ON HONG AND SZYMAŃSKI'S DESCRIPTION OF THE PRIMITIVE-IDEAL SPACE OF A GRAPH ALGEBRA

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ABSTRACT. In 2004, Hong and Szymański produced a complete description of the primitive-ideal space of the C^*-algebra of a directed graph. This article details a slightly different approach, in the simpler context of row-finite graphs with no sources, obtaining an explicit description of the ideal lattice of a graph algebra.

1. INTRODUCTION

The purpose of this paper is to present a new exposition, in a somewhat simpler setting, of Hong and Szymański's description of the primitive-ideal space of a graph C^*-algebra. Their analysis [8] relates the primitive ideals of $C^*(E)$ to the maximal tails T of E—subsets of the vertex set satisfying three elementary combinatorial conditions (see page 3). In previous work with Bates and Raeburn, Hong and Szymański had already studied the primitive ideals of $C^*(E)$ that are invariant for its gauge action. Specifically, [2, Theorem 4.7] shows that the gauge-invariant primitive ideals of $C^*(E)$ come in two flavours: those indexed by maximal tails in which every cycle has an entrance; and those indexed by *breaking vertices*, which receive infinitely many edges in E, but only finitely many in the maximal tail that they generate. Hong and Szymański completed this list by showing in [8, Theorem 2.10] that the non-gauge-invariant primitive ideals are indexed by pairs consisting of a maximal tail containing a cycle with no entrance, and a complex number of modulus 1.

The bulk of the work in [8] then went into the description of the Jacobson, or hull-kernel, topology on $\operatorname{Prim} C^*(E)$ in terms of the indexing set described in the preceding paragraph. Theorem 3.4 of [8] describes the closure of a subset of $\operatorname{Prim} C^*(E)$ in terms of the combinatorial data of maximal tails and breaking vertices, and the usual topology on the circle $\mathbb{T}$. (Gabe [7] subsequently pointed out and corrected a mistake in [8, Theorem 3.4], but there is no discrepancy for row-finite graphs with no sources.) The technical details and notation involved even in the statement of this theorem are formidable, with the upshot that applying Hong and Szymański's result requires discussion of

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a fair amount of background and notation. This is due to some extent to the complications introduced by infinite receivers in the graph (to see this, compare [8, Theorem 3.4] with the corresponding statement [8, Corollary 3.5] for row-finite graphs). But it is also caused in part by the numerous cases involved in describing how the different flavours of primitive ideals described in the preceding paragraph relate to one another topologically.

Here we restrict attention to the class of row-finite graphs with no sources originally considered in [11, 10, 3]; it is a well-known principal that results tend to be cleaner in this context. The C^*-algebra of an arbitrary graph E is a full corner of the C^*-algebra of a row-finite graph E_{ds} with no sources, called a Drinen–Tomforde desingularisation E [6], so in principal our results combined with the Rieffel correspondence can be used to describe the primitive-ideal space and the ideal lattice of any graph C^*-algebra. But in practice there is serious book-keeping hidden in this innocuous-sounding statement.

We take a somewhat different approach than Hong and Szymański. We start, as they do, by identifying all the primitive ideals (Theorem 3.7)—though we take a slightly different route to the result. Our next step is to state precisely when a given primitive ideal in our list belongs to the closure of some other set of primitive ideals (Theorem 4.1). We could then describe the closure operation along the lines of Hong and Szymanski's result, but here our approach diverges from theirs. We describe a list of (not necessarily primitive) ideals $J_{H,U}$ of $C^*(E)$ indexed by *ideal pairs*, consisting of a saturated hereditary set H and an assignment U of a proper open subset of the circle to every cycle with no entrance in the complement of H. We describe each $J_{H,U}$ concretely by providing a family of generators. We prove that the map $(H,U) \mapsto J_{H,U}$ is a bijection between ideal pairs and ideals, and describe the inverse assignment (Theorem 5.1). Finally, in Theorem 6.1, we describe the containment relation and the intersection and join operations on primitive ideals in terms of a partial ordering and a meet and a join operation on ideal pairs.

One can recover the closure of a subset $X \subseteq \operatorname{Prim} C^*(E)$, and so Hong and Szymański's result, either by using the characterisation of points in $\overline{X}$ from Theorem 4.1, or by computing $\bigcap X$ using Theorem 6.1 and listing all the primitive ideals that contain this intersection. To aid in doing the latter, we single out the ideal pairs that correspond to primitive ideals (Remark 5.3), and identify when a given $J_{H,U}$ is contained in a given primitive ideal (Lemma 5.2).

We hope that this presentation of the ideal structure of $C^*(E)$ when E is row-finite with no sources will provide a useful and gentle introduction to Hong and Szymański's beautiful result for arbitrary graphs; and in particular that it will be helpful to readers familiar with the usual listing of gauge-invariant ideals using saturated hereditary sets.

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1.1. **Background.** We assume familiarity with Raeburn's monograph [13] and take most of our notation and conventions from there. We have made an effort not to assume any further background.

We deal with row-finite directed graphs E with no sources; these consist of countable sets E^0, E^1 and maps $r,s : E^1 \to E^0$ such that r is surjective and finite-to-one. A Cuntz–Krieger family consists of projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$ and $p_v = \sum_{r(e)=v} s_e s_e^*$. We will use the convention where, for example, for $v \in E^0$ the notation vE^1 means $\{e \in E^1 : r(e) = v\}$. A path of length $n > 0$ is a string $\mu = e_1 \dots e_n$ of edges where $s(e_i) = r(e_{i+1})$, and E^n denotes the collection of paths of length n. We write E^* for the collection of all finite paths (including the vertices, regarded as paths of length 0), and set $vE^* := \{\mu \in E^* : r(\mu) = v\}$, $E^*w := \{\mu \in E^* : s(\mu) = w\}$ and $vE^*w = vE^* \cap E^*w$ when $v,w \in E^0$.

2. Infinite paths and maximal tails

Our first order of business is to relate maximal tails in a graph with the shift-tail equivalence classes of infinite paths (see also [9]).

Recall that a *maximal tail* in E^0 is a set $T \subseteq E^0$ such that:

(T1) if $e \in E^1$ and $s(e) \in T$, then $r(e) \in T$;
(T2) if $v \in T$ then there is at least one $e \in vE^1$ such that $s(e) \in T$; and
(T3) if $v,w \in T$ then there exist $\mu \in vE^*$ and $\nu \in wE^*$ such that $s(\mu) = s(\nu) \in T$.

If T is a maximal tail, there is a subgraph ET of E with vertices T and edges $E^1T := \{e \in E^1 : s(e) \in T\}$.

An *infinite path* in E is a string $x = e_1 e_2 e_3 \cdots$ of edges such that $s(e_i) = r(e_{i+1})$ for all i. We let $r(x) := r(e_1)$. Two infinite paths x and y are shift-tail equivalent if there exist $m,n \in \mathbb{N}$ such that

$$x_{i+m} = y_{i+n} \quad \text{for all } i \in \mathbb{N}.$$

This shift-tail equivalence is (as the name suggests) an equivalence relation, and we write $[x]$ for the equivalence class of an infinite path x.

Shift-tail equivalence classes $[x]$ of infinite paths correspond naturally to irreducible representations of $C^*(E)$ (see Lemma 3.2). However, the corresponding primitive ideals depend not on $[x]$, but only on the maximal tail consisting of vertices that are the range of an infinite path in $[x]$. The next lemma describes the relationship between shift-tail equivalence classes of infinite paths and maximal tails.

Lemma 2.1. *Let E be a row-finite graph with no sources. A set $T \subseteq E^0$ is a maximal tail if and only if there exists $x \in E^\infty$ such that $T = [x]^0 := \{r(y) : y \in [x]\}$.*

Proof. First suppose that T is a maximal tail. List $T = (v_1, v_2, \dots)$. Set $\lambda_1 = \mu_1 = v_1 \in E^*$, and then inductively, having chosen $\mu_{i-1} \in v_{i-2}E^*$ and $\lambda_{i-1} \in v_{i-1}E^*$ with $s(\lambda_{i-1}) = s(\mu)$, use (T3) to find $\mu_i \in v_{i-1}E^*$ and $\lambda_i \in v_iE^*$ such that $s(\mu_i) = s(\lambda_i) \in T$. We obtain an infinite path $x = \mu_1\mu_2\mu_3 \cdots$. Since each $\lambda_i\mu_{i+1}\mu_{i+2}\cdots$ belongs to $[x]$, we have $T \subseteq [x]^0$. For the reverse containment, observe that if $v \in [x]^0$, then there exists $y \in [x]$ such that $v = r(y_1)$. By definition of $[x]$ there are m,i such that $s(y_m) = s(\mu_i)$. Since $\mu_i \in T$, m applications of (T1) show that $r(y_1) \in T$. □

We divide the maximal tails in E into two sorts. Those which have a cycle with no entrance, and those which don't. The main point is that, as pointed out in [8], if T contains a cycle without an entrance, then it contains just one of them, and is completely determined by this cycle.

