

# EQUILIBRIUM STATES AND GROWTH OF QUASI-LATTICE ORDERED MONOIDS

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ABSTRACT. Each multiplicative real-valued homomorphism on a quasi-lattice ordered monoid gives rise to a quasi-periodic dynamics on the associated Toeplitz  $C^*$ -algebra; here we study the KMS equilibrium states of the resulting  $C^*$ -dynamical system. We show that, under a nondegeneracy assumption on the homomorphism, there is a critical inverse temperature  $\beta_c$  such that at each inverse temperature  $\beta \geq \beta_c$  there exists a unique KMS state. Strictly above  $\beta_c$ , the KMS states are generalised Gibbs states with density operators determined by analytic extension to the upper half-plane of the unitaries implementing the dynamics. These are faithful Type I states. The critical value  $\beta_c$  is the largest real pole of the partition function of the system and is related to the clique polynomial and skew-growth function of the monoid, relative to the degree map given by the logarithm of the multiplicative homomorphism. Motivated by the study of equilibrium states, we give a proof of the inversion formula for the growth series of a quasi-lattice ordered monoid in terms of the clique polynomial as in recent work of Albenque–Nadeau and McMullen for the finitely generated case, and in terms of the skew-growth series as in recent work of Saito. Specifically, we show that  $e^{-\beta_c}$  is the smallest pole of the growth series and thus is the smallest positive real root of the clique polynomial. We use this to show that equilibrium states in the subcritical range can only occur at inverse temperatures that correspond to roots of the clique polynomial in the interval  $(e^{-\beta_c}, 1)$ , but we are not aware of any examples in which such roots exist.

## INTRODUCTION

When  $C^*$ -algebras are used to model systems in quantum statistical mechanics, the time evolution of the system is modelled as a continuous action of the additive group of real numbers. Equilibrium states of the system at inverse temperature  $\beta$  are modelled by states satisfying a twisted trace-like condition formalised by Haag, Hugenholtz and Winnink [HHW], and called the *KMS condition* in honour of Kubo, Martin and Schwinger. Even when a given action  $\alpha$  of the reals on a  $C^*$ -algebra  $A$  does not represent a physical system, the associated simplices of KMS states at various inverse temperatures turn out to be a very interesting invariant of the pair  $(A, \alpha)$ . For example, for the periodic gauge action on a Cuntz–Krieger algebra, they encode the topological entropy and Perron-Frobenius measure of the underlying shift of finite type [EFW]; and for the Bost–Connes system they famously encode the Riemann zeta function as partition function and exhibit a phase transition related to explicit class field theory [BC, CMR, LLN]. As a result there has been

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very significant interest in the KMS structure of  $C^*$ -dynamical systems in recent years, particularly those related to combinatorial and algebraic objects such as graphs [CaLa, T], group actions [LRRW] and semigroups [BaHLR, LR2].

This paper explores the theme arising in the work of Enomoto–Fujii–Watatani [EFW] on the relationship between KMS states on Cuntz–Krieger algebras and the entropy of the underlying shift space. The topological entropy can be regarded as measuring the asymptotic complexity of the shift space; it measures, roughly speaking, the growth rate, with respect to  $n$ , of the number of allowed blocks of length  $n$  in the shift space. Analogously, we might expect that for an appropriate definition of the  $C^*$ -algebra of a monoid, and for a dynamics on this  $C^*$ -algebra encoded by an appropriate length function on the monoid itself, the resulting KMS data should reflect the growth rate of the semigroup relative to the given presentation and length.

Here we focus our investigation on the positive cones of quasi-lattice ordered groups. Recall that  $(G, P)$  is quasi-lattice ordered if  $G$  is a group with a distinguished submonoid  $P$  such that in the left order, defined by  $x \leq y$  iff  $x^{-1}y \in P$ , any finite collection of elements with a common upper bound has a unique least common upper bound. The Toeplitz  $C^*$ -algebra  $C_\lambda^*(P)$  is generated by the translation operators  $L_p : \varepsilon_x \mapsto \varepsilon_{px}$  on  $\ell^2(P)$ . We also consider the universal  $C^*$ -algebra  $C^*(P)$  for representations of  $P$  by isometries that satisfy the Nica-covariance relations, [Ni, LR1]. A multiplicative map  $N : P \rightarrow (0, \infty)$  from the monoid to the positive reals determines a quasi-periodic time evolution on  $C_\lambda^*(P)$  given on generators by  $\alpha_t(L_p) = N(p)^{it}L_p$ , and there is also a corresponding time evolution in  $C^*(P)$ . We assume that this homomorphism satisfies conditions that ensure its logarithm is a degree map in the sense of [S3] and study the KMS states of the associated  $C^*$ -dynamical system, establishing a relationship between equilibrium temperatures and rate of growth of the monoid.

In Section 1 we describe the semigroups and  $C^*$ -dynamical systems  $(C_\lambda^*(P), \alpha^N)$  that we study throughout the paper. Following the strategy laid out in [LR2], we next establish, in Section 2, an algebraic characterisation of those states that are  $\text{KMS}_\beta$  states for a given  $\beta$ . In Section 3 we identify the partition function of the system as a Dirichlet series (3.1) whose abscissa of convergence is the critical inverse temperature  $\beta_c$ . For every inverse temperature  $\beta > \beta_c$  we construct a natural generalised Gibbs state and we show that it is the unique  $\text{KMS}_\beta$ -state and that its GNS representation is a faithful, Type  $\text{I}_\infty$  representation. The system also has a unique  $\text{KMS}_{\beta_c}$  state and a unique ground state, which is a  $\text{KMS}_\infty$  state whenever  $\beta_c < \infty$ . The critical value  $\beta_c$  can be zero, but only if  $P$  is lattice ordered. In Section 4, we investigate the relationship between KMS states and the growth properties of  $P$ . We show that the projection  $Q_e$  onto the vacuum vector  $\varepsilon_e$  can be expressed as an operator-valued Euler-type product, which expands to an operator-valued analogue of the clique polynomial [AN, McM] and of the skew-growth series [S3]. When we evaluate the  $\text{KMS}_\beta$  states for  $\beta > \beta_c$  on  $Q_e$  through this expansion, and use the KMS condition, we obtain the inversion formula for the growth series from [S3, Remark 5.5]. We finish by showing that, for finitely generated quasi-lattice orders, any subcritical KMS states must occur at roots of the clique polynomial and factor through the boundary quotient of [CL2]. We have not been able to determine whether any such sporadic states do occur. Indeed, Saito [S3] poses the question of whether there exists a finitely generated quasi-lattice ordered monoid and a weight function satisfying his hypotheses such that that the clique polynomial admits any roots between its smallest root and 1. Our results show that establishing the existence of a subcritical KMS state would answer this question in the affirmative.

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### 1. QUASI-PERIODIC DYNAMICS ON THE $C^*$ -ALGEBRAS OF QUASI-LATTICE ORDERS.

Let  $G$  be a group and suppose  $P \subseteq G$  is a submonoid with  $P \cap P^{-1} = \{e\}$ . The relation  $\leq$  on  $G$  defined by  $x \leq y$  if  $x^{-1}y \in P$  is a (left) translation-invariant partial order on  $G$ . Following Nica [Ni] we say that the partially ordered group  $(G, P)$  is *quasi-lattice ordered* if every finite subset  $F$  of  $G$  that has an upper bound has a least upper bound in  $G$ . The least upper bound is necessarily unique, and is denoted by  $\vee F$ , or simply  $x \vee y$  if  $F = \{x, y\}$ . For such  $(G, P)$  we will also say that  $\leq$  is a quasi-lattice order on  $G$  (or on  $P$ ), and we often and somewhat loosely refer to  $P$  as a quasi-lattice ordered monoid. We follow [CL2] in that the partial order is defined on all of  $G$  and we do not require  $x \vee y$  to be in  $P$ . When  $x$  and  $y$  do not have a common upper bound we extend the notation by saying  $x \vee y = \infty$ .

