \in \big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) \cap \mathbb{N}\Lambda^0$, say $c = (1 - A_1^t)x + (1 - A_2^t)y$. Then $c + \mathrm{im}(1 - A_2^t) = (1 - A_1^t)x + \mathrm{im}(1 - A_2^t) = 0_{\mathrm{coker}(1 - A_2^t)}$ using (2). Now $c \in \mathrm{im}(1 - A_2^t) \cap \mathbb{N}\Lambda^0$. But Λ has no red cycles, so $(5) \implies (6)$ of Lemma 4.2 applied to the directed graph $\Lambda^{\mathbb{N}e_2}$ gives $\mathrm{im}(1 - A_2^t) \cap \mathbb{N}\Lambda^0 = \{0\}$. Thus $c = 0$, giving (3).

Finally, assume (3). Fix $n \in \mathbb{N}\Lambda^0$ and assume that

$$n + \mathrm{im}(1 - A_2^t) \in \big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) \cap \big(\mathbb{N}\Lambda^0 + \mathrm{im}(1 - A_2^t)\big).$$

Then there exist $x, y \in \mathbb{Z}\Lambda^0$ such that $n - (1 - A_1^t)x = (1 - A_2^t)y$. But now $n = (1 - A_1^t)x + (1 - A_2^t)y = 0$ by (3). Thus $n + \mathrm{im}(1 - A_2^t) = 0_{\mathrm{coker}(1 - A_2^t)}$. This gives (2). $\qquad\square$

Proof of Theorem 5.1. By symmetry, it suffices to prove the result when Λ has no red cycles. Let α be the automorphism of $B := C^*(\Lambda \times_c \mathbb{Z})$ described at (5.1), and recall that $B \times_\alpha \mathbb{Z}$ and $C^*(\Lambda)$ are Morita equivalent. Since $B \times_\alpha \mathbb{Z}$ and $C^*(\Lambda)$ are separable, they are stably isomorphic [4]. Since there are no red cycles, B is an AF algebra by Proposition 5.2. Thus Theorem 0.2 of [5] yields equivalence of the following: $B \times_\alpha \mathbb{Z}$ is AF-embeddable, $B \times_\alpha \mathbb{Z}$ is quasidiagonal, $B \times_\alpha \mathbb{Z}$ is stably finite, and $H_\alpha \cap K_0(B)^+ = \{0\}$. Now Lemma 2.1 gives equivalence of $(1), (2), (3)$ and $H_\alpha \cap K_0(B)^+ = \{0\}$. Finally, Lemma 5.4 gives the equivalence of $H_\alpha \cap K_0(B)^+ = \{0\}$ and (4). $\qquad\square$

6. Cofinal 2-graphs

In this section, we consider the structure of the C^*-algebra of a cofinal 2-graph that contains both blue and red cycles, none of which has an entrance. We use cofinality to establish that $C^*(\Lambda)$ is stably isomorphic to $C(\mathbb{T}^2)$, and hence AF-embeddable. This will be the final case in our proof of the implication $(2d) \implies (2a)$ in Theorem 1.1(2).

Proposition 6.1. *Let Λ be a row-finite, cofinal 2-graph with no sources. Suppose that Λ contains a blue cycle with no blue entrance and a red cycle with no red entrance. Then there exist a vertex v, a blue cycle $\zeta \in v\Lambda v$ and a red cycle $\xi \in v\Lambda v$ such that $v\Lambda^\infty = \{(\zeta\xi)^\infty\}$. We have $p_v C^*(\Lambda) p_v \cong C(\mathbb{T}^2)$, the projection p_v is full, and $C^*(\Lambda)$ is stably isomorphic to $C(\mathbb{T}^2)$. In particular Λ is not aperiodic, and $C^*(\Lambda)$ is AF-embeddable and nonsimple.*