A *cycle* in a graph E is a path $\mu = \mu_1 \dots \mu_n \in E^*$ such that $r(\mu_1) = s(\mu_n)$ and $s(\mu_i) \neq s(\mu_j)$ whenever $i \neq j$. Each cycle μ determines an infinite path $\mu^\infty := \mu\mu\mu\cdots$ and hence a maximal tail $T_\mu := [\mu^\infty]^0$; it is straightforward to check that

$$T_\mu = \{r(\lambda) : \lambda \in E^* r(\mu)\}.$$

Given a cycle $\mu \in E^*$ and a subset A of E^0 that contains $\{r(\mu_i) : i \leq |\mu|\}$, we say that *$\mu$ is a cycle with no entrance in A* if $\{e \in r(\mu_i)E^1 : s(e) \in A\} = \{\mu_i\}$ for each $1 \leq i \leq |\mu|$.

Lemma 2.2. *Let E be a row-finite graph with no sources. Suppose that $T \subseteq E^0$ is a maximal tail. Then either*

a) there is a cycle μ with no entrance in T such that $T = T_\mu$, and this μ is unique up to cyclic permutation of its edges; or

b) there is no cycle μ with no entrance in T.

Proof. Suppose that there is a cycle μ with no entrance in T. Lemma 2.1 implies that $T = [x]^0$ for some infinite path x. So there exists $y \in [x]$ such that $r(y) = r(\mu)$, and since shift-tail equivalence is an equivalence relation, we then have $T = [y]^0$. Since μ has no entrance in T, the only element of E^∞ lying entirely within T and with range $r(\mu)$ is μ^∞. So $y = \mu^\infty$, and $T = [\mu^\infty]^0 = T_\mu$.

If ν is another cycle with no entrance in $T = T_\mu$ then $r(\nu)E^* r(\mu) \neq \emptyset$, say $\lambda \in r(\nu)E^* r(\mu)$. Since ν has no entrance in T, we have $\lambda\mu = \nu_1^\infty \cdots \nu_k^\infty$ for some k. In particular $\nu_{k-|\mu|+1}^\infty \cdots \nu_k^\infty = \mu$, and we deduce that $\nu = \mu_i \cdots \mu_{|\mu|}\mu_1 \cdots \mu_{i-1}$, where $i \equiv k+1 \pmod{|\mu|}$. □

We call a maximal tail T satisfying (a) in Lemma 2.2 a *cyclic maximal tail* and write $\mathrm{Per}(T) := |\mu|$. We call a maximal tail T satisfying (b) in Lemma 2.2 a *aperiodic maximal tail*, and define $\mathrm{Per}(T) := 0$.

3. THE IRREDUCIBLE REPRESENTATIONS

In this section, we show that every primitive ideal of $C^*(E)$ naturally determines a corresponding maximal tail, and then construct a family of irreducible representations of $C^*(E)$ associated to each maximal tail of E.

The following lemma constructs a maximal tail from each primitive ideal of $C^*(E)$. It was proved for arbitrary graphs in [2, Lemma 4.1] using the relationship between ideals and saturated hereditary sets established there and that primitive ideals of separable C^*-algebras are prime. Here we present instead the direct representation-theoretic argument of [4, Theorem 5.3]. Recall that a saturated hereditary subset of E^0 is a subset whose complement satisfies axioms (T1) and (T2) of a maximal tail.

Lemma 3.1 ([2, Lemma 4.1])**.** *Let E be a row-finite graph with no sources. If I is a primitive ideal of $C^*(E)$, then $T := \{v \in E^0 : p_v \notin I\}$ is a maximal tail of E.*

Proof. The set of $v \in E^0$ such that $p_v \in I$ is a saturated hereditary set by [13, Lemma 4.5] (see also [3, Lemma 4.2]). So its complement T satisfies (T1) and (T2). To establish (T3), fix $v, w \in T$. Take an irreducible representation $\pi : C^*(E) \to \mathscr{B}(\mathscr{H})$ such that $\ker(\pi) = I$. Since $v \in T$, we have $p_v \notin I$, and so $\pi(p_v)\mathscr{H} \neq \{0\}$. Fix $\xi \in \pi(p_v)\mathscr{H}$ with $\|\xi\| = 1$. Since $p_w \notin I$, the space $\pi(p_w)\mathscr{H}$ is also a nontrivial subspace of $\mathscr{H}$. Since π is irreducible, ξ is cyclic for π, and so there exists $a \in C^*(E)$ such that $\pi(p_w)\pi(a)\xi = \pi(p_w a p_v)\xi$ is nonzero. In particular, we have $\pi(p_w a p_v) \neq 0$. Since $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu)\}$, and since $p_w s_\mu s_\nu^* p_v \neq 0$ only if $r(\mu) = w$ and $r(\nu) = v$, we have

$$\pi(p_w a p_v) \in \overline{\operatorname{span}}\{\pi(s_\mu s_\nu^*) : r(\mu) = w, r(\nu) = v, s(\mu) = s(\nu)\} \setminus \{0\}.$$

So there exist $\mu, \nu \in E$ with $r(\mu) = w$, $r(\nu) = v$, $s(\mu) = s(\nu)$, and $\pi(s_\mu p_{s(\mu)} s_\nu^*) = \pi(s_\mu s_\nu^*) \neq 0$. In particular, $\pi(p_{s(\mu)}) \neq 0$, giving $p_{s(\mu)} \notin I$. So $s(\mu) \in T$ satisfies $wE^*s(\mu), vE^*s(\mu) \neq \emptyset$. □

Next we show how to recover a family of primitive ideals from the shift-tail equivalence class of an infinite path.

Lemma 3.2. *Let E be a row-finite directed graph with no sources. For $x \in E^\infty$ and $z \in \mathbb{T}$, there is an irreducible representation $\pi_{x,z} : C^*(E) \to \mathscr{B}(\ell^2([x]))$ such that for all $y \in [x]$, $v \in E^0$ and $e \in E^1$, we have*

$$\pi_{x,z}(p_v)\delta_y = \begin{cases} \delta_y & \text{if } r(y_1) = v \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \pi_{x,z}(s_e)\delta_y = \begin{cases} z\delta_{ey} & \text{if } r(y_1) = s(e) \\ 0 & \text{otherwise.} \end{cases}$$

We have $\{v \in E^0 : p_v \notin \ker(\pi_{x,z})\} = [x]^0$.

Proof. It is easy to check that $\ell^2([x])$ is an invariant subspace of $\ell^2(E^\infty)$ for the infinite-path space representation of [13, Example 10.2] (with $k = 1$). So the infinite-path space representation reduces to a representation on $\mathscr{B}(\ell^2([x]))$. Precomposing with the gauge automorphism $\gamma_z : s_e \mapsto z s_e$ of [13, Proposition 2.1] yields a representation $\pi_{x,z}$ satisfying the desired formula.

To see that $\pi_{x,z}$ is irreducible, first observe that for each x, the rank-1 projection $\theta_{x,x}$ onto $\mathbb{C}\delta_x$ is equal to the strong limit

$$\theta_{x,x} = \lim_{n\to\infty} \pi_{x,z}(s_{x_1\cdots x_n} s_{x_1\cdots x_n}^*).$$

If $y, z \in [x]$, then $y = \mu w$ and $z = \nu w$ for some $\mu, \nu \in E^*$ and $w \in [x]$. Thus the rank-1 operator $\theta_{y,z}$ from $\mathbb{C}\delta_z$ to $\mathbb{C}\delta_y$ is in the strong closure of the image of $\pi_{x,z}$:

$$\theta_{y,z} = z^{|\nu|-|\mu|}\pi_{x,z}(s_\mu)\theta_{w,w}\pi_{x,z}(s_\nu^*) = \lim_{n\to\infty} \pi_{x,z}(z^{|\nu|-|\mu|} s_{\mu w_1\cdots w_n} s_{\nu w_1\cdots w_n}^*).$$

So $\mathscr{K}(\ell^2([x]))$ is contained in the strong closure of $\pi_{x,z}(C^*(E))$. Thus $\pi_{x,z}$ is irreducible.

If $v \notin [x]^0$, then $v \neq r(y_1)$ for any $y \in [x]$, and so the formula for $\pi_{x,z}$ shows that $p_v \in \ker(\pi_{x,z})$. On the other hand, if $v \in [x]^0$, then we can find $y \in [x]$ with $r(y_1) = v$, and then $\pi_{x,z}(p_v)\delta_y = \delta_y \neq 0$. □

Next we want to know when two of the irreducible representations constructed as in Lemma 3.2 have the same kernel. For the following, recall that if $H \subseteq E^0$ is a hereditary set (i.e., $E^0 \setminus H$ satisfies axiom (T1) of a maximal tail.), then $E \setminus EH$ is the subgraph of E with vertices $E^0 \setminus H$ and edges $E^1 \setminus E^1 H$. Note that if T is a maximal tail, then $H := E^0 \setminus T$ is a saturated hereditary set, and then $E \setminus EH = ET$.

