PURELY INFINITE SIMPLE C^*-ALGEBRAS THAT ARE PRINCIPAL GROUPOID C^*-ALGEBRAS

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ABSTRACT. From a suitable groupoid G, we show how to construct an amenable principal groupoid whose C^*-algebra is a Kirchberg algebra which is KK-equivalent to $C^*(G)$. Using this construction, we show by example that many UCT Kirchberg algebras can be realised as the C^*-algebras of amenable principal groupoids.

1. INTRODUCTION

Concrete models of Kirchberg algebras have proved extremely useful in classification theory—see for example [9, 30, 31]. In this paper, we develop a technique for realising many Kirchberg algebras as the C^*-algebras of amenable principal groupoids.

A priori, it is not clear that any Kirchberg algebra should admit such a model. Algebraically, a principal groupoid is just an equivalence relation; so in spirit at least, the C^*-algebras of amenable principal groupoids are akin to matrix algebras. Also, most of the existing groupoid models for Kirchberg algebras are based on graphs and their analogues [13, 30, 31, 33], and are not principal. But recent work of the third-named author with Rørdam shows that there are indeed examples of amenable principal groupoids whose C^*-algebras are Kirchberg algebras: [29, Theorem 6.11] shows that every nonamenable exact group admits a free and amenable action on the Cantor set for which the associated crossed-product is a Kirchberg algebra. Because the action is free, the corresponding transformation group groupoid is principal, and because the action is amenable, the corresponding groupoid is also amenable. Subsequent work of the third-named author with Elliott [9] shows that more than one Kirchberg algebra is achievable via this construction, but a complete range result has yet to be established. Suzuki also shows in [32] that many Kirchberg algebras can be constructed as crossed products arising from Cantor minimal systems.

Here we investigate a range of Kirchberg algebras that can be modelled using amenable principal groupoids. Our approach is as follows: We start out with a principal groupoid G such that $C^*(G)$ is simple and the K-theory of $C^*(G)$ is known. Then, given an appropriate automorphism α of G, we construct a *twisted product groupoid* G_α^∞ and prove that G_α^∞ is principal and amenable when G is, and that $C^*(G_\alpha^\infty)$ is a Kirchberg algebra. We then show that $C^*(G)$ embeds into $C^*(G_\alpha^\infty)$ via a homomorphism that induces a KK-equivalence, and in particular induces an isomorphism in K-theory.

By applying the Kirchberg–Phillips classification theorem, we therefore reduce the question of which Kirchberg algebras have amenable principal groupoid models to the question of which pairs of K-groups can be obtained from simple C^*-algebras of amenable principal groupoids G that admit a suitable automorphism α (see Theorem 5.5). By showing that every simple AF algebra, and every simple A$\mathbb{T}$-algebra arising from a rank-2 Bratteli diagram as in [18], admits such a model, we show that a very broad range of Kirchberg algebras can be modelled by amenable principal groupoids. In fact, all of these examples can be modelled by amenable principal groupoids that are ample; that is, étale groupoids that have a basis of compact open bisections.

We outline the necessary background in Section 2. In Section 3, we show how to form a twisted product $H \times_{c,\alpha} G$ of groupoids G and H given a continuous cocycle $c : H \to \mathbb{Z}$ and an automorphism α of G. We then investigate how properties of G and H relate to properties of $H \times_{c,\alpha} G$. Most of our results are about what happens when H is the standard groupoid model H_∞ for the Cuntz algebra $\mathcal{O}_\infty$ and $c : H_\infty \to \mathbb{Z}$ is the canonical cocycle. In this case we write G_α^∞ for $H_\infty \times_{c,\alpha} G$.

In Section 4, borrowing an idea from [27], we show that $C^*(G_\alpha^\infty)$ is isomorphic to the Toeplitz algebra $\mathcal{T}_X$ of a C^*-correspondence X over $C^*(G)$. As a set, the groupoid G_α^∞ is just the cartesian product $H_\infty \times G$. We show that the map $f \mapsto 1_{H_\infty^{(0)}} \times f$ from $C_c(G)$ to $C_c(G_\alpha^\infty)$ extends to an embedding $\iota_G : C^*(G) \to C^*(G_\alpha^\infty)$. We then show that the isomorphism $C^*(G_\alpha^\infty) \cong \mathcal{T}_X$ intertwines ι_G and the canonical inclusion $i_{C^*(G)} : C^*(G) \to \mathcal{T}_X$. Combining this with Pimsner's results [21], we deduce that the KK-class of ι_G is a KK-equivalence, and in particular, $K_*(\iota_G)$ is an isomorphism. Though we don't need it for our later results, we also prove that $\mathcal{T}_X$—and hence also $C^*(G_\alpha^\infty)$—coincides with $\mathcal{O}_X$.