Proof. Let μ be a blue cycle with no blue entrance and ν a red cycle with no red entrance. We claim that the cycle μ has no entrance in the sense of [11]: that is, that for every $\eta \in r(\mu)\Lambda$ we have $\mathrm{MCE}(\mu, \eta) \neq \emptyset$. To see this, fix $\eta \in r(\mu)\Lambda$. Since Λ has no sources, there exists $\beta \in s(\eta)\Lambda^{d(\mu)}$. Factor $\eta\beta = \beta'\eta'$ where $d(\beta') = d(\beta) = d(\mu)$. Since $r(\mu_i)\Lambda^{e_1} = \{\mu_i\}$ for each i, we have $\beta' = \mu$ and then $\eta\beta$ is a common extension of μ and η. In particular $\mathrm{MCE}(\mu, \eta) \neq \emptyset$ as required. Similarly, ν has no entrance.

Now fix an infinite path x in Λ. Since Λ is cofinal, there exists $n \in \mathbb{N}^2$ such that $r(\mu)\Lambda x(n)$ and $r(\nu)\Lambda x(n)$ are nonempty. Fix $\lambda \in r(\mu)\Lambda x(n)$. Factorise $\lambda = \lambda_b\lambda_r$ where $\lambda_r \in \Lambda^{\mathbb{N}e_2}$ and $\lambda_b \in \Lambda^{\mathbb{N}e_1}$. Since each $r(\mu_i)\Lambda^{e_1} = \{\mu_i\}$, we have $\lambda_b = (\mu^\infty)(0, d(\lambda_b))$; so replacing μ with $(\mu^\infty)(d(\lambda_b), d(\lambda_b) + d(\mu))$, we may assume that $\lambda \in \Lambda^{\mathbb{N}e_2}$. Since μ has no entrance, and since $d(\lambda_r) \wedge d(\mu) = 0$, Lemma 5.6 of [11] shows that each $s(\lambda_r)\Lambda^{pe_1}$ is a singleton, and that there exists $p > 0$ such that the unique element ζ_0 of $s(\lambda_r)\Lambda^{pe_1}$ is a cycle. We have $kp \geq d(\lambda_b)$ for some k and then since $\lambda_b \in s(\lambda_r)\Lambda^{|\lambda_b|e_1}$, we deduce that λ_b is an initial segment of ζ_0^∞. It follows that $(\zeta_0^\infty)(d(\lambda_b), d(\lambda_b) + pe_1)$ is a blue cycle with no entrance whose range is $x(n)$. Let ζ be the shortest nontrivial blue cycle with no entrance such that $r(\zeta) = x(n)$. Applying the same reasoning with the colours reversed shows that there is also a shortest red cycle ξ with no entrance such that $r(\xi) = x(n)$.

Let $v := x(n)$. We claim that $v\Lambda^\infty = \{(\zeta\xi)^\infty\}$. Since $(l \cdot d(\zeta\xi))_{l=1}^\infty$ is a cofinal sequence in $\mathbb{N}^2$, Remarks 2.2 of [15] implies that we just need to show that each $v\Lambda^{l \cdot d(\zeta\xi)} = \{(\zeta\xi)^l\}$. We argue by induction. Our base case is $l = 1$. Fix $\eta \in v\Lambda^{d(\zeta\xi)}$. Express $\eta = \eta_b\eta_r$ where $d(\eta_b) = d(\zeta)$ and $d(\eta_r) = d(\xi)$. Since ζ has no entrance, we have $r(\zeta)\Lambda^{d(\zeta)} = \{\zeta\}$, so that $\eta_b = \zeta$. Now $r(\eta_r) = s(\zeta) = r(\xi)$, and since ξ has no entrance, we deduce that $\eta_r = \xi$. So $\eta = \eta_b\eta_r = \zeta\xi$. Now for the inductive hypothesis, suppose that $v\Lambda^{l \cdot d(\zeta\xi)} = \{(\zeta\xi)^l\}$. Fix $\alpha \in v\Lambda^{(l+1) \cdot d(\zeta\xi)}$. Factorise $\alpha = \alpha'\alpha''$ with $d(\alpha') = d(\zeta\xi)$. Since $r(\alpha') = v$, the base case gives $\alpha' = \zeta\xi$. Hence $r(\alpha'') = s(\xi) = r(\xi) = v$, and the inductive hypothesis gives $\alpha'' = (\zeta\xi)^l$. Hence $\alpha = \alpha'\alpha'' = (\zeta\xi)(\zeta\xi)^l = (\zeta\xi)^{l+1}$. So $v\Lambda^{(l+1) \cdot d(\zeta\xi)} = \{(\zeta\xi)^{l+1}\}$.