**Lemma 1.1.** *Let  $(G, P)$  be a quasi-lattice ordered group and suppose  $x, y, z \in G$ . Then  $x$  and  $y$  have a common upper bound iff  $zx$  and  $zy$  have a common upper bound, in which case  $zx \vee zy = z(x \vee y)$ .*

*Proof.* Suppose  $x$  and  $y$  have a common upper bound; then  $zx, zy \leq z(x \vee y)$ , so  $zx$  and  $zy$  have a common upper bound. The converse follows on switching from  $x, y, z$  to  $zx, zy, z^{-1}$ . To prove the equality, notice that  $zx, zy \leq zx \vee zy$  implies that  $x, y \leq z^{-1}(zx \vee zy)$ , so  $x \vee y \leq z^{-1}(zx \vee zy)$ . Thus we have  $zx, zy \leq z(x \vee y) \leq zx \vee zy$  and since  $zx \vee zy$  is the least common upper bound of  $zx$  and  $zy$  we must have  $z(x \vee y) = zx \vee zy$ .  $\square$

Let  $(G, P)$  be a quasi-lattice ordered group and let  $\{\varepsilon_x : x \in P\}$  be the canonical orthonormal basis for  $\ell^2(P)$ . For each  $p \in P$ , the map  $L_p : \varepsilon_x \mapsto \varepsilon_{px}$  extends to an isometry on  $\ell^2(P)$ . This gives a representation  $L : P \rightarrow \mathcal{B}(\ell^2(P))$  by isometries, called the *left regular representation*. The reduced  $C^*$ -algebra  $C_\lambda^*(P)$  is the  $C^*$ -algebra generated by the image of the left regular representation:

$$C_\lambda^*(P) := C^*(\{L_p : p \in P\}) \subseteq \mathcal{B}(\ell^2(P)).$$

A fundamental observation made by Nica in [Ni] is that, in addition to the obvious multiplicativity, the isometries in the left regular representation also satisfy the relations

$$L_p L_p^* L_q L_q^* = \begin{cases} L_{p \vee q} L_{p \vee q}^* & \text{if } p \vee q < \infty \\ 0 & \text{otherwise,} \end{cases} \quad (1.1)$$

which we refer to as Nica-covariance. As a notational convenience we will write  $L_\infty = 0$  so (1.1) becomes  $L_p L_p^* L_q L_q^* = L_{p \vee q} L_{p \vee q}^*$  for all  $p, q \in P$ .

This observation led Nica to define the (full)  $C^*$ -algebra of  $P$  to be the universal unital  $C^*$ -algebra  $C^*(P)$  with presentation

$$\langle \{v_p\}_{p \in P} : v_p^* v_p = 1, v_p v_q = v_{pq}, v_p v_p^* v_q v_q^* = v_{p \vee q} v_{p \vee q}^*, p, q \in P \rangle \quad (1.2)$$

where again  $v_\infty = 0$  by convention. Since the  $v_p$  are isometries, multiplying the relation  $v_{p \vee q} v_{p \vee q}^*$  on the left by  $v_p^*$  and on the right by  $v_q$  gives

$$v_p^* v_q = v_{p^{-1}(p \vee q)} v_{q^{-1}(p \vee q)}^* \quad \text{for all } p, q \in P. \quad (1.3)$$

This gives a Wick-ordering of products and implies that the set of products of the form  $v_p v_q^*$  is closed under multiplication and adjoints, hence

$$C^*(P) = \overline{\text{span}}\{v_p v_q^* : p, q \in P\}. \quad (1.4)$$

For the details of these constructions, in particular the existence of the universal  $C^*$ -algebra and its relationship with the reduced  $C^*$ -algebra of  $P$ , see [Ni, LR1]. For more on the theory of Toeplitz algebras of monoids, see also [CL1, CL2, Li1, Li2].

Every multiplicative map  $N : P \rightarrow (0, \infty)$  on a quasi-lattice ordered monoid gives rise to a time evolution on  $C^*(P)$  for which the generating isometries  $v_p$  are eigenvectors.

**Proposition 1.2.** *Let  $(G, P)$  be a quasi-lattice ordered group and suppose  $N : P \rightarrow (0, \infty)$  is a multiplicative map. Then there exists a strongly continuous one-parameter group  $\{\alpha_t\}_{t \in \mathbb{R}}$  of automorphisms of  $C^*(P)$  satisfying*

$$\alpha_t(v_p) = N(p)^{it} v_p \quad \text{for all } p \in P \text{ and } t \in \mathbb{R}.$$

*Proof.* For each  $t \in \mathbb{R}$ , the collection  $\{N(p)^{it} v_p : p \in P\}$  consists of isometries that also satisfy the defining relations in the presentation (1.2). So the universal property of  $C^*(P)$  gives homomorphisms  $\alpha_t : C^*(P) \rightarrow C^*(P)$  that satisfy  $\alpha_s \alpha_t = \alpha_{s+t}$  and have inverses  $\alpha_{-t}$ , so they are automorphisms. Continuity of the maps  $t \mapsto \sigma_t(a)$  for each  $a \in C^*(P)$  follows from a standard “ $\epsilon/3$  argument”.  $\square$

*Remark 1.3.* The automorphism group  $\alpha_t^N$  from Proposition 1.2 descends to a spatially implemented dynamics on  $C_\lambda^*(P)$ . Indeed, for each  $t \in \mathbb{R}$  the map  $U_t : \varepsilon_x \mapsto N(x)^{it} \varepsilon_x$  for  $x \in P$  extends to a unitary operator on  $\ell^2(P)$ . A routine argument shows that  $\{U_t : t \in \mathbb{R}\}$  is a strongly continuous one-parameter unitary group on  $\ell^2(P)$ , and the left regular representation  $L : P \rightarrow \mathcal{B}(\ell^2(P))$  satisfies

$$U_t L_p U_t^* = N(p)^{it} L_p \quad (p \in P) \quad (t \in \mathbb{R}).$$

We continue to write  $\alpha^N$  for this dynamics on  $C_\lambda^*(P)$ .

*Remark 1.4.* If the group  $G$  is abelian, the dynamics  $\alpha^N$  is a continuous one-parameter subgroup of the canonical dual action of the compact group  $\hat{G}$  on  $C^*(P)$ , and is quasi-periodic in the usual sense of converging averages, see e.g. [La2, Corollary 8]. In general, when  $G$  is nonabelian, there is a coaction of  $G$  on  $C^*(P)$ , see e.g. [LR1, Section 6], and thus also a corresponding quotient coaction of the (abelian) multiplicative subgroup of  $\mathbb{R}_+^*$  generated by  $N(P)$ .