Proposition 3.3. *Let E be a row-finite graph with no sources. Fix $x, y \in E^\infty$ and $w, z \in \mathbb{T}$. The irreducible representations $\pi_{x,w}$ and $\pi_{y,z}$ have the same kernel if and only if $[x]^0 = [y]^0$ and $w^{\operatorname{Per}([x]^0)} = z^{\operatorname{Per}([x]^0)}$.*

The crux of the proof of Proposition 3.3 is Lemma 3.5, which we state separately because it is needed again later to prove that every primitive ideal is of the form $I_{\pi,z}$. Our proof of Lemma 3.5 in turn relies on the following standard fact about kernels of irreducible representations; we thank the anonymous referee for suggesting the following elementary proof.

Lemma 3.4. *Let A be a C^*-algebra, let J be an ideal of A, and let π_1 and π_2 be irreducible representations of A that do not vanish on J. Then $\ker(\pi_1) = \ker(\pi_2)$ if and only if $\ker(\pi_1) \cap J = \ker(\pi_2) \cap J$.*

Proof. The "$\implies$" direction is obvious. Suppose that $\ker(\pi_1) \cap J = \ker(\pi_2) \cap J$. By symmetry, it suffices to show that $\ker(\pi_1) \subseteq \ker(\pi_2)$. Since π_2 is irreducible, $\ker(\pi_2)$ is primitive, and hence prime (see, for example, [12, Proposition 3.13.10]). By assumption, we have $\ker(\pi_1) \cap J = \ker(\pi_2) \cap J \subseteq \ker(\pi_2)$. Since π_2 does not vanish on J, we have $J \not\subseteq \ker(\pi_2)$. So primeness of $\ker(\pi_2)$ forces $\ker(\pi_1) \subseteq \ker(\pi_2)$. □

Lemma 3.5. *Let E be a row-finite graph with no sources, and suppose that T is a maximal tail of E. Let $H := E^0 \setminus T$.*

(1) Suppose that T is an aperiodic tail and π is an irreducible representation of $C^*(E)$ such that $\{v \in E^0 : \pi(p_v) \neq 0\} = T$. Then $\ker \pi$ is generated as an ideal by $\{p_v : v \in H\}$.

(2) Suppose that T is a cyclic tail and that μ is a cycle with no entrance in T. Suppose that π_1 and π_2 are irreducible representations of $C^*(E)$ such that

$$\{v : \pi_1(p_v) \neq 0\} = T = \{v : \pi_2(p_v) \neq 0\}.$$

Then each π_i restricts to a 1-dimensional representation of $C^*(s_\mu)$, and $\ker \pi_1 = \ker \pi_2$ if and only if $\pi_1(s_\mu) = \pi_2(s_\mu)$ as complex numbers. Each $\ker \pi_i$ is generated as an ideal by $\{p_v : v \in H\} \cup \{\pi_i(s_\mu) p_{r(\mu)} - s_\mu\}$.

Proof. We start with some setup that is needed for both statements. Let I be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. This H is a saturated hereditary set. If π is an irreducible representation such that $\{v \in E^0 : \pi(p_v) \neq 0\} = T$, then I is contained in $\ker \pi$ by definition. By [13, Remark 4.12], there is an isomorphism $C^*(E)/I \cong C^*(E \setminus EH)$ that carries $p_v + I$ to p_v for $v \in E^0 \setminus H$. Since $I \subseteq \ker \pi$, the representation π descends to an irreducible representation of $C^*(E)/I$, and hence determines a representation $\tilde{\pi}$ of

$C^*(E \setminus EH)$ such that
$$\tilde{\pi}(p_v) = \pi(p_v) \quad \text{for } v \in E^0 \setminus H.$$

Now, for (1), if T is an aperiodic maximal tail, and π is as above, then every cycle in $E \setminus H$ has an entrance in $E \setminus H$, and $\tilde{\pi}$ is a representation of $C^*(E \setminus EH)$ such that $\tilde{\pi}(p_v) \neq 0$ for all $v \in (E \setminus EH)^0$. So the Cuntz–Krieger uniqueness theorem [13, Theorem 2.4] implies that $\tilde{\pi}$ is faithful. Hence $\ker \pi = I$, proving (1).

For (2), consider the ideal J of $C^*(E \setminus EH)$ generated by $p_{r(\mu)}$. Then $\tilde{\pi}_i(J) \neq \{0\}$ for $i = 1,2$. So Lemma 3.4 implies that $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have the same kernel if and only if $\ker(\tilde{\pi}_1) \cap J = \ker(\tilde{\pi}_2) \cap J$. Since J is generated as an ideal by $p_{r(\mu)}$, the corner $p_{r(\mu)} J p_{r(\mu)} = \overline{\operatorname{span}}\{s_{\mu^n} s_{\mu^m}^* : m,n \in \mathbb{N}\}$ is full in J. Rieffel induction from a C^*-algebra to a full corner is implemented by restriction of representations [14, Proposition 3.24]. Since Rieffel induction carries irreducible representations to irreducible representations and induces a bijection between primitive-ideal spaces, we deduce that each $\tilde{\pi}_i$ is an irreducible representation of $C^*(s_\mu) \subseteq J$, and that
$$\ker \tilde{\pi}_1 = \ker \tilde{\pi}_2 \qquad \Longleftrightarrow \qquad \ker(\tilde{\pi}_1) \cap p_{r(\mu)} J p_{r(\mu)} = \ker(\tilde{\pi}_2) \cap p_{r(\mu)} J p_{r(\mu)}.$$
Since μ has no entrance, s_μ is a unitary element of $p_{r(\mu)} J p_{r(\mu)}$, so $C^*(s_\mu) \cong C(\sigma(s_\mu))$. Since the irreducible representations of a commutative C^*-algebra are 1-dimensional, we deduce that each $\tilde{\pi}_i$ is a 1-dimensional representation of $C^*(s_\mu) \subseteq C^*(E \setminus EH)$ and hence each π_i is a 1-dimensional representation of $C^*(s_\mu) \subseteq C^*(E)$. Moreover, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have the same kernel if and only if they are implemented by evaluation at the same point z in $\sigma(s_\mu)$, and hence if and only if $\pi_1(s_\mu) = \pi_2(s_\mu)$.

For the final statement fix $i \in \{1,2\}$. Since I is contained in the ideal J' generated by $\{p_v : v \in H\} \cup \{\pi_i(s_\mu) p_{r(\mu)} - s_\mu\}$, we have $J' = \ker \pi_i$ if and only if $\ker \tilde{\pi}_i$ is equal to the image J'' of J'/I in $C^*(E \setminus EH)$. Since Rieffel induction induces a bijection on ideal-spaces, $\ker \tilde{\pi}_i = J''$ if and only if $p_{r(\mu)} \ker \tilde{\pi}_i p_{r(\mu)} = p_{r(\mu)} J'' p_{r(\mu)}$. Both of these ideals coincide with the maximal ideal corresponding to the complex number $\pi_i(s_\mu) \in \sigma(s_\mu)$, so we are done. □

Proof of Proposition 3.3. The final statement of Lemma 3.2 implies that if $\ker \pi_{x,w} = \ker \pi_{y,z}$, then $[x]^0 = [y]^0$. So it suffices to prove that if $[x]^0 = [y]^0$, then
$$\ker \pi_{x,w} = \ker \pi_{y,z} \text{ if and only if } w^{\operatorname{Per}([x]^0)} = z^{\operatorname{Per}([x]^0)}. \tag{1}$$

For this we consider two cases. First suppose that $[x]^0$ is an aperiodic maximal tail. Then Lemma 3.5(1) implies that each of $\ker \pi_{x,w}$ and $\ker \pi_{y,z}$ is generated by $\{p_v : v \notin T\}$, and in particular the two are equal. Also, $w^{\operatorname{Per}([x]^0)} = w^0 = 1 = z^0 = z^{\operatorname{Per}([x]^0)}$, so the equivalence (1) holds.