Let $y = \sigma^n(x)$ so that $v\Lambda^\infty = \{y\}$. Since $v\Lambda^\infty = \{y\}$, and since $\sigma^{d(\xi)}(y) \in v\Lambda^\infty$, we see that $\sigma^{d(\xi)}(y) = y$, and $\sigma^{d(\zeta)}(y) = y$ by the same reasoning. We also see that Λ is not aperiodic since $\sigma^{d(\xi)}(z) = z$ for all $z \in v\Lambda^\infty = \{y\}$.

Let $\{S_\lambda\} \subseteq \mathcal{B}(\ell^2(\Lambda^\infty))$ be the partial isometries defining the infinite-path space representation of $C^*(\Lambda)$, and let $\mathrm{lt} : \mathbb{Z}^2 \to \mathcal{B}(\ell^2(\mathbb{Z}^2))$ be the left-regular representation. By [29, Theorem 4.7.6], there is a faithful representation π of $C^*(\Lambda)$ on $\ell^2(\Lambda^\infty) \otimes \ell^2(\mathbb{Z}^2)$ such that $\pi(s_\lambda) = S_\lambda \otimes \mathrm{lt}_{d(\lambda)}$ for all λ. We have $p_v C^*(\Lambda) p_v = \overline{\mathrm{span}}\{s_\mu s_\nu^* : r(\mu) = r(\nu) = v\}$. Fix a spanning element $s_\mu s_\nu^*$ of $p_v C^*(\Lambda) p_v$. Then

$$\pi(s_\mu s_\nu^*)(h_z \otimes h_m) = \begin{cases} h_{\mu\sigma^{d(\nu)}(z)} \otimes h_{m-d(\mu)+d(\nu)} & \text{if } z(0, d(\nu)) = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Since $v\Lambda^\infty = \{y\}$, we have $v\Lambda^{d(\nu)} = \{y(0, d(\nu))\}$, and so $z(0, d(\nu)) = \nu$ if and only if $z = y$, and then $\mu\sigma^{d(\nu)}(z) = \mu\sigma^{d(\nu)}(y) \in v\Lambda^\infty = \{y\}$. Writing θ_{h_y, h_y} for the rank-1 projection onto $\mathbb{C}h_y \in \ell^2(\Lambda^\infty)$, we obtain

$$\pi(s_\mu s_\nu^*)(h_z \otimes h_m) = \delta_{z,y}(h_y \otimes h_{m+(d(\mu)-d(\nu))}) = \left(\theta_{h_y, h_y} \otimes \mathrm{lt}_{d(\mu)-d(\nu)}\right)(h_z \otimes h_m).$$

So each $\pi(s_\mu s_\nu^*) = \theta_{h_y, h_y} \otimes \mathrm{lt}_{d(\mu)-d(\nu)})$. Since π is faithful, we deduce that $p_v C^*(\Lambda) p_v$ is isomorphic to $C^*(\{\mathrm{lt}_{d(\mu)-d(\nu)} : \mu, \nu \in v\Lambda, s(\mu) = s(\nu)\})$, and hence to the C^*-algebra of the subgroup H of $\mathbb{Z}^2$ generated by the elements $\{d(\mu) - d(\nu) : \mu, \nu \in v\Lambda, s(\mu) = s(\nu)\}$. In particular, $d(\xi) = d(\xi) - d(v)$ and $d(\eta) = d(\eta) - d(v)$ both belong to H, and since

$d(\xi) \in \mathbb{N}e_1$ and $d(\eta) \in \mathbb{N}e_2$, we see that the rank of H is 2, and so $H \cong \mathbb{Z}^2$. Hence $p_v C^*(\Lambda) p_v \cong C(\mathbb{T}^2)$.