## 2. CHARACTERISATION OF KMS STATES OF $C_\lambda^*(P)$

Let  $A$  be a  $C^*$ -algebra with a time evolution  $\alpha : \mathbb{R} \rightarrow \text{Aut}(A)$ . We say that  $a \in A$  is analytic for  $\alpha$  if  $t \mapsto \alpha_t(a)$  extends to an entire function  $z \mapsto \alpha_z(a)$  from  $\mathbb{C}$  to  $A$ . Let  $\beta \in \mathbb{R}$ . By definition [BR, Definition 5.3.1], a state  $\varphi$  of  $A$  is  $\text{KMS}_\beta$  for  $\alpha$  if

$$\varphi(ab) = \varphi(b\alpha_{i\beta}(a)) \quad (2.1)$$

for all  $a, b$  in some dense  $\alpha$ -invariant  $*$ -subalgebra of the analytic elements of  $A$ . When  $(A, \alpha)$  represents a quantum dynamical system, these are the equilibrium states of  $(A, \alpha)$  at inverse temperature  $\beta$ . If  $t \mapsto \alpha_t(a)$  and  $t \mapsto \alpha_t(b)$  have analytic extensions, then  $z \mapsto \alpha_z(a) + \lambda \alpha_z(b)$  is an analytic extension of  $t \mapsto \alpha_t(a + \lambda b)$ . So uniqueness of analytic extensions shows that the KMS condition passes from a given set of analytic elements to its linear span. Hence, to show that a state is  $\text{KMS}_\beta$ , it suffices to prove that (2.1) holds

for every  $a$  and  $b$  in an  $\alpha$ -invariant subset of analytic elements of  $A$  whose linear span is a dense  $*$ -subalgebra of  $A$ . The  $\text{KMS}_0$  states are defined to be the  $\alpha$ -invariant traces on  $A$ .

We are interested in computing  $\text{KMS}$  states of  $C^*(P)$  for the quasi-periodic dynamics associated to a multiplicative homomorphism  $N : P \rightarrow (0, \infty)$ . Since the function  $z \mapsto (N_p/N_q)^{iz} v_p v_q^*$  is a  $C^*$ -algebra valued entire function for every  $p, q \in P$ , the elements  $v_p v_q^*$  are analytic; and they span an  $\alpha$ -invariant dense  $*$ -subalgebra of  $C^*(P)$  by (1.4). Thus, we will carry out our computations on this spanning set.

We begin by showing that if we are interested in studying  $\text{KMS}$  states, then the range of  $N$  has to lie either entirely within  $[1, \infty)$  or entirely within  $(0, 1]$ .

**Proposition 2.1.** *Assume there exists a  $\text{KMS}_\beta$  state of  $(C^*(P), \alpha^N)$ .*

- (a) *if  $\beta > 0$ , then  $N(P) \subset [1, \infty)$  and*
- (b) *if  $\beta < 0$ , then  $N(P) \subset (0, 1]$ .*

*Proof.* Let  $\varphi$  be a  $\text{KMS}_\beta$  state and let  $p \in P$ . Since  $v_p$  is an isometry,

$$0 \leq \varphi(1 - v_p v_p^*) = 1 - \varphi(v_p v_p^*) = 1 - N(p)^{-\beta} \varphi(v_p^* v_p) = 1 - N(p)^{-\beta}.$$

Hence  $N(p)^{-\beta} \leq 1$ , from which (a) and (b) follow.  $\square$

Next we obtain a characterisation of  $\text{KMS}$  states of  $C^*(P)$  in terms of their values on the spanning elements. To include the case of infinitely generated  $P$  we need to assume that  $\inf\{N(p) : p \in P \setminus \{e\}\} > 1$ ; this is automatically satisfied when  $P$  is generated by a finite set  $\mathcal{S} \subset P \setminus \{e\}$  and  $N(s) > 1$  for every  $s \in \mathcal{S}$ .

**Proposition 2.2.** *Let  $(G, P)$  be a quasi-lattice ordered group, let  $N : P \rightarrow [1, \infty)$  be a multiplicative map such that  $\inf\{N(p) : p \in P \setminus \{e\}\} > 1$ , and let  $\alpha^N$  be the associated dynamics on  $C^*(P)$ . Suppose that  $\beta \in (0, \infty)$ . A state  $\varphi$  is a  $\text{KMS}_\beta$  state for  $\alpha^N$  if and only if for every  $p, q \in P$ ,*

$$\varphi(v_p v_q^*) = \begin{cases} N(p)^{-\beta} & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

*In particular, for each  $\beta > 0$  there is at most one  $\text{KMS}_\beta$ -state for  $\alpha^N$ .*

*Proof.* Suppose first  $\varphi$  is a  $\text{KMS}_\beta$  state for  $\alpha^N$  and let  $p, q \in P$ . When  $p = q$  the  $\text{KMS}$  condition implies  $\varphi(v_p v_q^*) = N(p)^{-\beta} \varphi(v_p^* v_p) = N(p)^{-\beta}$ . It remains to prove that  $\varphi(v_p v_q^*) = 0$  when  $p \neq q$ . Since  $p \neq q$ , we can write  $\{p, q\} = \{p_0, q_0\}$  with  $p_0 \neq e$ . Using first that

$$|\varphi(v_p v_q^*)| = |\overline{\varphi((v_p v_q^*)^*)}| = |\varphi(v_q v_p^*)|, \quad (2.3)$$

and then second the  $\text{KMS}$  condition, we see that

$$|\varphi(v_p v_q^*)| = |\varphi(v_{p_0} v_{q_0}^*)| = N(p_0)^{-\beta} |\varphi(v_{q_0}^* v_{p_0})|. \quad (2.4)$$

If  $q_0 \vee p_0 = \infty$ , then Nica-covariance gives  $v_{q_0}^* v_{p_0} = 0$  and we are done. If  $q_0 \vee p_0 < \infty$ , then Nica-covariance gives  $v_{q_0}^* v_{p_0} = v_{q_0^{-1}(p_0 \vee q_0)} v_{p_0^{-1}(p_0 \vee q_0)}^*$ . Since  $q_0 \neq p_0$ , cancellation in  $P$  ensures that  $q_0^{-1}(p_0 \vee q_0) \neq p_0^{-1}(p_0 \vee q_0)$ , so again we may write  $\{q_0^{-1}(p_0 \vee q_0), p_0^{-1}(p_0 \vee q_0)\} = \{p_1, q_1\}$  with  $p_1 \neq e$ . Again using (2.3) with  $(p_1, q_1)$  and the  $\text{KMS}_\beta$  condition on (2.4) yields

$$|\varphi(v_p v_q^*)| = N(p_0)^{-\beta} N(p_1)^{-\beta} |\varphi(v_{q_1}^* v_{p_1})|. \quad (2.5)$$

If  $q_1 \vee p_1 = \infty$ , then Nica-covariance gives  $v_{q_1}^* v_{p_1} = 0$  and we are done. If  $q_1 \vee p_1 < \infty$ , then we continue the process by induction. If at some stage  $p_i \vee q_i = \infty$ , we obtain  $|\varphi(v_p v_q^*)| = 0$

directly. Otherwise we generate an infinite sequence  $\{(p_n, q_n)\}$  of pairs of elements of  $P$  such that  $p_n \neq e$  and

$$|\varphi(v_p v_q^*)| = \left( \prod_{i=0}^n N(p_i)^{-\beta} \right) |\varphi(v_{p_{n+1}} v_{q_{n+1}}^*)|.$$

Since  $|\varphi(v_{p_{n+1}} v_{q_{n+1}}^*)| \leq 1$  and  $\prod_{i=0}^n N(p_i)^{-\beta} \leq (\inf_{p \in P \setminus \{e\}} N(p))^{-n\beta} \rightarrow 0$  this finishes the proof that  $\varphi(v_p v_q^*) = 0$  for  $p \neq q$ . Hence  $\varphi$  satisfies (2.2).