Now suppose that $[x]^0$ is cyclic, and let μ be a cycle with no entrance in $[x]^0$. We must show that $\ker \tilde{\pi}_{x,w} = \ker \tilde{\pi}_{y,z}$ if and only if $w^{|\mu|} = z^{|\mu|}$. Since μ has no entrance, both $\pi_{x,w}(p_{r(\mu)})\ell^2([x])$ and $\pi_{y,z}(p_{r(\mu)})\ell^2([y])$ are equal to the 1-dimensional space $\mathbb{C}\delta_{\mu^\infty}$, and we have
$$\pi_{x,w}(s_\mu)\delta_{\mu^\infty} = w^{|\mu|}\delta_{\mu^\infty} \qquad \text{and} \qquad \pi_{y,z}(s_\mu)\delta_{\mu^\infty} = z^{|\mu|}\delta_{\mu^\infty}.$$

So, identifying the image of $\pi_{x,w}(C^*(s_\mu))$ with $\mathbb{C}$, we have $\pi_{x,w}(s_\mu) = w^{|\mu|}$ and similarly $\pi_{y,z}(s_\mu) = z^{|\mu|}$. So Lemma 3.5(2) shows that $\ker \pi_{x,w} = \ker \pi_{y,z}$ if and only if $z^{|\mu|} = w^{|\mu|}$. □

We are now ready to state and prove our first main result—a catalogue of the primitive ideals of $C^*(E)$. Proposition 3.3 says that the following definition makes sense.

Definition 3.6. Let E be a row-finite directed graph with no sources. Suppose that T is a maximal tail in E^0 and that $z \in \{w^{\operatorname{Per}(T)} : w \in \mathbb{T}\} \subseteq \mathbb{T}$. We define

$$I_{T,z} := \ker \pi_{x,w} \text{ for any } (x,w) \in E^\infty \times \mathbb{T} \text{ such that } [x]^0 = T \text{ and } w^{\operatorname{Per}(T)} = z.$$

Theorem 3.7. *The map $(T,z) \mapsto I_{T,z}$ is a bijection from*

$$\{(T, w^{\operatorname{Per}(T)}) : T \text{ is a maximal tail}, w \in \mathbb{T}\}$$

to $\operatorname{Prim} C^*(E)$.

Proof. Lemma 3.2 shows that each $I_{T,z}$ is a primitive ideal. Proposition 3.3 shows that $(T,z) \mapsto I_{T,z}$ is injective. So we just have to show that it is surjective. Fix a primitive ideal J of $C^*(E)$, let $T = \{v : p_v \notin J\}$, and let π be an irreducible representation of $C^*(E)$ with kernel J. Then T is a maximal tail according to Lemma 3.1. We must show that J has the form $I_{T,z}$.

If T is aperiodic, then Lemma 3.5(1) shows that $J = \ker \pi = \ker \pi_{x,1} = I_{[x]^0,1}$ for any x such that $[x]^0 = T$.

If T is cyclic, let μ be a cycle with no entrance in T. Lemma 3.5(2) shows that $\pi(C^*(s_\mu))$ is one-dimensional, so we can identify $\pi(s_\mu)$ with a nonzero complex number z. Since s_μ is an isometry, $|z| = 1$. Now Lemma 3.5(2) implies that any $w \in \mathbb{T}$ with $w^{|\mu|} = z$ satisfies $\ker \pi = \ker \pi_{[\mu^\infty]^0,w} = I_{[x]^0,z}$. □

4. The closure operation

The Jacobson, or hull-kernel, topology on $\operatorname{Prim} C^*(E)$ is the one determined by the closure operation $\overline{X} = \{I \in \operatorname{Prim} C^*(E) : \bigcap_{J \in X} J \subseteq I\}$. The ideals of $C^*(E)$ are in bijection with the closed subsets of $\operatorname{Prim} C^*(E)$: the ideal I_X corresponding to a closed subset X is

$$I_X := \bigcap_{J \in X} J.$$

So the first step in describing the ideals of $C^*(E)$ is to say when a primitive ideal I belongs to the closure of a set X of primitive ideals. We do so with the following theorem.

Theorem 4.1. *Let E be a row-finite graph with no sources. Let X be a set of pairs (T,z) consisting of a maximal tail T and an element z of $\{w^{\operatorname{Per}(T)} : w \in \mathbb{T}\}$. Consider another such pair (S,w). Then $\bigcap_{(T,z) \in X} I_{T,z} \subseteq I_{S,w}$ if and only if both of the following hold:*

a) $S \subseteq \bigcup_{(T,z) \in X} T$, and *b) if S is a cyclic tail and the cycle μ with no entrance in S also has no entrance in $\bigcup_{(T,z)\in X} T$, then*

$$w \in \overline{\{z : (S,z) \in X\}}.$$

We will need the following simple lemma in the proof Theorem 4.1, and at a number of other points later in the paper.

Lemma 4.2. *Let E be a row-finite graph with no sources, let H be a saturated hereditary subset of $C^*(E)$ and let μ be a cycle with no entrance in $E^0 \setminus H$. Let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. Then there is an isomorphism*

$$(p_{r(\mu)}C^*(E)p_{r(\mu)})/(p_{r(\mu)}I_H p_{r(\mu)}) \cong p_{r(\mu)}C^*(E \setminus EH)p_{r(\mu)}$$

carrying $s_\mu + p_{r(\mu)}I_H p_{r(\mu)}$ to s_μ, and there is an isomorphism of $p_{r(\mu)}C^*(E \setminus EH)p_{r(\mu)}$ onto $C(\mathbb{T})$ carrying s_μ to the generating monomial function $z \mapsto z$.

Proof. Remark 4.12 of [13] shows that there is an isomorphism $C^*(E)/I_H \cong C^*(E \setminus EH)$ that carries $s_e + I_H$ to s_e if $e \in E^1 \setminus E^1 H$ and to zero otherwise. This restricts to the desired isomorphism $p_{r(\mu)}C^*(E)p_{r(\mu)}/p_{r(\mu)}I_H p_{r(\mu)} \cong p_{r(\mu)}C^*(E \setminus EH)p_{r(\mu)}$. The element $s_\mu \in p_{r(\mu)}C^*(E \setminus EH)p_{r(\mu)}$ satisfies $s_\mu^* s_\mu = p_{r(\mu)} = s_\mu s_\mu^*$ because μ has no entrance in $E^0 \setminus H$. So it suffices to show that the spectrum of s_μ calculated in $p_{r(\mu)}C^*(E \setminus EH)p_{r(\mu)}$ is $\mathbb{T}$. To see this, observe that the gauge action γ satisfies $\gamma_w(s_\mu) = w^{|\mu|}(s_\mu)$. So for $\lambda, w \in \mathbb{T}$, $\lambda p_{r(\mu)} - s_\mu$ is invertible if and only if $\gamma_w(\lambda p_{r(\mu)} - s_\mu) = w^{|\mu|}(w^{-|\mu|}\lambda p_{r(\mu)} - s_\mu)$. That is, $\sigma(\mu)$ is invariant under rotation by elements of the form $w^{|\mu|}$, which is all of $\mathbb{T}$. Since the spectrum is nonempty, it follows that it is the whole circle. □

Proof of Theorem 4.1. We first prove the "if" direction. So suppose that (a) and (b) are satisfied. We consider two cases. First suppose that S is an aperiodic tail. Then $\mathrm{Per}(S) = \{0\}$, and so $w = 1$. For each maximal tail T of E, let

$$T_- := T \setminus \{v : v \text{ lies on a cycle with no entrance in } T\},$$

and let I_{T_-} be the ideal generated by $\{p_v : v \notin T_-\}$. If T is a cyclic maximal tail and μ is a cycle with no entrance in T, and if $z \in \{w^{\mathrm{Per}(T)} : w \in \mathbb{T}\}$, then Lemma 3.5(2) shows that $I_{T,z}$ is generated by $\{p_v : v \notin T\} \cup \{zp_{r(\mu)} - s_\mu\}$. So $I_{T,z} \subseteq I_{T_-}$. So it suffices to show that

$$\textstyle\bigcap_{(T,z)\in X} I_{T_-} \subseteq I_{S,1}.$$

For this it suffices to show that $\bigcup_{(T,z)\in X} T_- \supseteq S$. We fix $v \in E^0 \setminus \bigcup_{(T,z)\in X} T_-$ and show that $v \notin S$. If $v \notin T$ for all $(T,z) \in X$, then it follows from (a) that $v \notin S$. So we may assume that $v \in \big(\bigcup_{(T,z)\in X} T\big) \setminus \big(\bigcup_{(T,Z)\in X} T_-\big)$. In particular, there exist pairs $(T,z) \in X$ such that $v \in T$. Fix any such pair. Since $v \notin T_-$, it must lie in a cycle μ in T with no entrance in T. Property (T1) shows that μ is contained entirely in T, and then Lemma 2.2 then gives $T = [\mu^\infty]^0 = r(E^*v)$. So μ has no entrance in $r(E^*v)$, and the only pairs $(T,z) \in X$ with $v \in T$ satisfy $T = r(E^*v)$. Thus μ has no entrance in $\bigcup_{(T,z)\in X} T$. Since $S \subseteq \bigcup_{(T,z)\in X} T$, and every cycle in S has an entrance in S, we deduce that μ does not lie in S and hence $v \notin S$ as required.