Since Λ is cofinal, Proposition 3.4 of [25] shows that p_v is full, and then $C^*(\Lambda)$ is stably isomorphic to $p_v C^*(\Lambda) p_v$ by [4]. It follows immediately that $C^*(\Lambda)$ is not simple. There is a continuous surjection of the Cantor set 2^ω onto $\mathbb{T}^2$ (see, for example, [14, page 66]) and hence an embedding of $C(\mathbb{T}^2)$ into $C(2^\omega)$. Since $C(2^\omega)$ is AF [9, page 77], it follows that $C(\mathbb{T}^2)$ is AF-embeddable. Thus $C^*(\Lambda)$ is AF-embeddable by Lemma 2.1. $\qquad\square$

The final observation that we need to complete the proof of our main theorem is that if a k-graph contains a cycle with an entrance in any of its coordinate subgraphs, then its C^*-algebra is not stably finite.

Lemma 6.2. *Let Λ be a row-finite k-graph with no sources, and fix $j \leq k$. If there exists a cycle with an entrance in the jth coordinate graph of Λ, then $C^*(\Lambda)$ is not stably finite.*

Proof. This follows from the argument of [16, Theorem 2.4] or from [11, Corollary 3.8]: if $\mu = \mu_1 \ldots \mu_n$ is a cycle with an entrance f in the jth coordinate graph of Λ, then $S := \sum_{i=1}^{n} s_{\mu_i}$ satisfies $S^*S \geq SS^* + s_f s_f^* > SS^*$, so S^*S is an infinite projection. $\qquad\square$

We can now finish the proof of our main theorem.

Proof of Theorem 1.1(2). Let Λ be a cofinal row-finite 2-graph with no sources. If Λ has no red cycle or no blue cycle, then the result follows immediately from Theorem 5.1. Thus we may assume that Λ has a red and a blue cycle. We consider two cases:

(1) No red cycle has a red entrance and no blue cycle has a blue entrance;
(2) at least one blue cycle has a blue entrance or at least one red cycle has a red entrance.

First, suppose that no red cycle has a red entrance and no blue cycle has a blue entrance. Since Λ is cofinal, $C^*(\Lambda)$ is AF-embeddable by Proposition 6.1. Thus $C^*(\Lambda)$ is quasidiagonal and stably finite as well. So for the equivalence of the four conditions, it suffices to show the matrix condition

$$\left(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t) \right) \cap \mathbb{N}\Lambda^0 = \{0\}.$$

holds. But this follows from Theorem 1.1(1) with $k = 2$.

Second, suppose that Λ has a blue cycle $\mu = \mu_1 \ldots \mu_n$ with a blue entrance $e \in r(\mu_i)\Lambda^{e_1} \setminus \{\mu_i\}$. By Lemma 6.2, $C^*(\Lambda)$ is not stably finite, hence not quasidiagonal and not AF-embeddable. So we need to show the matrix condition fails, which follows from Lemma 4.2 applied to the blue graph $\Lambda^{\mathbb{N}e_1}$. $\qquad\square$

7. Examples

In this section we reconcile our results with what is known about the C^*-algebras of rank-2 Bratteli diagrams [21] and a key example from [11], and indicate how to apply them to another class of concrete examples.