Next suppose that  $\varphi$  is a state on  $C^*(P)$  that satisfies (2.2) for every  $p, q \in P$ . In order to show that  $\varphi$  is a  $\text{KMS}_\beta$  state, it suffices to show that

$$\varphi(v_{p_1} v_{q_1}^* v_{p_2} v_{q_2}^*) = N(p_1)^{-\beta} N(q_1)^\beta \varphi(v_{p_2} v_{q_2}^* v_{p_1} v_{q_1}^*)$$

for every  $p_1, q_1, p_2, q_2 \in P$ . Fix  $p_i, q_i \in P$  for  $i = 1, 2$ . By Nica-covariance, when  $v_{p_2} v_{q_2}^* v_{p_1} v_{q_1}^* \neq 0$ , it can be written as  $v_x v_y^*$  for some  $x, y \in P$ . So our assumption on  $\varphi$  implies that the right-hand side vanishes unless  $v_{p_2} v_{q_2}^* v_{p_1} v_{q_1}^*$  is self-adjoint, and hence equal to  $v_{q_1} v_{p_1}^* v_{q_2} v_{p_2}^*$ . Thus, in order to conclude that  $\varphi$  is a  $\text{KMS}_\beta$  state, it suffices to prove that

$$N(p_1)^\beta \varphi(v_{p_1} v_{q_1}^* v_{p_2} v_{q_2}^*) = N(q_1)^\beta \varphi(v_{q_1} v_{p_1}^* v_{q_2} v_{p_2}^*). \quad (2.6)$$

The Nica-covariance relation for  $v_{q_1}^* v_{p_2}$  shows that

$$v_{p_1} v_{q_1}^* v_{p_2} v_{q_2}^* = \begin{cases} v_{p_1 q_1^{-1}(q_1 \vee p_2)} v_{q_2 p_2^{-1}(q_1 \vee p_2)}^* & \text{if } q_1 \vee p_2 \neq \infty \\ 0 & \text{otherwise,} \end{cases}$$

so (2.2) gives

$$\varphi(v_{p_1} v_{q_1}^* v_{p_2} v_{q_2}^*) = \begin{cases} N(p_1 q_1^{-1}(q_1 \vee p_2))^{-\beta} & \text{if } q_1 \vee p_2 \neq \infty \text{ and } p_1 q_1^{-1} = q_2 p_2^{-1} \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

Similarly, Nica-covariance for  $v_{p_1}^* v_{q_2}$  followed by (2.2) shows that

$$\varphi(v_{q_1} v_{p_1}^* v_{q_2} v_{p_2}^*) = \begin{cases} N(q_1 p_1^{-1}(p_1 \vee q_2))^{-\beta} & \text{if } p_1 \vee q_2 \neq \infty \text{ and } q_1 p_1^{-1} = p_2 q_2^{-1} \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

To see that the two cases in (2.7) match up with those in (2.8) first note that

$$p_1 q_1^{-1}(q_1 \vee p_2) = (p_1 \vee p_1 q_1^{-1} p_2) = p_1 \vee q_2 p_2^{-1} p_2 = p_1 \vee q_2,$$

where the first equality holds by Lemma 1.1 and the second one because  $p_1 q_1^{-1} = q_2 p_2^{-1}$ . This gives

$$q_1^{-1}(q_1 \vee p_2) = p_1^{-1}(p_1 \vee q_2).$$

Since  $N$  is a homomorphism, we then have

$$\begin{aligned} N(p_1)^\beta N(p_1 q_1^{-1}(q_1 \vee p_2))^{-\beta} &= N(q_1^{-1}(q_1 \vee p_2))^{-\beta} \\ &= N(p_1^{-1}(p_1 \vee q_2))^{-\beta} \\ &= N(q_1)^\beta N(q_1 p_1^{-1}(p_1 \vee q_2))^{-\beta}. \end{aligned}$$

Combining this with (2.7) and (2.8) gives (2.6). That there is at most one state satisfying (2.2) follows from (1.4) together with linearity and continuity of states.  $\square$

*Remark 2.3.* The same result, with the same proof, holds for the KMS states of  $C_\lambda^*(P)$ . From (2.2), it is clear what the values of a KMS state have to be on a dense subalgebra. The upshot of Proposition 2.2 is that in order to decide whether  $C^*(P)$  and  $C_\lambda^*(P)$  admit a KMS state at inverse temperature  $\beta$ , we need to decide whether formula (2.2) determines a positive linear functional on  $C^*(P)$ .

*Remark 2.4.* We point out that there exist quasi-lattice ordered groups that do not admit homomorphisms  $N$  with the properties assumed in Proposition 2.2. This stems from the fact that such homomorphisms must factor through the abelianisation of the monoid, and semidirect products provide easy examples without any such homomorphisms. For instance, any real valued homomorphism  $N$  of the affine monoid  $\mathbb{N} \rtimes \mathbb{N}^\times$  must be trivial on the additive part  $\mathbb{N}$ , so Proposition 2.2 cannot apply to  $C^*(\mathbb{N} \rtimes \mathbb{N}^\times)$ . In this case, if one makes the obvious choice  $N : \mathbb{N} \rtimes \mathbb{N}^\times \rightarrow [1, \infty)$  given by  $N(r, a) = a$ , then for each  $\beta$  above the critical inverse temperature, the extremal  $\text{KMS}_\beta$  states are indexed by  $\mathbb{T} = \hat{\mathbb{Z}}$ , [LR2, Theorem 7.1].

### 3. CRITICAL TEMPERATURE AND GENERALISED GIBBS STATES

Recall that the unitaries  $\{U_t\}_{t \in \mathbb{R}}$  determined by  $U_t \varepsilon_p = N(p)^{it} \varepsilon_p$  implement the dynamics  $\alpha^N$  spatially on  $C_\lambda^*(P)$  and are diagonal with respect to the standard orthonormal basis  $\{\varepsilon_x\}$  of  $\ell^2(P)$ . Let  $H$  denote the unbounded, diagonal self-adjoint operator on  $\ell^2(P)$  with eigenvalues  $w_p = \log N(p) \geq 0$  and corresponding eigenvectors  $\varepsilon_p$ . Applying the analytic functional calculus, we can write  $U_t = \exp(itH)$ .

We define  $\exp(-\beta H)$  to be the diagonal operator with eigenvalues  $N(p)^{-\beta}$  with respect to the standard basis. Then  $\beta \mapsto \exp(-\beta H)$  is a semigroup of contractions, which can be obtained as the restriction  $U_{i\beta}$  to the positive imaginary axis  $\beta \geq 0$  of the operator-valued analytic extension to the upper half plane of the unitary group  $t \mapsto U_t$ . We emphasise that the operator  $\exp(-\beta H)$  is a bona-fide contractive linear operator that can be defined directly and without any reference to the unbounded operator  $H$  because  $N(p)^{-\beta} \leq 1$  for every  $p \in P$ .