Now suppose that S is cyclic and μ is a cycle with no entrance in S. Let V be the set of vertices on μ. Lemma 2.2 gives $S=\{r(\alpha):s(\alpha)\in V\}$. Since $S\subseteq\bigcup_{(T,z)\in X}T$, there exists $(T,z)\in X$ with $r(\mu)\in T$. Since T satisfies (T1), we deduce that the cycle μ lies in the subgraph ET of E. So there exists $(T,z)\in X$ such that $V\subseteq T$, and then $S\subseteq T$ because $S=\{r(\alpha):s(\alpha)\in V\}$ and T satisfies (T1). So it suffices to show that

$$\bigcap_{(T,z)\in X,S\subseteq T}I_{T,z}\subseteq I_{S,\omega}.$$

For this, first suppose that there exists $(T,z)\in X$ such that T is a proper superset of S; say $v\in T\setminus S$. Since $S=\{r(\alpha):s(\alpha)\in V\}$, we see that $vE^*V=\emptyset$, and hence $vE^*S=\emptyset$. So there exists $w\in T\setminus S$ such that VE^*w and vE^*w are both nonempty. Hence

$$T\supseteq\{r(\alpha):s(\alpha)=w\}\supseteq\{r(\alpha):s(\alpha)\in V\}=S.$$

If T is a cyclic tail, the cycle with no entrance that it contains lies outside of S, so the final statement of Lemma 3.5(2) shows that all the generators of $I_{T,z}$ belong to $I_{S,w}$; and if T is aperiodic, then all the generators of $I_{T,z}$ belong to $I_{S,w}$ by Lemma 3.5(1). In either case, we conclude that $I_{T,z}\subseteq I_{S,w}$, and hence $\bigcap_{(T,z)\in X,S\subseteq T}I_{T,z}\subseteq I_{S,\omega}$.

So it now suffices to show that $\bigcap_{z:(S,z)\in X}I_{S,z}\subseteq I_{S,w}$. Let I_S be the ideal generated by $\{p_v:v\notin S\}$. Then each $I_{S,z}$ contains I_S, as does $I_{S,w}$, so we need only show that in the quotient $C^*(E)/I_S\cong C^*(ES)$, the intersection of the images J_z of the $I_{S,z}$ is contained in J_w. Each J_z is generated by $zp_{r(\mu)}-s_\mu$ and is therefore contained in the ideal generated by $p_{r(\mu)}$, and similarly for J_w. Since the ideal generated by $p_{r(\mu)}$ is Morita equivalent to the corner determined by $p_{r(\mu)}$, it suffices to show that $\bigcap_{(S,z)\in X}p_{r(\mu)}J_zp_{r(\mu)}\subseteq p_{r(\mu)}J_wp_{r(\mu)}$. The isomorphism $p_{r(\mu)}C^*(ES)p_{r(\mu)}\cong C(\mathbb{T})$ of Lemma 4.2 carries each $p_{r(\mu)}J_zp_{r(\mu)}$ to $\{f\in C(\mathbb{T}):f(z)=0\}$. So $\bigcap_{(S,z)\in X}p_{r(\mu)}J_zp_{r(\mu)}$ is carried to $\big\{f\in C(\mathbb{T}):f\equiv 0\text{ on }\overline{\{z:(S,z)\in X\}}\big\}$, and in particular is contained in the image of $p_{r(\mu)}J_wp_{r(\mu)}$.

We now prove the "only if" direction. To do this, we prove the contrapositive. So we first suppose that (a) does not hold. Then there is some $v\in S\setminus\bigcup_{(T,z)}T$. This implies that $p_v\in I_{(T,z)}$ for all (T,z), but $p_v\notin I_{S,w}$, and so $\bigcap_{(T,z)}I_{T,zi}\not\subseteq I_{S,w}$ as required.

Now suppose that $S\subseteq\bigcup_{(T,z)}T$, that μ is a cycle with no entrance in S and that μ also has no entrance in $\bigcup_{(T,z)\in X}T$, and that $w\notin\overline{\{z:(S,z)\in X\}}$. As above, $S=\{r(\alpha):s(\alpha)=r(\mu)\}$, and since μ has no entrance in any T, for each (T,z) we have either $T=S$ or $r(\mu)\notin T$. Whenever $r(\mu)\notin T$, we have $p_{r(\mu)}\in I_{(T,z)}$, and so $\bigcap_{(T,z)}p_{r(\mu)}I_{T,z}p_{r(\mu)}=\bigcap_{(S,z)\in X}p_{r(\mu)}I_{S,z}p_{r(\mu)}$. Once again taking quotients by I_S, it suffices to show that

$$\bigcap_{(S,z)\in X}p_{r(\mu)}J_zp_{r(\mu)}\not\subseteq p_{r(\mu)}J_wp_{r(\mu)}.$$

Since $w\notin\overline{\{z:(S,z)\in X\}}$, there exists $f\in C(\mathbb{T})$ such that $f(w)=0$ and $f(z)=1$ whenever $(S,z)\in X$. Let $g=1-f\in C(\mathbb{T})$. Then the images of the elements f and g belong to $\bigcap_{(S,z)\in X}p_{r(\mu)}J_zp_{r(\mu)}$ and $p_{r(\mu)}J_wp_{r(\mu)}$ respectively. Their sum is the identity element

$p_{r(\mu)}$, which does not belong to J_w. Thus

$$p_{r(\mu)}J_w p_{r(\mu)} + \bigcap_{(S,z)\in X} p_{r(\mu)}J_z p_{r(\mu)} \neq J_w.$$

Consequently, $\bigcap_{(S,z)\in X} p_{r(\mu)}J_z p_{r(\mu)} \not\subseteq p_{r(\mu)}J_w p_{r(\mu)}$. □

5. The ideals of $C^*(E)$

We use Theorem 4.1 above to describe all the ideals of $C^*(E)$. We index them by what we call ideal pairs for E. To define these, given a saturated hereditary set H of E^0, we will write $\mathscr{C}(H)$ for the set

$$\mathscr{C}(H) := \{\mu : \mu \text{ is a cycle with no entrance in } E^0 \setminus H\}.$$

An *ideal pair* for E is then a pair (H,U) where H is a saturated hereditary set, and U is a function assigning to each $\mu \in \mathscr{C}(H)$ a proper open subset $U(\mu)$ of $\mathbb{T}$, with the property that $U(\mu) = U(\nu)$ whenever $[\mu^\infty] = [\nu^\infty]$.

Observe that if the maximal tail $E^0 \setminus H$ is aperiodic, so that $\mathscr{C}(H) = \emptyset$, then there is exactly one ideal pair of the form (H,U): the function U is the unique (trivial) function from the empty set to the collection of proper open subsets of $\mathbb{T}$.

To see how to obtain an ideal of $C^*(E)$ from an ideal pair, we need to do a little bit of background work.

For each open subset $U \subseteq \mathbb{T}$, we fix a function $h_U \in C(\mathbb{T})$ such that

$$\{z \in \mathbb{T} : h_U(z) \neq 0\} = U.$$

For example, we could take

$$h_U(z) := \inf\{|z-w| : w \notin U\}.$$

Let $\pi : C(\mathbb{T}) \to \ell^2(\mathbb{Z})$ be the faithful representation that carries the generating monomial $z \mapsto z$ to the bilateral shift operator $U : e_n \mapsto e_{n+1}$. The classical theory of Toeplitz operators says that if $P_+ : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ denotes the orthogonal projection onto the Hardy space $\overline{\operatorname{span}}\{e_n : n \geq 0\}$, then there is an isomorphism ρ from $P_+\pi(C(\mathbb{T}))P_+$ to the Toeplitz algebra $\mathscr{T} \subseteq \ell^2(\mathbb{N})$ generated by the unilateral shift operator S, such that if $q : \mathscr{T} \to C(\mathbb{T})$ is the quotient map that divides out the ideal of compact operators, then $q(\rho(P_+\pi(f)P_+)) = f$ for every $f \in C(\mathbb{T})$.

If $H \subseteq E^0$ is saturated and hereditary, then for each $\mu \in \mathscr{C}(H)$, we have $s_\mu s_\mu^* \leq p_{r(\mu)} = s_\mu^* s_\mu$, with equality precisely if μ has no entrance in E^0. So if μ has no entrance in E^0, then s_μ is unitary in $p_{r(\mu)}C^*(E)p_{r(\mu)}$, and we can apply the functional calculus in the corner to define a nonzero element $h_U(s_\mu) \in C^*(E)$. If μ has an entrance in E^0, then $s_\mu s_\mu^* < s_\mu^* s_\mu$, so Coburn's theorem [5] gives an isomorphism $\psi : \mathscr{T} \cong C^*(s_\mu)$ that carries S to s_μ.