7.1. **Rank-2 Bratteli diagrams.** Recall from [21] that a row-finite 2-graph Λ with no sources is called a rank-2 Bratteli diagram if there is a decomposition $\Lambda^0 = \bigcup_{n=1}^{\infty} V_n$ such that: (1) each V_n is finite; (2) $\Lambda^{e_1} = \bigsqcup_{n=1}^{\infty} V_n \Lambda^{e_1} V_{n+1}$; (3) $\Lambda^{e_1} v \neq \emptyset$ for every $v \in \Lambda^0 \setminus V_1$; (4) $\Lambda^{e_2} = \bigsqcup_{n=1}^{\infty} V_n \Lambda^{e_2} V_n$; and (5) each $v\Lambda^{e_2}$ is a singleton.

It then follows that there exist c_n such that each V_n decomposes as

$$V_n = \bigsqcup_{j=1}^{c_n} V_{n,j}$$

such that the vertices in each $V_{n,j}$ are connected together in a red cycle with no entrance and $\Lambda^{e_2} = \bigsqcup_{n,j} V_{n,j} \Lambda^{e_2} V_{n,j}$. Theorem 3.1 of [21] implies that $C^*(\Lambda)$ is an A$\mathbb{T}$-algebra, and is therefore AF-embeddable. We show how to recover this from Theorem 5.1.

Since Λ has no blue cycles, we must show that

$$\big(\operatorname{im}(1 - A_1^t) + \operatorname{im}(1 - A_2^t) \big) \cap \mathbb{N}\Lambda^0 = \{0\}.$$

Suppose that $a, b \in \mathbb{Z}\Lambda^0$ satisfy $(1 - A_1^t)a + (1 - A_2^t)b \geq 0$. The matrix A_2^t is block-diagonal with respect to the decomposition $\Lambda^0 = \bigsqcup_{n,j} V_{n,j}$, and the diagonal blocks are all permutation matrices. So for each n, j we have $\sum_{v \in V_{n,j}}((1 - A_2^t)b)_v = 0$. Thus $\sum_{v \in V_{n,j}}((1 - A_1^t)a)_v \geq 0$ for all n, j.

It suffices to prove that $(1 - A_1^t)a = 0$; for then Since $(1 - A_1^t)a + (1 - A_2^t)b \geq 0$ forces $(1 - A_2^t)b \geq 0$, and then since $\sum_{v \in V_{n,j}}((1 - A_2^t)b)_v = 0$, we obtain $(1 - A_2^t)b = 0$.

For each n, j, let $x_{n,j} = \sum_{v \in V_{n,j}} a_v$. Lemma 4.2 of [21] shows that for $v, w \in V_{n,j}$ and for $l \leq c_{n+1}$, the sets $v\Lambda^{e_1}V_{n+1,l}$ and $w\Lambda^{e_1}V_{n+1,l}$ have the same cardinality $C_n(j,l)$. (Unfortunately our notation and our convention for connectivity matrices are not compatible with [21], so our $C_n(j,l)$ is denoted $A_n(l,j)$ there.) It follows that

$$(7.1) \qquad 0 \leq \sum_{w \in V_{n+1,l}} ((1 - A_1^t)a)_w = x_{n+1,l} - \sum_{j \leq c_n} C_n(j,l)x_{n,j}$$

for all n, l. Let F be the directed graph with vertices $\{(n,j) : n \in \mathbb{N}, j \leq c_n\}$ and $|(n,j)F^1(n+1,l)| = C_n(j,l)$ for all n, j, l. The vertex matrix A_F satisfies $A_F^t((n,j),(n+1,l)) = C_n(j,l)$, and so $(1 - A_F^t)x \in \mathbb{N}F^0$. Since F has no cycles, (5) $\Longrightarrow$ (6) gives $(1 - A_F)x = 0$. So (7.1) gives $(1 - A_1^t)a = 0$.