In Section 5, we investigate the structure of G_α^∞. In particular we identify conditions on G that ensure that $C^*(G_\alpha^\infty)$ is a Kirchberg algebra. We summarise our results about G_α^∞ in our main theorem, Theorem 5.5. We finish Section 5 with a slight modification to our construction of G_α^∞ that shows that the class of Kirchberg algebras achievable via our construction is closed under stabilisation.

In the last two sections, we provide two classes of examples. In Section 6, we show that the graph groupoids G of suitable Bratteli diagrams for simple AF algebras admit automorphisms satisfying our hypotheses. We deduce that for every simple dimension group $D \neq \mathbb{Z}$ the stable Kirchberg algebra with K-theory $(D, \{0\})$ can be realised as the C^*-algebra of an amenable principal groupoid. We show further that for every element d of the positive cone of D, the unital Kirchberg algebra with the same K-theory and with the class of the unit equal to d can also be constructed in this way. In Section 7, we apply a similar analysis to the groupoids associated to the rank-2 Bratteli diagrams of [18]. Thus for a large class of pairs (D_0, D_1) of simple dimension groups with $D_0 \neq \mathbb{Z}$ and D_1 a subgroup of D_0 for which the quotient is entirely torsional, the stable Kirchberg algebra with K-theory (D_0, D_1) can be realised as the C^*-algebra of an amenable principal groupoid, and for many possible order units d for D_0, the unital Kirchberg algebra with K-theoretic data (D_0, d, D_1) can too.

2. Preliminaries

In this section, we collect the background needed for the remainder of the paper.

2.1. **Groupoids and C^*-algebras.** A groupoid is a small category G with inverses. We denote the collection of identity morphisms of G by $G^{(0)}$, and identify it with the object set, so that the domain and codomain maps become maps $s, r : G \to G^{(0)}$ such that $r(g)g = g = gs(g)$ for all $g \in G$. We are interested in topological groupoids: groupoids endowed with a topology under which inversion from G to G and composition from $\{(g, h) \in G \times G : s(g) = r(h)\}$ (under the relative topology) to G are continuous. By an *automorphism* α of a topological groupoid G we mean a structure-preserving homeomorphism $\alpha : G \to G$. If α preserves the algebraic, but not necessarily the topological, structure, we will call it an *algebraic* automorphism.

A groupoid is said to be *étale* if the range map (equivalently the source map) is a local homeomorphism. An open set $U \subseteq G$ on which r and s are both homeomorphisms is called an *open bisection*; so an étale groupoid is a groupoid that has a basis consisting of open bisections. A groupoid is said to be *ample* if its topology admits a base of compact open bisections. An ample groupoid is necessarily étale.

We think of a groupoid G as determining a partially defined action on its unit space: for $g \in G$ and $u \in G^{(0)}$, g can act on u if $s(g) = u$, and then $g \cdot u = r(g)$. We write $[u]$ for the orbit $\{g \cdot u : s(g) = u\}$ of u under G. We will use the standard notation in the literature where

$\{g \in G : s(g) = u\}$ is denoted G_u, and $\{g \in G : r(g) = u\}$ is denoted G^u. So

$$[u] = r(G_u) = s(G^u).$$

The groupoid G is said to be *minimal* if $\overline{[u]} = G^{(0)}$ for every $u \in G^{(0)}$. It is said to be *principal* if the map $g \mapsto (r(g), s(g))$ is injective. Equivalently, G is principal if, for every $u \in G^{(0)}$, $\{g \in G : s(g) = u = r(g)\} = \{u\}$.

Each locally compact, Hausdorff, étale groupoid has two associated C^*-algebras, which were introduced in [24]: the full and reduced algebras $C^*(G)$ and $C_r^*(G)$. Both are completions of the convolution *-algebra $C_c(G)$ in which

$$(\xi * \eta)(g) = \sum_{hk=g} \xi(h)\eta(k) \quad \text{and} \quad \xi^*(g) = \overline{\xi(g^{-1})}$$

for $\xi, \eta \in C_c(G)$. The full C^*-algebra is the completion with respect to a suitable universal norm. The reduced C^*-algebra is described as follows. For each $u \in G^{(0)}$, there is a representation $R_u : C_c(G) \to \mathcal{B}(\ell^2(G_u))$ such that, for $\xi \in C_c(G)$ and $g \in G_u$,

$$(2.1) \qquad\qquad R_u(\xi)\delta_g = \sum_{h \in G_{r(g)}} \xi(h)\delta_{hg}.$$

The reduced C^*-algebra is the completion of the image of $C_c(G)$ under $\bigoplus_{u \in G^{(0)}} R_u$.