Evaluation of the usual trace on  $\mathcal{B}(\ell^2(P))$  at  $\exp(-\beta H)$  using the standard orthonormal basis gives an infinite sum of positive terms,

$$\text{Tr}(\exp(-\beta H)) = \sum_{p \in P} N(p)^{-\beta}, \quad (3.1)$$

which takes values in  $[0, \infty]$  and is decreasing as a function of  $\beta \in [0, \infty]$ . We define the *critical inverse temperature* to be the abscissa of convergence,

$$\beta_c := \inf\{\beta \in (0, \infty) : \text{Tr}(\exp(-\beta H)) < \infty\}, \quad (3.2)$$

with the usual convention that  $\inf \emptyset = \infty$ . We collect below the consequences of the above considerations that will be needed for our analysis of KMS states, and in particular the  $\text{KMS}_{\beta_c}$  states when  $\beta_c < \infty$ . Notice that  $\text{Tr}(\exp(-\beta H)) < \infty$  for every  $\beta > \beta_c$  and  $\sum_{p \in P} N(p)^{-\beta} = \infty$  for every  $\beta < \beta_c$ .

**Lemma 3.1.** *Let  $N : P \rightarrow [1, \infty)$  be a multiplicative morphism such that  $N(p) = 1$  only if  $p = e$  and consider the following statements.*

- (a)  $\beta_c < \infty$ ;
- (b)  $\{p \in P : N(p) \leq r\}$  is finite for every  $r \geq 1$ ;
- (c)  $\inf\{N(p) : p \in P, p \neq e\} > 1$ .

Then (a)  $\implies$  (b)  $\implies$  (c).

Suppose, in addition, that  $P$  is finitely generated. Let  $\mathcal{S}$  be a finite set of nontrivial generators and define  $\eta := \min\{N(s) : s \in \mathcal{S}\}$ . Then  $\beta_c \leq \frac{\log|\mathcal{S}|}{\log\eta} < \infty$ .

*Proof.* If condition (c) fails, then there exists a sequence  $p_n \in P$  such that  $1 < N(p_{n+1}) < N(p_n)$ , so condition (b) also fails. Suppose now that (b) fails and fix  $x > 1$  such that  $\{p \in P : N(p) \leq x\}$  is infinite. Then the series in (3.1) has infinitely many terms satisfying  $N(p)^{-\beta} \geq x^{-\beta}$ , hence diverges for every  $\beta \in (0, \infty)$  so condition (a) fails.

Assume now  $\mathcal{S} \subset P \setminus \{e\}$  is a finite set of generators for  $P$  and define  $\eta := \min\{N(s) : s \in \mathcal{S}\}$ . Each  $p \in P$  can be factored as  $p = s_1 s_2 \cdots s_n$  with  $s_k \in \mathcal{S}$ . Then  $N(p)^{-\beta} \leq \eta^{-\beta n}$ , giving

$$\sum_{p \in P} N(p)^{-\beta} \leq \sum_{x \in \mathbb{F}_{\mathcal{S}}^+} \eta^{-\beta|x|}.$$

Since the series on the right converges if and only if  $|\mathcal{S}|\eta^{-\beta} < 1$ , we have  $\beta_c \leq \frac{\log|\mathcal{S}|}{\log\eta}$ .  $\square$

*Remark 3.2.* Let  $\mathbb{F}_{\mathcal{S}}^+$  be the free monoid generated by a countable set  $\mathcal{S} = \{s_1, s_2, \dots\}$ .

- (1) The usual word length gives an example that satisfies (c) but not (b) in Lemma 3.1.
- (2) The homomorphism  $N : \mathbb{F}_{\mathcal{S}}^+ \rightarrow [1, \infty)$  determined by  $N(s_k) = \log(k+2)$  gives an example that satisfies (b) but not (a) in Lemma 3.1.

**Proposition 3.3.** *Suppose  $(G, P)$  is a quasi-lattice ordered group and let  $N : P \rightarrow [1, \infty)$  be a multiplicative map such that  $\{p \in P : N(p) \leq x\}$  is finite for every  $x \geq 1$ . Let  $(\lambda_n)_{n \in \mathbb{N}}$  be the strictly increasing listing of the elements of the set  $\{\log N(p) : p \in P\}$ . Then*

- (1)  $\exp(-H)$  is a positive compact operator on  $\ell^2(P)$  with decreasing eigenvalue list  $1 = e^{-\lambda_0} > \dots > e^{-\lambda_n} > e^{-\lambda_{n+1}} \dots$ , and finite multiplicities  $a_n := \#\{p \in P : N(p) = e^{\lambda_n}\}$ ;
- (2)  $\beta_c = \limsup_{n \rightarrow \infty} \frac{1}{\lambda_n} \log(\#\{p : N_p \leq e^{\lambda_n}\})$
- (3) if  $\beta_c < \infty$ , the Dirichlet series  $\sum_{n=0}^{\infty} a_n e^{-s\lambda_n}$  is absolutely convergent on the region  $\Re s > \beta_c$  to an analytic function  $s \mapsto \text{Tr}(\exp(-sH))$  for which  $s = \beta_c$  is a singular point.

*Proof.* Since  $N$  is multiplicative,  $N^{-1}(1)$  is a subsemigroup of  $P$ , and it is finite by assumption. Since  $P$  has no torsion, it follows that  $N^{-1}(1) = \{e\}$ . So  $a_0 = 1$ .

The set  $\{\log N(p) : p \in P\}$  has finite intersection with any bounded segment  $[0, \lambda]$ , so its elements can be listed in increasing order, yielding a sequence  $(\lambda_n) \nearrow \infty$ . Part (1) then follows from the spectral theorem for self-adjoint compact operators. Since  $\#\{p : N_p \leq e^{\lambda_n}\} = \sum_{j=0}^n a_j$ , parts (2) and (3) follow from [HR, Theorems 8 and 10].  $\square$

**Definition 3.4.** In situations where  $\beta_c < \infty$ , we define the (*generalised*) *Gibbs state at inverse temperature*  $\beta > \beta_c$  to be the state on  $\mathcal{B}(\ell^2(P))$  defined by

$$\psi_{\beta}(X) := \text{Tr}(\exp(-\beta H))^{-1} \text{Tr}(X \exp(-\beta H)) \quad X \in \mathcal{B}(\ell^2(P)).$$

This gives a state on  $C_{\lambda}^*(P)$ . Let  $\lambda : C^*(P) \rightarrow C_{\lambda}^*(P)$  be the left-regular representation  $\lambda(v_p) = L_p$ . Then  $\psi_{\beta} \circ \lambda$  is a state on  $C^*(P)$ , which we again call the generalised Gibbs state on  $C^*(P)$  at inverse temperature  $\beta$ .

For the next result, recall that a *ground state* for  $\alpha$  is a state  $\phi$  such that the entire function

$$z \mapsto \phi(y\alpha_z(x)), \quad z \in \mathbb{C}, \quad (3.3)$$



is bounded on the upper half plane for every  $x, y \in C_\lambda^*(P)$  with  $x$  analytic.