7.2. **An example of Evans.** In his thesis [10], Evans investigates conditions under which a k-graph algebra is AF. He described two particularly vexing examples that indicate the difficulties involved in answering this question. The examples in question have common skeleton illustrated in Figure 1.
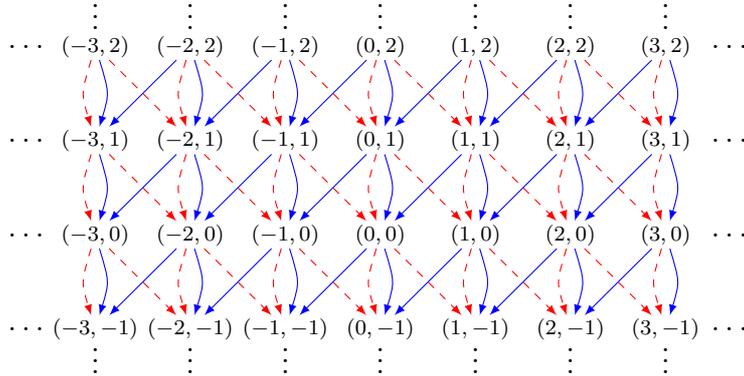

FIGURE 1. The common skeleton of two examples studied in [10].

In [11], Evans and Sims proved that the C^*-algebra of one of the two examples discussed in [10] is the UHF algebra of type 2^∞, and that the other is AF-embeddable. Since the

skeleton in Figure 1 contains no cycles of either colour, Theorem 5.1 applies. We show here how to see that the C^*-algebra of any 2-graph with the skeleton in Figure 1 is AF-embeddable.

We must show that the coordinate matrices $A_1, A_2 \in M_{\Lambda^0}(\mathbb{N})$ for the blue and red graphs in the skeleton satisfy $\big(\operatorname{im}(1 - A_1^t) + \operatorname{im}(1 - A_2^t)\big) \cap \mathbb{N}\Lambda^0 = \{0\}$.

Fix $j \in \mathbb{N}$. For all $i \in \mathbb{N}$, the matrix A_1^t satisfies $A_1^t \delta_{(i,j)} = \delta_{(i,j+1)} + \delta_{(i+1,j+1)}$, and A_2^t satisfies $A_2^t \delta_{(i,j)} = \delta_{(i,j+1)} + \delta_{(i-1,j+1)}$. We deduce that for $a, b \in \mathbb{Z}\Lambda^0$, we have

$$(7.2) \qquad \sum_{i=-\infty}^{\infty} \big((1 - A_1^t)a + (1 - A_2^t)b\big)_{(i,j)} = \sum_{i=-\infty}^{\infty} (a+b)_{(i,j)} - 2\sum_{i=-\infty}^{\infty} (a+b)_{(i,j-1)}.$$

Now suppose that $a, b \in \mathbb{Z}\Lambda^0$ satisfy $(1 - A_1^t)a + (1 - A_2^t)b \geq 0$. Then

$$\sum_{i=-\infty}^{\infty} (a+b)_{(i,j)} \geq 2\sum_{i=-\infty}^{\infty} (a+b)_{(i,j-1)}.$$

Since $a + b$ is finitely supported, this forces $\sum_{i=-\infty}^{\infty}(a+b)_{(i,j)} = 0$. Putting this back into (7.2) forces $\sum_{i=-\infty}^{\infty} \big((1 - A_1^t)a + (1 - A_2^t)b\big)_{i,j} = 0$. Since $(1 - A_1^t)a + (1 - A_2^t)b$ is nonnegative, we deduce that it is zero.