There are various notions of amenability for groupoids, but we won't give the formal definitions of here. They all imply that $C^*(G)$ and $C_r^*(G)$ coincide. For our purposes, it suffices to recall that when G is locally compact, Hausdorff and étale, [2, Corollary 6.2.14(ii)] (also [5, Theorem 5.6.18]) shows that G is amenable if and only if $C_r^*(G)$ is nuclear — and then $C^*(G)$ automatically coincides with $C_r^*(G)$.

2.2. **Graph groupoids.** A key example for us will be the graph groupoids introduced in [16], and subsequently extended in [19] to allow for non-row-finite graphs. For the majority of the paper, we will be interested only in the groupoid of the graph E with one vertex v and infinitely many edges $\{e_i : i \in \mathbb{N}\}$; but we will need greater generality in Section 6.

A directed graph E consists of two countable sets E^0 and E^1 and two maps $r, s : E^1 \to E^0$. We will assume that our graphs E have no sources in the sense that $r^{-1}(v) \neq \emptyset$ for all $v \in E^0$; but we will allow them to be non-row-finite in the sense that $r^{-1}(v)$ may be infinite.

A *path* in a directed graph E is a vertex, or a word $\mu = \mu_1 \ldots \mu_n$ over the alphabet E^1 with the property that $s(\mu_i) = r(\mu_{i+1})$ for all i. We write $s(\mu)$ for $s(\mu_n)$ and $r(\mu)$ for $r(\mu_1)$. If μ is a vertex v, then we define $s(\mu) = r(\mu) = v$. An *infinite path* is a right-infinite word $x = x_1 x_2 \ldots$ over the alphabet E^1 satisfying $s(x_i) = r(x_{i+1})$ for all i. The space of all finite paths is denoted E^* and the space of all infinite paths is denoted E^∞. We write $|\mu|$ for the number of edges in a path μ, with the convention that $|v| = 0$ for $v \in E^0$ and $|\mu| = \infty$ when μ is an infinite path. For a path μ (either finite or infinite) and an integer $n \leq |\mu|$, we write $\mu(0, n) := \mu_1 \ldots \mu_n \in E^*$. We write P_E, or just P for the collection

$$E^\infty \sqcup \{\mu \in E^* : r^{-1}(s(\mu)) \text{ is infinite}\}.$$

For each $\mu \in E^*$ we define

$$Z(\mu) := \{\alpha \in P : |\alpha| \geq |\mu| \text{ and } \alpha(0, |\mu|) = \mu\}$$

and then for each finite set $F \subseteq r^{-1}(s(\mu))$ we define

$$Z(\mu \setminus F) := Z(\mu) \setminus \bigcup_{e \in F} Z(\mu e).$$

The collection of all such $Z(\mu \setminus F)$ form a base of compact sets that generate a locally compact Hausdorff topology on P. In particular, for $\mu \in E^*$, a base of neighbourhoods of μ is the collection $\{Z(\mu \setminus F) : F \subseteq r^{-1}(s(\mu)) \text{ is finite}\}$, and for $\mu \in E^\infty$ a base of neighbourhoods of μ is $\{Z(x(0, n)) : n \in \mathbb{N}\}$.

We borrow some notation from the k-graph literature. For $\mu \in E^*$ and $F \subseteq P$, we write μF for $\{\mu\lambda : \lambda \in F, r(\lambda) = s(\mu)\}$, similarly for $F \subseteq E^*$, we write $F\mu = \{\lambda\mu : \lambda \in F, s(\lambda) = r(\mu)\}$. So, for example, μP is equal to $Z(\mu)$, and $vE^1 = r^{-1}(v)$.

For $n \in \mathbb{N}$, there is map $\sigma^n : \{\mu \in P : |\mu| \geq n\} \to P$ given by $\sigma^n(\mu) = s(\mu)$ for $\mu \in E^n$ and $\sigma^n(\mu) = \mu_{n+1} \cdots \mu_{|\mu|}$ for $|\mu| > n$. Observe that σ^n restricts to a homeomorphism on $Z(\mu)$ whenever $|\mu| \geq n$. Also observe that the domain of σ^{m+n} is $(\sigma^m)^{-1}(\mathrm{dom}\,\sigma^n)$ and on this domain, $\sigma^{m+n} = \sigma^n \circ \sigma^m$.