Using the preceding paragraph, given an ideal pair (H,U) and given $\mu \in \mathscr{C}(H)$, we obtain an element $\tau_\mu^U \in C^*(s_\mu) \subseteq p_{r(\mu)}C^*(E)p_{r(\mu)}$ given by

$$\tau_\mu^U := \begin{cases} h_{U(\mu)}(s_\mu) & \text{if } \mu \text{ has no entrance in } E^0 \\ \psi(\rho(P_+\pi(h_{U(\mu)})P_+)) & \text{otherwise.} \end{cases}$$

Theorem 5.1. *Let E be a row-finite graph with no sources. Let $\mathscr{I}_E$ denote the set of all ideal pairs for E. For each $(H,U) \in \mathscr{I}_E$, let $J_{H,U}$ be the ideal of $C^*(E)$ generated by*

$$\{p_v : v \in H\} \cup \{\tau_\mu^U : \mu \in \mathscr{C}(H)\}.$$

(1) The map $(H,U) \mapsto J_{H,U}$ is a bijection of $\mathscr{I}_E$ onto the collection of all closed 2-sided ideals of $C^*(E)$.
(2) Given an ideal I of $C^*(E)$, let $H_I := \{v \in E^0 : p_v \in I\}$, and for $\mu \in \mathscr{C}(H_I)$, let $U_I(\mu) = \mathbb{T} \setminus \operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$. Then (H_I,U_I) is an ideal pair and $I = J_{H_I,U_I}$.

Before proving the theorem, we need the following lemma.

Lemma 5.2. *Let E be a row-finite directed graph with no sources. Let (H,U) be an ideal pair for E, let T be a maximal tail of E and take $z \in \{w^{\operatorname{Per}(T)} : w \in \mathbb{T}\}$. Then $J_{H,U} \subseteq I_{T,z}$ if and only if both of the following hold:*
a) $H \subseteq E^0 \setminus T$; and
b) if T is cyclic and the cycle μ with no entrance in T belongs to $\mathscr{C}(H)$, then $z \notin U(\mu)$.

In particular, we have $\{v : p_v \in J_{H,U}\} = H$.

Proof. For the "if" direction, fix $x \in E^\infty$ such that $T = [x]^0$ and $w \in \mathbb{T}$ such that $w^{\operatorname{Per}(T)} = z$. We just have to show that $\pi_{x,w}$ annihilates all the generators of $J_{H,U}$. For this, first fix $v \in H$. Then the final statement of Lemma 3.2 shows that $p_v \in \ker \pi_{x,w}$. Now fix $\mu \in \mathscr{C}(H)$. If $r(\mu) \notin T$, then $\pi_{x,w}(p_{r(\mu)}) = 0$ as above and then since $\tau_\mu^U \in p_{r(\mu)}C^*(E)p_{r(\mu)}$, it follows that $\pi_{x,w}(\tau_\mu^U) = 0$. So suppose that $r(\mu) \in T$. Since μ has no entrance in $E^0 \setminus H$ and since $T \subseteq E^0 \setminus H$, the cycle μ has no entrance in T. So T is a cyclic maximal tail, and $[x]^0 = [\mu^\infty]^0$ by Lemma 2.2. We then have $z \notin U(\mu)$ by hypothesis. The ideal I_H generated by $\{p_v : v \in H\}$ is contained in $\ker(\pi_{x,w})$, so $\pi_{x,w}$ descends to a representation $\tilde{\pi}_{x,w}$ of $C^*(E)/I_H$. Lemma 4.2 shows that $p_{r(\mu)}C^*(E)p_{r(\mu)}/p_{r(\mu)}Ip_{r(\mu)} \cong C(\mathbb{T})$, and this isomorphism carries the restriction of $\tilde{\pi}_{x,w}$ to the 1-dimensional representation ε_z given by evaluation at z. The isomorphism of Lemma 4.2 also carries $\tau_\mu^U + p_{r(\mu)}Ip_{r(\mu)}$ to $h_{U(\mu)}$. Since $z \notin U(\mu)$, we have $\varepsilon_z(h_{U(\mu)}) = 0$, and so $\pi_{x,w}(\tau_\mu^U) = 0$. So all of the generators of $J_{H,U}$ belong to $\ker \pi_{x,w}$ as required.

For the "only if" implication, we prove the contrapositive. Again fix $x \in E^\infty$ such that $T = [x]^0$ and $w \in \mathbb{T}$ such that $w^{\operatorname{Per}(T)} = z$, so that $I_{T,z} = \ker \pi_{x,w}$. First suppose that $H \not\subseteq E^0 \setminus T$; say $v \in T \cap H$. Then $p_v \in J_{H,U}$ by definition, but $p_v \notin \ker \pi_{x,w}$ by the final statement of Lemma 3.2, giving $J_{H,U} \not\subseteq \ker \pi_{x,w}$. Now suppose that $H \subseteq E^0 \setminus T$, that T

is cyclic and that the cycle μ with no entrance in T belongs to $\mathscr{C}(H)$, but that $z \in U(\mu)$. Arguing as in the preceding paragraph, we see that $\pi_{x,w}(h_{U(\mu)}(z)p_{r(\mu)} - \tau_\mu^U) = 0$. Since $\tau_\mu^U \in J_{H,U}$, we deduce that $p_{r(\mu)} \in J_{H,U} + \ker \pi_{x,w}$. Since $p_{r(\mu)} \notin \ker \pi_{x,w}$ by Lemma 3.2, we deduce that $J_{H,U} \not\subseteq \ker \pi_{x,w}$.

For the final statement, observe that $H \subseteq \{v : p_v \in J_{H,U}\}$ by definition. For the reverse containment, recall that by definition of an ideal pair, each $U(\mu)$ is a proper subset of $\mathbb{T}$. So for each $\mu \in \mathscr{C}(H)$, we can choose $z_\mu \in \mathbb{T} \setminus U(\mu)$. By the preceding paragraphs, we have $J_{H,U} \subseteq I_{[\mu]^0, z_\mu}$ for each $\mu \in \mathscr{C}(H)$. For each $v \in E^0 \setminus H$ that does not belong to $[\mu^\infty]^0$ for any $\mu \in \mathscr{C}(H)$, we can choose an infinite path x^v in $E^0 \setminus H$ with $r(x_1^v) = v$. This $x^v \notin [\mu^\infty]$ for $\mu \in \mathscr{C}(H)$ because v does not belong to any $[\mu^\infty]^0$. So each $[x^v]^0$ is a maximal tail contained in the complement of H and the preceding paragraphs show that $J_{H,U} \subseteq I_{[x^v]^0,1}$. We now have

$$J_{H,U} \subseteq \Big(\bigcap\nolimits_\mu I_{[\mu]^0, z_\mu}\Big) \cap \Big(\bigcap\nolimits_v I_{[x^v]^0,1}\Big).$$

By construction, the right-hand side does not contain p_v for any $v \notin H$, and so we deduce that $v \notin H$ implies $p_v \notin J_{H,U}$ as required. □

Proof of Theorem 5.1. To prove the theorem, it suffices to show that the assignment $(H,U) \mapsto J_{H,U}$ is injective, and then prove statement (2).

The general theory of C^*-algebras says that every ideal of a C^*-algebra A is equal to the intersection of all of the primitive ideals that contain it. By definition, the topology on $\operatorname{Prim}(A)$ is the weakest one in which $\{I \in \operatorname{Prim}(A) : J \subseteq I\}$ is closed for every ideal J of A, and the map which sends J to this closed subset of $\operatorname{Prim}(A)$ is a bijection. So to prove that $(H,U) \mapsto J_{H,U}$ is injective, we just have to show that the closed sets $Y_{H,U} := \{I \in \operatorname{Prim} C^*(E) : J_{H,U} \subseteq I\}$ are distinct for distinct pairs (H,U).

By Lemma 5.2, we have

$$\begin{aligned} Y_{H,U} = \{I_{T,z} : T \subseteq E^0 \setminus H \text{ is a maximal tail, and}\\ &\text{if } T \text{ is cyclic and the cycle } \mu \text{ with no entrance in } T\\ &\text{also has no entrance in } H\text{, then } z \notin U(\mu)\}. \end{aligned}$$

Suppose that (H_1,U_1) and (H_2,U_2) are distinct ideal pairs of E. We consider two cases. First suppose that $H_1 \neq H_2$. Without loss of generality, there exists $v \in H_1 \setminus H_2$. Since H_2 is saturated, there exists $e_1 \in vE^1$ such that $s(e_1) \notin H_2$. Since H_1 is hereditary, we have $s(e) \in H_1$. Repeating this argument we obtain edges $e_i \in s(e_{i-1})E^1$ with $s(e_i) \in H_1 \setminus H_2$, and hence an infinite path x lying in $(E \setminus EH_1) \setminus (E \setminus EH_2)$. Now $[x]^0$ is a maximal tail contained in $H_1 \setminus H_2$. If $[x]^0$ is an aperiodic tail or is a cyclic tail such that the cycle with no entrance in $[x]^0$ has an entrance in $E \setminus EH_2$, we set $z = 1$. If $[x]^0 = [\mu^\infty]^0$ for some $\mu \in \mathscr{C}(H_2)$, we choose any $z \in \mathbb{T} \setminus U_2(\mu)$. Then Lemma 5.2 shows that $I_{[x]^0,z} \in Y_{H_2,U_2} \setminus Y_{H_1,U_1}$.