7.3. A class of acyclic 2-graphs.

In this section we illustrate our results with a class of examples for which the question of AF-embeddability has not been settled by previous results. They have skeletons of the following form (the numbers $|v_n\Lambda^{e_1}v_{n+1}|$ and $|v_m\Lambda^{e_1}v_{m+1}|$ of blue edges connecting distinct pairs of consecutive vertices are not assumed to be equal, and likewise for red edges).

$$(7.3)$$
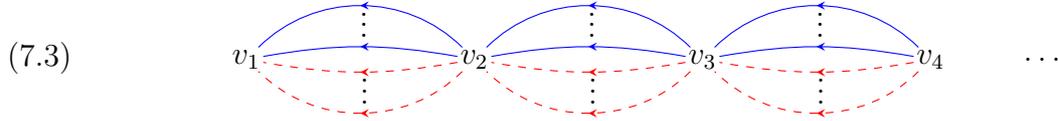
In any such 2-graph Λ, the factorisation property is completely determined by bijections $\theta_n : v_n\Lambda^{e_1}v_{n+1} \times v_{n+1}\Lambda^{e_2}v_{n+2} \to v_n\Lambda^{e_2}v_{n+1} \times v_{n+1}\Lambda^{e_1}v_{n+2}$: specifically, $\alpha\beta = \beta'\alpha'$ where $\theta_n(\alpha, \beta) = (\beta', \alpha')$ (see [15, Section 6] or [13]).

Since these 2-graphs have no cycles, Theorem 5.1 applies to characterise when the associated C^*-algebras are AF-embeddable. To apply it, we first show that the ratios $|v_n\Lambda^{e_1}|/|v_n\Lambda^{e_2}|$ are all equal.

Lemma 7.1. *Suppose that Λ is a 2-graph with skeleton of the form* (7.3). *Then*
$$|v_n\Lambda^{e_1}|/|v_n\Lambda^{e_2}| = |v_m\Lambda^{e_1}|/|v_m\Lambda^{e_2}| \quad \text{for all } m, n \in \mathbb{N} \setminus \{0\}.$$

Proof. We need only show that $|v_n\Lambda^{e_1}|/|v_n\Lambda^{e_2}| = |v_{n+1}\Lambda^{e_1}|/|v_{n+1}\Lambda^{e_2}|$ for each n. The factorisation property gives $|v_n\Lambda^{e_1}||v_{n+1}\Lambda^{e_2}| = |v_n\Lambda^{(1,1)}v_{n+2}| = |v_n\Lambda^{e_2}||v_{n+1}\Lambda^{e_1}|$. Since all the quantities involved are finite and nonzero, we can rearrange to obtain $|v_n\Lambda^{e_1}|/|v_n\Lambda^{e_2}| = |v_{n+1}\Lambda^{e_1}|/|v_{n+1}\Lambda^{e_2}|$. $\square$

We can now characterise AF-embeddability of $C^*(\Lambda)$ for any 2-graph Λ with skeleton of the form (7.3) as an immediate consequence of Theorem 5.1. We could also deduce from the same result that when $C^*(\Lambda)$ is not AF-embeddable, it is not stably finite. But with a little extra work, we can prove that when it is not AF-embeddable, $C^*(\Lambda)$ is purely infinite.

Proposition 7.2. *Suppose that Λ is a 2-graph with skeleton of the form (7.3). Then $C^*(\Lambda)$ is AF-embeddable if $|v_1\Lambda^{e_1}| = |v_1\Lambda^{e_2}|$, and is purely infinite otherwise.*

Proof. First suppose that $|v_1\Lambda^{e_1}| = |v_1\Lambda^{e_2}|$. Then $|v_n\Lambda^{e_1}| = |v_n\Lambda^{e_2}|$ for all n by Lemma 7.1. So $A_1^t = A_2^t$. Hence $\big(\mathrm{im}(1 - A_1^t) + \mathrm{im}(1 - A_2^t)\big) = \mathrm{im}(1 - A_1^t)$. Since the blue subgraph of Λ has no cycles, we have $\mathrm{im}(1 - A_1^t) \cap \mathbb{N}\Lambda^0 = \{0\}$ by (5) $\Longrightarrow$ (6) of Lemma 4.2. Hence $C^*(\Lambda)$ is AF-embeddable by Theorem 5.1.