**Theorem 3.5.** *Let  $(G, P)$  be a quasi-lattice ordered group and suppose  $N : P \rightarrow [1, \infty)$  is a multiplicative map such that  $N(p) = 1$  only if  $p = e$ , let  $\alpha^N$  be the associated dynamics on  $C_\lambda^*(P)$ , and assume  $0 < \beta_c < \infty$ . Then*

- (1) *for each  $\beta > \beta_c$ , the Gibbs state  $\psi_\beta$  is the unique  $\text{KMS}_\beta$  state of the dynamical system  $(C_\lambda^*(P), \alpha^N)$  and its GNS representation is a faithful, type  $I_\infty$  representation of  $C_\lambda^*(P)$ ;*
- (2) *if  $Q_e$  denotes the projection onto the vacuum subspace  $\mathbb{C}\varepsilon_e$  of  $\ell^2(P)$ , then*

$$\text{Tr}(\exp(-\beta H)) \psi_\beta(Q_e) = 1 \quad (\beta > \beta_c); \quad \text{and}$$

- (3) *there is a unique  $\text{KMS}_{\beta_c}$  state for  $(C_\lambda^*(P), \alpha^N)$ .*

Whether or not  $\beta_c < \infty$ , the Fock state  $\phi_e(\cdot) = \langle \pi_L(\cdot)\varepsilon_e, \varepsilon_e \rangle$  corresponding to the vacuum vector  $\varepsilon_e$  is the unique ground state for  $(C_\lambda^*(P), \alpha^N)$ . This ground state is a  $\text{KMS}_\infty$  state if  $\beta_c < \infty$ .

*Proof.* Assume first  $\beta > \beta_c$ , so that  $Z_N(\beta) := \text{Tr}(\exp(-\beta H)) < \infty$ . By definition of  $\psi_\beta$ , we have

$$\psi_\beta(v_p v_q^*) := \frac{1}{Z_N(\beta)} \sum_{x \in P} N(x)^{-\beta} \langle L_p L_q^* \varepsilon_x, \varepsilon_x \rangle.$$

Since

$$\langle L_p L_q^* \varepsilon_x, \varepsilon_x \rangle = \langle L_q^* \varepsilon_x, L_p \varepsilon_x \rangle = \begin{cases} 1 & \text{if } p = q \leq x \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

we have,

$$\begin{aligned} \psi_\beta(v_p v_q^*) &= \frac{1}{Z_N(\beta)} \sum_{x \in P} N(x)^{-\beta} \langle L_q^* \varepsilon_x, L_p \varepsilon_x \rangle \\ &= \delta_{p,q} \frac{1}{Z_N(\beta)} \sum_{w \in P} N(pw)^{-\beta} = \delta_{p,q} N(p)^{-\beta}. \end{aligned} \tag{3.4}$$

Hence  $\psi_\beta$  satisfies (2.2), and is therefore the unique  $\text{KMS}_\beta$  state by Proposition 2.2. It factors through  $C_\lambda^*(P)$  by construction.

Let  $L$  be the left regular representation of  $P$  on  $\ell^2(P)$ . Then

$$Q_e := \prod_{p \in P \setminus \{e\}} (1 - L_p L_p^*) \tag{3.5}$$

is the rank-one projection onto the basis vector in  $\ell^2(P)$  corresponding to the identity  $e \in P$ . The infinite product (3.5) is the limit of a decreasing net of projections indexed by the finite subsets of  $P \setminus \{e\}$  and is in the von Neumann algebra  $C_\lambda^*(P)''$ . For  $p, q \in P$  the rank-one operator  $\langle \cdot, \varepsilon_q \rangle \varepsilon_p$  is the product  $L_p Q_e L_q^*$ , which is also in  $C_\lambda^*(P)''$ . Thus the compact operators are contained in  $C_\lambda^*(P)''$  and the left regular representation  $\pi_L$  is irreducible.

The assertion about faithfulness and type holds because the range of the density operator defining  $\psi_\beta$  is generating for  $\pi_L$ , so the GNS representation of  $\psi_\beta$  is quasi-equivalent to  $\pi_L$ , see e.g. [La1, Lemma 3.2].

A computation analogous to (3.4) shows that  $\psi_\beta(L_p Q_e L_p^*) = N_p^{-\beta} \psi_\beta(Q_e)$  and since the series  $\sum_p L_p Q_e L_p^*$  converges strongly and monotonically to the identity and  $\psi_\beta$  is a normal state on  $\mathcal{B}(\ell^2(P))$ , we have

$$1 = \psi_\beta(1) = \sum_p \psi_\beta(L_p Q_e L_p^*) = \sum_p N_p^{-\beta} \psi_\beta(Q_e) = Z_N(\beta) \psi_\beta(Q_e).$$

To see that a  $\text{KMS}_\beta$  state exists at  $\beta = \beta_c < \infty$ , recall that the states of the unital  $C^*$ -algebra  $C^*(P)$  form a weak\* compact set so there exists a weak\* convergent sequence  $\psi_{\beta_n}$  with  $\beta_n \rightarrow \beta_c^+$ , which by [BR, Proposition 5.3.25] converges to a  $\text{KMS}_{\beta_c}$  state  $\psi_{\beta_c}$ . The uniqueness follows from Proposition 2.2.

To prove the final two assertions, suppose that  $\phi$  is a ground state. We have

$$\phi(L_p \alpha_{i\beta}(L_q^*)) = N_q^\beta \phi(L_p L_q^*),$$

so  $\phi$  vanishes on  $L_p L_q^*$  if  $q \neq e$ , and hence (by taking adjoints) also if  $p \neq e$ . Thus  $\phi$  is the Fock state. Conversely, direct computation shows that the Fock state is a ground state. If  $\beta_c < \infty$ , we may take limits as  $\beta \rightarrow \infty$  in (3.4), which shows that the Fock state is a  $\text{KMS}_\infty$  state.  $\square$

*Remark 3.6.* What happens below the critical temperature, when it is strictly positive, is somewhat mysterious. But we will see later in Proposition 4.5 that for most values  $\beta < \beta_c$  there are no  $\text{KMS}_\beta$  states.

The case  $\beta_c = 0$  requires special consideration. Recall first that, by definition, a  $\text{KMS}_0$  state (or *chaotic equilibrium state*) is an  $\alpha^N$ -invariant trace.

**Proposition 3.7.** *Let  $(G, P)$  be a quasi-lattice ordered group and suppose  $N : P \rightarrow [1, \infty)$  is a multiplicative map such that  $N(p) = 1$  only if  $p = e$ . Consider the following conditions:*

- (1)  $\beta_c = 0$ ;
- (2) the  $C^*$ -algebra  $C^*(P)$  has a  $\text{KMS}_0$  state that vanishes at  $v_p v_q^*$  for  $p \neq q$ ;
- (3) the  $C^*$ -algebra  $C^*(P)$  has a tracial state;
- (4)  $P$  is lattice ordered.

Then (1)  $\implies$  (2)  $\iff$  (3)  $\iff$  (4)

*Proof.* Suppose first that  $\beta_c = 0$ . Applying Theorem 3.5(1) to each  $\beta \in \{1/n : n = 1, 2, \dots\}$  gives a sequence  $\psi_n$  of  $\text{KMS}_{1/n}$  states that all vanish on  $v_p v_q^*$  for  $p \neq q$ . These are automatically  $\alpha^N$ -invariant because each  $1/n > 0$ . Taking the limit of any weak\*-convergent subsequence of the  $\psi_n$  gives (2) by [BR, Proposition 5.3.25]. This proves (1) implies (2). That (2) implies (3) is straightforward.

Suppose now  $\tau$  is a tracial state on  $C^*(P)$  and let  $p, q \in P$ . Then

$$\tau(v_p v_p^* + v_q v_q^*) = \tau(v_p v_p^*) + \tau(v_q v_q^*) = \tau(v_p^* v_p) + \tau(v_q^* v_q) = 2,$$

so  $v_p v_p^* + v_q v_q^*$  cannot be a projection, which means that  $v_p v_p^*$  and  $v_q v_q^*$  are not orthogonal. Hence  $p \vee q < \infty$  for every  $p, q \in P$  and  $P$  is lattice ordered. This proves (3) implies (4).