Now suppose that $H_1 = H_2$. Then $U_1 \neq U_2$, so we can find $\mu \in \mathscr{C}(H_1) = \mathscr{C}(H_2)$ such that $U_1(\mu) \neq U_2(\mu)$. Again without loss of generality, we can assume that there exists

$z \in U_1(\mu) \setminus U_2(\mu)$, and then we have $I_{[\mu^\infty]^0,z} \in Y_{H_2,U_2} \setminus Y_{H_1,U_1}$. This completes the proof that the $Y_{H,U}$ are distinct.

It remains to prove (2). Given an ideal I, the set $H := H_I$ is a saturated hereditary set by [13, Lemma 4.5]. Since the ideal I_H generated by $\{p_v : v \in H\}$ is contained in I, Lemma 4.2 shows that $s_\mu + I$ is unitary in $(p_{r(\mu)} + I)C^*(E)/I(p_{r(\mu)} + I)$ for each $\mu \in C(H)$; so its spectrum is a closed subset of $\mathbb{T}$, showing that $U_I(\mu)$ is an open subset of $\mathbb{T}$. If $\mu, \nu \in C(H)$ with $[\mu^\infty] = [\nu^\infty]$, then $\mu^\infty = \alpha\nu^\infty$ for some initial segment α of μ^∞. The Cuntz–Krieger relations show that $s_\alpha^* s_\mu s_\alpha + I = s_\nu + I$ and $s_\alpha s_\nu s_\alpha^* + I = s_\mu + I$; so conjugation by $s_\alpha + I$ gives an isomorphism $C^*(s_\mu) + I \cong C^*(s_\nu) + I$, and in particular

$$\operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I) = \operatorname{spec}_{(p_{r(\nu)}+I)(C^*(E)/I)(p_{r(\nu)}+I)}(s_\nu + I),$$

giving $U_I(\mu) = U_I(\nu)$. So (H,U) is an ideal pair.

To see that $I = J_{H_I,U_I}$, we first check the containment $\supseteq$. For this, it suffices to show that every generator of J_{H_I,U_I} belongs to I. We have $p_v \in I$ for all $v \in H_I$ by definition. Fix $\mu \in \mathscr{C}(H_I)$; we must show that $\tau_\mu^{U_I} \in I$. For this, let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. Since I_H is contained in both I and J_{H_I,U_I} we just have to show that $J_{H_I,U_I}/I_H$ is contained in I/I_H. For this, let $\pi : p_{r(\mu)}C^*(E)p_{r(\mu)} \to C(\mathbb{T})$ be the composition of the isomorphism of Lemma 4.2 with the canonical surjection $p_{r(\mu)}C^*(E)p_{r(\mu)} \to (p_{r(\mu)} + I_H)(C^*(E)/I_H)(p_{r(\mu)} + I_H)$. Then $\pi(\tau_\mu^{U_I}) = h_{U_I(\mu)}$ vanishes on $\mathbb{T} \setminus U_I(\mu)$, which is $\operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$. Since the quotient map by the image of I under π is given by restriction of functions to $\operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$, it follows that $\tau_\mu^{U_I} + I_H \in I/I_H$ as required.

For the reverse containment, recall that every ideal of $C^*(E)$ is the intersection of the primitive ideals that contain it, so it suffices to show that if $I_{S,w} \in Y_{H_I,U_I}$, then $I \subseteq I_{S,w}$. Fix $I_{S,w} \in Y_{H_I,U_I}$. We can express I as an intersection of primitive ideals and therefore, by Theorem 3.7, we have $I = \bigcap_{(T,z)\in X} I_{T,z}$ for some set X of pairs consisting of a maximal tail T and an element $z \in \{u^{\operatorname{Per}(T)} : u \in \mathbb{T}\}$. We then have

$$v \in H_I \iff p_v \in I \iff p_v \in \textstyle\bigcap_{(T,z)\in X} I_{T,z} \iff v \in \bigcap_{(T,z)\in X} E^0 \setminus T,$$

and we deduce that $H_I = E^0 \setminus \bigcup_{(T,z)\in X} T$. Since $I_{S,w} \in Y_{H_I,U_I}$, we have $S \subseteq E^0 \setminus H_I = \bigcup_{(T,z)\in X} T$. So if S is an aperiodic tail, or is a cyclic tail such that the cycle μ with no entrance in S has an entrance in $\bigcup_{(T,z)\in X} T$, then Theorem 4.1 immediately gives $I = \bigcap_{(T,z)\in X} I_{T,z} \subseteq I_{S,w}$. So suppose that S is cyclic, and the cycle μ with no entrance in S has no entrance in $\bigcup_{(T,z)\in X} T$. Again using that $I_{S,w} \in Y_{H_I,U_I}$, we see that $w \notin U_I(\mu)$. Hence $w \in \operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$. So if $\pi : p_{r(\mu)}C^*(E)p_{r(\mu)} \to C(\mathbb{T})$ is the map described in the preceding paragraph, we have $f(w) = 0$ for all f in $\pi(I) = \bigcap_{(S,z)\in X} \pi(I_{S,z})$. Each $\pi(I_{S,z})$ is the set of functions that vanishes at z, so we deduce that every function vanishing at every z for which $(S,z) \in X$ also vanishes at w; that is $w \in \overline{\{z : (S,z) \in X\}}$. Now Theorem 4.1 again gives $I = \bigcap_{(T,z)\in X} I_{T,z} \subseteq I_{S,w}$. $\square$

Remark 5.3. To see where the primitive ideals of $C^*(E)$ fit into the catalogue of Theorem 5.1, first let us establish the convention that if $\mathscr{C}(H) = \emptyset$, then $\emptyset$ denotes the unique (trivial) function from $\mathscr{C}(H)$ to the collection of open subsets of $\mathbb{T}$, and that if $\mathscr{C}(H)$ is a singleton, then $\check{z}$ denotes the function on $\mathscr{C}(H)$ that assigns the value $\mathbb{T} \setminus \{z\}$ to the unique element of $\mathscr{C}(H)$. Now if T is a maximal tail and $z \in \{w^{\mathrm{Per}(T)} : w \in \mathbb{T}\}$, then Lemma 3.5 and the definition of the ideals $J_{H,U}$ show that

$$I_{T,z} = \begin{cases} J_{E^0\setminus T,\emptyset} & \text{if } T \text{ is aperiodic} \\ J_{E^0\setminus T,\check{z}} & \text{if } T \text{ is cyclic.} \end{cases}$$

Remark 5.4. The ideal $J_{H,U}$ is gauge invariant (i.e., $\gamma_z(J_{H,U}) = J_{H,U}$ for every $z \in \mathbb{T}$) if and only if $U(\mu) = \emptyset$ for every $\mu \in \mathscr{C}(H)$, in which case $J_{H,U} = I_H$. Thus, we recover from Theorem 5.1 the description of the gauge invariant ideals of $C^*(E)$ presented in [2, Theorem 4.1].

6. The lattice structure

To finish off the description of the lattice of ideals of $C^*(E)$, we describe the complete-lattice structure in terms of ideal pairs.

We define $\preceq$ on the set $\mathscr{I}_E$ of ideal pairs for a row-finite graph E with no sources by

$$(H_1,U_1) \preceq (H_2,U_2) \qquad \Longleftrightarrow \qquad H_1 \subseteq H_2 \text{ and } U_1(\mu) \subseteq U_2(\mu) \text{ for all } \mu \in \mathscr{C}(H_1) \cap \mathscr{C}(H_2).$$

In the following, given $X \subseteq \mathbb{T}$, we write $\mathrm{Int}(X)$ for the interior of X.

Theorem 6.1. *Let E be a row-finite graph with no sources.*

(1) Given ideal pairs (H_1,U_1) and (H_2,U_2) for E, we have $J_{H_1,U_1} \subseteq J_{H_2,U_2}$ if and only if $(H_1,U_1) \preceq (H_2,U_2)$.

(2) Given a set $K \subseteq \mathscr{I}_E$ of ideal pairs for E, we have $\bigcap_{(H,U)\in K} J_{H,U} = J_{H_K,U_K}$ where $H_K = \bigcap_{(H,U)\in K} H$, and $U_K(\mu) = \mathrm{Int}\big(\bigcap_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu)\big)$.