Now suppose that $|v_1\Lambda^{e_1}| \neq |v_1\Lambda^{e_2}|$. We may assume (by reversing the roles of the colours if necessary) that $|v_1\Lambda^{e_1}| < |v_1\Lambda^{e_2}|$. So $R := |v_1\Lambda^{e_1}|/|v_1\Lambda^{e_2}| < 1$, and Lemma 7.1 gives $|v_n\Lambda^{e_1}|/|v_n\Lambda^{e_2}| = R$ for all n. We claim that Λ is aperiodic. To see this, we suppose that Λ is not aperiodic, and derive a contradiction. Parts (2) and (3) of [8, Theorem 4.2] show that there exist $p \neq q \in \mathbb{N}^2$ and an element $n \in \mathbb{N}$ for which there is a source-preserving bijection $\psi : v_n\Lambda^p \to v_n\Lambda^q$. Let $|p| := p_1 + p_2$ be the sum of the coordinates of $p \in \mathbb{N}^2$, and likewise for q. If $\lambda \in v_n\Lambda^p$, then

$$v_{n+|p|} = s(\lambda) = s(\psi(\lambda)) = v_{n+|q|},$$

and in particular, $|p| = |q|$. Since $p \neq q$, we have $p_1 \neq q_1$, so without loss of generality, we may assume that $p_1 > q_1$. Using the factorisation property, we see that

$$|v_n\Lambda^p| = \prod_{i=0}^{p_1-1} |v_{n+i}\Lambda^{e_1}| \prod_{i=p_1}^{|p|-1} |v_{n+i}\Lambda^{e_1}|, \quad \text{and}$$

$$|v_n\Lambda^q| = \prod_{i=0}^{q_1-1} |v_{n+i}\Lambda^{e_1}| \prod_{i=q_1}^{|q|-1} |v_{n+i}\Lambda^{e_2}| = \prod_{i=0}^{q_1-1} |v_{n+i}\Lambda^{e_1}| \prod_{i=q_1}^{|p|-1} |v_{n+i}\Lambda^{e_2}|.$$

Hence

$$|v_n\Lambda^p|/|v_n\Lambda^q| = \prod_{i=q_1}^{p_1-1} \frac{|v_{n+i}\Lambda^{e_1}|}{|v_{n+i}\Lambda^{e_2}|} = R^{p_1-q_1} < 1,$$

which contradicts $|v_n\Lambda^p| = |v_n\Lambda^q|$. So Λ is aperiodic as claimed.

Now fix $n \in \mathbb{N}$. Using that $|v_n\Lambda^{e_1}| < |v_n\Lambda^{e_2}|$, choose an injection $\phi : v_n\Lambda^{e_1} \to v_n\Lambda^{e_2}$ and an element $\beta \in v_n\Lambda^{e_2} \setminus \phi(v_n\Lambda^{e_1})$. The element $V := \sum_{\alpha \in v_n\Lambda^{e_1}} s_{\phi(\alpha)} s_\alpha^*$ satisfies

$$V^*V = \sum_{\alpha \in v_n\Lambda^{e_1}} s_\alpha s_\alpha^* = \sum_{\alpha \in v_n\Lambda^{e_1}} s_\alpha s_\alpha^* = p_{v_n}, \text{ and}$$

$$VV^* = \sum_{\alpha \in v_n\Lambda^{e_1}} s_{\phi(\alpha)} s_{\phi(\alpha)}^* \leq \sum_{\eta \in v_n\Lambda^{e_2} \setminus \beta} s_\eta s_\eta^* = p_{v_n} - s_\beta s_\beta^* < p_{v_n}.$$

So each p_{v_n} is an infinite projection.

So Λ is cofinal and aperiodic, and all vertex projections are infinite in $C^*(\Lambda)$. Thus [3, Corollary 5.1] implies that $C^*(\Lambda)$ is purely infinite. $\qquad\square$

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