To prove (4) implies (2), assume  $P$  is lattice ordered. Then the restriction to  $P$  of any unitary representation of  $G$  trivially satisfies Nica-covariance. So the universal property of  $C^*(P)$  gives a homomorphism  $C^*(P) \rightarrow C^*(G)$  that carries generators to generators. Post-composing this map with the canonical trace on  $C^*(G)$  yields a trace on  $C^*(P)$ .  $\square$

*Remark 3.8.* In the proof of Proposition 3.7, we cannot conclude that  $C^*(P)$  admits a unique  $\alpha^N$ -invariant trace because Proposition 2.2 does not apply at  $\beta = 0$ . To see what goes wrong, consider  $P = \mathbb{N}^2$  with its usual order and  $N : \mathbb{N}^2 \rightarrow \mathbb{R}$  given by  $N(n) = n_1 + n_2$ . This yields the dynamics on  $C^*(\mathbb{Z}^2)$  given by  $\alpha_t(U_m) = e^{it(m_1+m_2)}$ , which fixes the subalgebra  $\overline{\text{span}}\{U_{(j,-j)} : j \in \mathbb{Z}\} \cong C(\mathbb{T})$ . So every probability measure on  $\mathbb{T}$  determines an  $\alpha$ -invariant trace on  $C^*(P)$ .

*Remark 3.9.* For each integer  $n \geq 2$  the free monoid  $P = \mathbb{F}_n^+$  on  $n$  generators gives rise to  $C^*(\mathbb{F}_n^+) \cong \mathcal{TO}_n$ , the Toeplitz extension of the Cuntz algebra  $\mathcal{O}_n$ . The dynamics corresponding to the choice  $N_s = 1$  for each generator  $s$  of  $\mathbb{F}_n^+$  is the usual periodic gauge action. In this case we have  $\beta_c = \log n$  [OP, E]. Consideration of the free monoid on infinitely many generators leads to  $\mathcal{TO}_\infty$ , which has no  $\text{KMS}_\beta$  states at any finite  $\beta$ . On the other hand, taking, for example,  $N(i) = 2^i$  for each generator  $i$  of  $\mathbb{F}_\infty^+$  gives a dynamics on  $\mathcal{TO}_\infty$  that admits  $\text{KMS}_\beta$  states at many finite inverse temperatures.

#### 4. GROWTH RATE AND INVERSION FORMULA

The partition function  $Z_N(\beta) := \text{Tr}(\exp(-\beta H)) = \sum_{p \in P} N_p^{-\beta}$  of the system  $(C_\lambda^*(P), \alpha^N)$  and in particular its abscissa of convergence are intrinsically related to the growth of  $P$  relative to  $N$ . Indeed,  $\beta \mapsto Z_N(\beta) = \text{Tr}(\exp(-\beta H))$  is the growth series for  $P$  relative to the weight  $w = \log N$  evaluated at  $t = e^{-\beta}$ , [McM, AN, S1, S2, S3]. We explore next this interesting connection of the classification of equilibrium states and recent work on monoid growth.

Recall from the proof of Theorem 3.5 that  $Q_e = \prod_{p \in P} (1 - L_p L_p^*)$ , where the infinite product is viewed as the monotone limit of a decreasing net of projections indexed by the finite subsets of  $F$  of  $P \setminus \{e\}$ . We explore the relationship between  $Q_e$  and the skew growth series and the clique polynomial of  $P$ , cf. [AN, McM, S3]. Recall that a *clique* in a quasi-lattice ordered monoid  $P$  is a finite subset  $F$  that has a least upper bound  $\vee F$  in  $P$ . The motivating example is that of a right angled Artin monoid, generated by the vertices of a simplicial graph  $\Gamma$ , in which a pair of generators commutes if and only if they are joined by an edge. In this case the cliques are the subsets of generators that correspond to finite full subgraphs of  $\Gamma$ . By convention, we shall admit the empty set as a clique and write  $\vee \emptyset = e$ . For any subset  $A \subseteq P$ , we define  $\text{cl}(A) := \{F \subseteq A : F \text{ is a clique}\}$ .

Let  $P$  be a quasi-lattice ordered monoid with a homomorphism  $N : P \rightarrow [1, \infty)$  as in Theorem 3.5. Then  $w = \log N$  is a degree map as defined in [S3, Section 4] and  $P$  satisfies the descending chain condition. It follows that the set  $\mathcal{S}$  of minimal elements in  $P$  generates  $P$ , in fact  $\mathcal{S}$  is the smallest generating subset. When  $\mathcal{S}$  is finite, the *clique polynomial* is

$$C_{\mathcal{S},w}(t) := \sum_{F \in \text{cl}(\mathcal{S})} (-1)^{|F|} t^{w(\vee F)},$$

see [CF, AN, McM]. The *skew-growth series* defined by Saito in [S3] provides a vast generalization for cancelative monoids endowed with a degree map, and consists of alternating sums over towers. Since in the present situation  $P$  has the least upper bound property, Saito's towers have height at most 1, and are indexed by cliques themselves, see [S3, Example 2]. Thus, when the set  $\mathcal{S}$  of minimal elements is infinite, the skew-growth function  $N_{P,\text{deg}}$ , relative to the degree map  $\text{deg} = w$ , is given by the same formula, now interpreted

as a formal infinite *clique series*, which for  $t = e^{-\beta}$  becomes

$$C_{\mathcal{S},w}(e^{-\beta}) = \sum_{F \in \text{cl}(\mathcal{S})} (-1)^{|F|} e^{-\beta w(\vee F)}. \quad (4.1)$$

We show that the vacuum projection  $Q_e$  has an expression as an operator-valued product over generators that is analogous to (the inverse of) the familiar Euler product over the prime numbers. This product has an obvious expansion for finite  $\mathcal{S}$ , namely

$$\sum_{F \in \text{cl}(\mathcal{S})} (-1)^{|F|} L_{\vee F} L_{\vee F}^*, \quad (4.2)$$

which is a finite sum and belongs to  $C_\lambda^*(P)$ . When  $\mathcal{S}$  is infinite, (4.2) is an infinite sum, which we interpret as the strong operator limit of the decreasing net  $\{\sum_{F \in \text{cl}(\mathcal{F})} (-1)^{|F|} L_{\vee F} L_{\vee F}^*\}_{\mathcal{F}}$ , indexed by the finite subsets  $\mathcal{F}$  of  $\mathcal{S}$  directed by inclusion, and is an element of  $C_\lambda^*(P)''$ .

**Proposition 4.1.** *Suppose  $(G, P)$  is a quasi-lattice ordered group, and let  $\mathcal{S}$  be the set of minimal elements of  $P \setminus \{e\}$ . Then*

$$Q_e = \prod_{s \in \mathcal{S}} (1 - L_s L_s^*) = \sum_{F \in \text{cl}(\mathcal{S})} (-1)^{|F|} L_{\vee F} L_{\vee F}^*. \quad (4.3)$$

*Proof.* Since  $\mathcal{S} \subset P \setminus \{e\}$  it is clear that  $Q_e \leq \prod_{s \in \mathcal{S}} (1 - L_s L_s^*)$ . For every  $p \in P \setminus \{e\}$  there exists  $s \in \mathcal{S}$  such that  $p = sp'$  for some  $p' \in P$ , so  $L_p L_p^* = L_s L_{p'} L_{p'}^* L_s^* \leq L_s L_s^*$ , and thus  $L_p L_p^* (1 - L_s L_s^*) = 0$ . Hence  $(1 - L_p L_p^*) (1 - L_s L_s^*) = 1 - L_s L_s^*$ , from which we see that  $Q_e \geq \prod_{s \in \mathcal{S}} (1 - L_s L_s^*)$ . This proves that  $Q_e = \prod_{s \in \mathcal{S}} (1 - L_s L_s^*)$ , as a strong limit when  $\mathcal{S}$  is infinite.