(3) Fix a set $K \subseteq \mathscr{I}_E$ of ideal pairs of E. Let A be the saturated hereditary closure of $\bigcup_{(H,U)\in K} H$. Let $B = \{r(\mu) : \mu \in \mathscr{C}(A)$ and $\bigcup_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu) = \mathbb{T}\}$. Let H^K be the saturated hereditary closure of $A \cup B$ in E^0, and for each $\mu \in \mathscr{C}(H^K)$, let $U^K(\mu) = \bigcup_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu)$. Then $\overline{\mathrm{span}}\big(\bigcup_{(H,U)\in K} J_{H,U}\big) = J_{H^K,U^K}$.

Proof. (1): First suppose that $(H_1,U_1) \preceq (H_2,U_2)$. We show that every generator of J_{H_1,U_1} belongs to J_{H_2,U_2}. For each $v \in H_1$ we have $v \in H_2$ and therefore $p_v \in J_{H_2,U_2}$. Suppose that $\mu \in \mathscr{C}(H_1)$. If $r(\mu) \in H_2$, then $p_{r(\mu)} \in J_{H_2,U_2}$ and so $\tau_\mu^{U_1} \in p_{r(\mu)}C^*(E)p_{r(\mu)}$ belongs to J_{H_2,U_2} as well. So we may suppose that $r(\mu) \notin H_2$. Since $H_1 \subseteq H_2$ and since μ has no entrance in $E^0 \setminus H_1$, it cannot have an entrance in $E^0 \setminus H_2$, so it belongs to $\mathscr{C}(H_2)$. The ideal I_{H_1} generated by $\{p_v : v \in H_1\}$ is contained in both J_{H_1,U_1} and J_{H_2,U_2}. By Lemma 4.2, we have $(p_{r(\mu)} + I_{H_1})(C^*(E)/I_{H_1})(p_{r(\mu)} + I_{H_1}) \cong C(\mathbb{T})$ and this isomorphism carries $\tau_\mu^{U_1}$ to $h_{U_1(\mu)}$ and carries the image of J_{H_2,U_2} to $\{f \in C(\mathbb{T}) : f^{-1}(\mathbb{C} \setminus \{0\}) \subseteq U_2(\mu)\}$. Since $U_1(\mu) \subseteq U_2(\mu)$, it follows that the image of $\tau_\mu^{U_1}$ in the corner $(p_{r(\mu)}+I_{H_1})(C^*(E)/I_{H_1})(p_{r(\mu)}+I_{H_1})$ belongs to the image of J_{H_2,U_2}, and therefore $\tau_\mu^{U_1}+I_{H_1}\subseteq J_{H_2,U_2}$, giving $\tau_\mu^{U_1}\in J_{H_2,U_2}$.

Now suppose that $J_{H_1,U_1}\subseteq J_{H_2,U_2}$. The final statement of Lemma 5.2 shows that $H_1\subseteq H_2$, so we must show that whenever $\mu\in\mathscr{C}(H_1)\cap\mathscr{C}(H_2)$, we have $U_1(\mu)\subseteq U_2(\mu)$. Theorem 5.1 (2) shows that

$$U_i(\mu)=\mathbb{T}\setminus\operatorname{spec}_{(p_{r(\mu)}+J_{H_i,U_i})(C^*(E)/J_{H_i,U_i})(p_{r(\mu)}+J_{H_i,U_i})}(s_\mu+J_{H_i,U_i}).$$

Since $J_{H_1,U_1}\subseteq J_{H_2,U_2}$, there is a homomorphism $q:C^*(E)/J_{H_1,U_1}\to C^*(E)/J_{H_2,U_2}$ that carries $s_\mu+J_{H_1,U_1}$ to $s_\mu+J_{H_2,U_2}$. In particular, q carries $p_{r(\mu)}+J_{H_1,U_1}$ to $p_{r(\mu)}+J_{H_2,U_2}$, and so induces a unital homomorphism between the corners determined by these projections. Since unital homomorphisms decrease spectra, we obtain

$$\begin{aligned}&\operatorname{spec}_{(p_{r(\mu)}+J_{H_2,U_2})(C^*(E)/J_{H_2,U_2})(p_{r(\mu)}+J_{H_2,U_2})}(s_\mu+J_{H_2,U_2})\\&\qquad\subseteq\operatorname{spec}_{(p_{r(\mu)}+J_{H_1,U_1})(C^*(E)/J_{H_1,U_1})(p_{r(\mu)}+J_{H_1,U_1})}(s_\mu+J_{H_1,U_1}),\end{aligned}$$

and hence $U_1(\mu)\subseteq U_2(\mu)$.

(2): The ideal $\bigcap_{(H,U)\in K}J_{H,U}$ is the largest ideal that is contained in $J_{H,U}$ for every (H,U) in K. Since the map $(H,U)\to J_{H,U}$ is a bijection carrying $\preceq$ to $\subseteq$, it suffices to show that $(H_K,U_K)\preceq(H,U)$ for all $(H,U)\in K$, and is maximal with respect to $\preceq$ amongst pairs (H'',U'') satisfying $(H'',U'')\preceq(H,U)$ for all $(H,U)\in K$. The pair (H_K,U_K) satisfies $(H_K,U_K)\preceq(H,U)$ for all $(H,U)\in K$ by definition of H_K and U_K. Suppose that $(H'',U'')\preceq(H,U)$. Then $H''\subseteq H$ for all $(H,U)\in K$, and hence $H''\subseteq H_K$; and if $\mu\in\mathscr{C}(H'')\cap\mathscr{C}(H_K)$, and if $(H,U)\in K$ satisfies $\mu\in\mathscr{C}(H)$, then $U''(\mu)\subseteq U(\mu)$ because $(H'',U'')\preceq(H,U)$. So $U''(\mu)$ is an open subset of $\bigcap_{(H,U)\in K,\mu\in\mathscr{C}(H)}U(\mu)$, and therefore belongs to $\operatorname{Int}\big(\bigcap_{(H,U)\in K,\mu\in\mathscr{C}(H)}U(\mu)\big)=U_K$.

(3): The ideal $\overline{\operatorname{span}}\big(\bigcup_{(H,U)\in K}J_{H,U}\big)$ is the smallest ideal containing $J_{H,U}$ for every (H,U) in K. So as above it suffices to show that $(H,U)\preceq(H^K,U^K)$ for all $(H,U)\in K$, and that (H^K,U^K) is minimal with respect to $\preceq$ amongst pairs (H'',U'') satisfying $(H,U)\preceq(H'',U'')$ for all $(H,U)\in K$. The pair (H^K,U^K) satisfies $(H,U)\preceq(H^K,U^K)$ for all $(H,U)\in K$ by construction. Suppose that (H'',U'') is another ideal pair satisfying $(H,U)\preceq(H'',U'')$ for all $(H,U)\in K$. We just have to show that $(H^K,U^K)\preceq(H'',U'')$. We have $H\subseteq H''$ for every $(H,U)\in K$, and since H'' is saturated and hereditary, it follows that $A\subseteq H''$. If $v\in B$, then there exists $\mu\in\mathscr{C}(A)$ such that $\bigcup_{(H,U)\in K,\mu\in\mathscr{C}(H)}U(\mu)=\mathbb{T}$, and then by compactness of $\mathbb{T}$, there are finitely many pairs $(H_1,U_1),\dots,(H_n,U_n)\in K$ such that $\mu\in\mathscr{C}(H_i)$ for each i, and $\bigcup_{i=1}^n U(\mu)=\mathbb{T}$. Choose a partition of unity $\{f_1,\dots,f_n\}\in C(\mathbb{T})$ subordinate to the U_i. Let I_A be the ideal of $C^*(E)$ generated by $\{p_v:v\in A\}$. Then each f_i belongs to the image of $p_{r(\mu)}J_{(H_i,U_i)}p_{r(\mu)}$ under the isomorphism of Lemma 4.2, and so $1=\sum_i f_i$ belongs to the image of $\sum_{i=1}^n p_{r(\mu)}J_{H_i,U_i}p_{r(\mu)}$. Since each $(H_i,U_i)\preceq(H'',K'')$, it follows that 1 belongs to the image of $J_{(H'',K'')}$. But the preimage of 1 is $p_{r(\mu)}+I_A$, and we deduce that $p_{r(\mu)}\in J_{(H'',K'')}$. The final statement of Lemma 5.2 therefore implies that $v\in H''$. So $A\cup B\subseteq H''$, and since H'' is saturated and hereditary, it follows that $H^K\subseteq H''$. Now suppose that $\mu \in \mathscr{C}(H^K) \cap \mathscr{C}(H'')$. For each $z \in U^K(\mu)$, there exists $(H,U) \in K$ such that $\mu \in \mathscr{C}(H)$ and $z \in U(\mu)$. Since $(H,U) \preceq (H'',U'')$ and $\mu \in \mathscr{C}(H'') \cap \mathscr{C}(H)$, we deduce that $z \in U''(\mu)$. So $U^K(\mu) \subseteq U''(\mu)$. So we have $(H^K,U^K) \preceq (H'',U'')$ as required. □

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