Suppose now that  $\mathcal{F}$  is a finite subset of  $\mathcal{S}$ . Then

$$\prod_{s \in \mathcal{F}} (1 - L_s L_s^*) = \sum_{F \subset \mathcal{F}} (-1)^{|F|} \prod_{s \in F} L_s L_s^*.$$

By Nica-covariance,  $\prod_{s \in F} L_s L_s^*$  is equal to  $L_{\vee F} L_{\vee F}^*$  or to 0, according to whether  $F$  is a clique or not. Since  $\prod_{s \in \mathcal{S}} (1 - L_s L_s^*) = \lim_{\mathcal{F} \nearrow \mathcal{S}} \prod_{s \in \mathcal{F}} (1 - L_s L_s^*)$ , this finishes the proof.  $\square$

**Corollary 4.2.** *If  $\beta > \beta_c$ , then we have the inversion formula*

$$\left( \sum_{p \in P} N_p^{-\beta} \right) \left( \sum_{F \in \text{cl}(\mathcal{S})} (-1)^{|F|} N_{\vee F}^{-\beta} \right) = 1.$$

*Proof.* For each  $\beta > \beta_c$ , the generalized Gibbs state  $\psi_\beta$  on  $\mathcal{B}(\ell^2(P))$  is normal in the left regular representation, and since the right hand side of (4.3) is the strong limit of a bounded net indexed by finite sets, we obtain

$$\psi_\beta(Q_e) = \sum_{F \in \text{cl}(\mathcal{S})} (-1)^{|F|} N_{\vee F}^{-\beta},$$

where the right hand side is the limit of sums over cliques of finite subsets of  $\mathcal{S}$ , and is a conditionally convergent numerical series. The claim now follows from Theorem 3.5(2).  $\square$

*Remark 4.3.* The convergence of the clique series in Corollary 4.2 is conditional and depends on our convention of adding in stages over the cliques of finite subsets of  $\mathcal{S}$ . One could list  $\mathcal{S}$ , for instance so that  $w(s_n)$  is nondecreasing, and obtain a proper series of partial sums given by initial segments of this list, but convergence would remain conditional. In general, the terms cannot be rearranged to write the clique series with  $t = e^{-\beta}$  as a Dirichlet series,

but the inversion formula itself shows that the limit function has a meromorphic extension to the half plane  $\Re(s) > \beta_c$ , with poles at the zeros of  $Z_N(s)$ .

*Remark 4.4.* If the clique series converges absolutely on a region  $\Re s > \sigma_0$  for some  $\sigma_0 < \infty$ , then Corollary 4.2 gives the analytic inversion formula (\*\*\*) in [S3, Remark 5.5],

$$\sum_{p \in P} e^{-sw(p)} C_{S,w}(e^{-s}) = 1, \quad \Re(s) > \max\{\beta_c, \sigma_0\}$$

in the particular case of  $P$  quasi-lattice ordered with degree function  $w = \log N$  and minimal set  $\mathcal{S}$ .

For finitely generated  $P$  the skew growth series is the clique polynomial, and we have the following result.

**Proposition 4.5** (cf. Theorem 4.2 of [McM] and Example 1 of [S3]). *Resume the hypotheses of Theorem 3.5, and suppose further that  $P$  is finitely generated. Then*

- (1)  $e^{-\beta_c}$  is the smallest root of the clique polynomial  $C_{S,w}(t)$  and the  $\text{KMS}_{\beta_c}$  state  $\varphi_{\beta_c}$  vanishes on the compact operators  $\mathcal{K}(\ell^2(P))$ ;
- (2) if  $\varphi$  is a  $\text{KMS}_\beta$  state for some  $\beta < \beta_c$ , then  $e^{-\beta}$  is a root of  $C_{S,w}$  and  $\varphi$  vanishes on the compact operators  $\mathcal{K}(\ell^2(P))$ .

*Proof.* If  $C_{S,w}(e^{-\beta_c}) \neq 0$ , then the inversion formula of Corollary 4.2 shows that  $Z_N(s)$  has an analytic extension to a neighborhood of  $\beta_c$ , contradicting the last assertion of Proposition 3.3. This shows that  $Z_N(s)$  has a pole at  $\beta_c$ .

For each  $p \in P$ , we know  $V_p Q_e V_p^* = \theta_{p,p} \in \mathcal{K}(\ell^2(P))$ , so  $I - \sum_{p \in F} V_p Q_e V_p^* \geq 0$  for each finite subset  $F \subseteq P$ . If  $\varphi$  is a  $\text{KMS}_\beta$  state, then

$$1 \geq \sum_{p \in F} \varphi(V_p Q_e V_p^*) = \sum_{p \in F} e^{-\beta w(p)} \varphi(Q_e) = \sum_{p \in F} e^{-\beta w(p)} C_{S,w}(e^{-\beta}). \quad (4.4)$$

For  $\beta < \beta_c$ , the sum  $\sum_{p \in P} e^{-\beta w(p)}$  diverges, and hence (4.4) implies that  $e^{-\beta}$  is a root of the polynomial  $C_{S,w}(t)$ . Hence there are no  $\text{KMS}_\beta$  states for any  $\beta < \beta_c$  such that  $C_{S,w}(e^{-\beta}) \neq 0$ .

Now suppose that  $e^{-\beta}$  is a root of the clique polynomial and that  $\varphi$  is a  $\text{KMS}_\beta$  state for  $\alpha^N$ . Then  $\varphi(Q_e) = C_{S,w}(e^{-\beta}) = 0$ . Since  $Q_e \perp L_p L_p^*$  for all  $p \neq e$ , (1.4) shows that the ideal  $\langle Q_e \rangle$  generated by  $Q_e$  is equal to  $\overline{\text{span}}\{V_p Q_e V_q^* : p, q \in P\}$ . For  $p, q \in P$ , the KMS condition and the Cauchy–Schwartz inequality show that

$$|\varphi(V_p Q_e V_q^*)|^2 = e^{-2\beta w(p)} |\varphi(V_q^* V_p Q_e)|^2 \leq e^{-2\beta w(p)} |\varphi(V_q^* V_p V_p^* V_q) \varphi(Q_e)| = 0,$$

so  $\varphi$  vanishes on the ideal  $\langle Q_e \rangle$ . An elementary calculation shows that for  $p, q \in P$  the product  $L_p Q_e L_q^*$  is the rank-1 operator  $\theta_{p,q}$ , so  $\langle Q_e \rangle = \mathcal{K}(\ell^2(P))$ .

Since  $Z(s)$  has no poles on  $\Re(s) > \beta_c$ , the function  $C_{S,w}(e^{-s})$  has no zeros there, hence no root of  $C_{S,w}$  can have an absolute value smaller than  $e^{-\beta_c}$ .  $\square$

*Remark 4.6.* We expect a similar result for infinitely generated quasi-lattice monoids, with the skew growth function in place of the clique polynomial. However, the situation here is more delicate: such a result would depend on analytic or meromorphic continuation of the skew growth function up to the abscissa of convergence of  $Z(\beta)$ .

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