As a set, the *graph groupoid* G_E of E is equal to

$$(2.2) \qquad G_E = \{(x, m-n, y) : x, y \in P \text{ and } \sigma^m(x) = \sigma^n(y)\}.$$

Its unit space is $\{(x, 0, x) : x \in P\}$, which we identify with P, and multiplication and inverse are $(x, m, y)(y, n, z) = (x, m+n, z)$ and $(x, m, y)^{-1} = (y, -m, x)$. The topology has basic compact open sets

$$Z((\alpha, \beta) \setminus F) = \{(\alpha x, |\alpha| - |\beta|, \beta x) : x \in Z(s(\alpha) \setminus F)\}$$

indexed by pairs $(\alpha, \beta) \in E^*$ with $s(\alpha) = s(\beta)$ and finite sets $F \subseteq s(\alpha)E^1$.

Remark 2.1. Under this topology G_E is an amenable, second-countable, locally compact, Hausdorff, ample groupoid [19]. If $C^*(E)$ denotes the graph algebra of E, then there is an isomorphism $C^*(E) \cong C^*(G_E)$ that carries each generator s_e to the characteristic function $1_{Z(e,s(e))}$.

2.3. C^*-**correspondences and** C^*-**algebras.** Following [11], given a C^*-algebra A, we say that a right A-module X is a *right inner-product module* if it is endowed with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ that is linear and A-linear in the second variable, satisfies $\langle x, y \rangle_A^* = \langle y, x \rangle_A$, and satisfies $\langle x, x \rangle_A \geq 0$ for all x, with equality only for $x = 0$. The formula $\|x\| := \|\langle x, x \rangle_A\|^{1/2}$ defines a norm on X, and X is a *right Hilbert A-module* if it is complete in this norm. An operator T on X is *adjointable* if there is an operator T^* on X such that $\langle Tx, y \rangle_A = \langle x, T^*y \rangle_A$ for all x, y; T is then linear, bounded and A-linear and T^* is unique. The space $\mathcal{L}(X)$ of all adjointable operators is a C^*-algebra under the operator norm. For $x, y \in X$, the formula $\Theta_{x,y}(z) := x \cdot \langle y, z \rangle_A$ determines an adjointable operator with adjoint $\Theta_{x,y}^* = \Theta_{y,x}$. The set $\overline{\mathrm{span}}\{\Theta_{x,y} : x, y \in X\}$ is an ideal of $\mathcal{L}(X)$ denoted $\mathcal{K}(X)$. For more details, see for example [23, Section 2.2].

A right Hilbert A-module X over a C^*-algebra A becomes a C^*-*correspondence over* A when endowed with a homomorphism $\phi : A \to \mathcal{L}(X)$, which we then regard as defining a left action of A on X: $a \cdot x := \phi(a)x$. A *Toeplitz representation*, or just a *representation*, of X in a C^*-algebra B is a pair (ψ, π) consisting of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\pi(a)\psi(x) = \psi(a \cdot x), \quad \psi(x)\pi(a) = \psi(x \cdot a) \quad \text{and} \quad \psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A)$$

for all $a \in A$ and $x, y \in X$.

Each C^*-correspondence X over A has two associated C^*-algebras: the *Toeplitz algebra* $\mathcal{T}_X$ and the *Cuntz–Pimsner algebra* $\mathcal{O}_X$ [21]. The Toeplitz algebra $\mathcal{T}_X$ is the universal C^*-algebra generated by a Toeplitz representation (i_X, i_A) of X. Every representation of X induces a homomorphism $\psi^{(1)} : \mathcal{K}(X) \to B$ satisfying $\psi^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$. If the homomorphism $\phi : A \to \mathcal{L}(X)$ implementing the left action is injective, then we say that the representation (ψ, π) is *Cuntz–Pimsner covariant* if

$$\psi^{(1)}(\phi(a)) = \pi(a) \quad \text{whenever } \phi(a) \in \mathcal{K}(X).$$

The Cuntz–Pimsner algebra $\mathcal{O}_X$ of X is the quotient of $\mathcal{T}_X$ that is universal for Cuntz–Pimsner covariant representations of X.

3. A twisted product of groupoids

Let H be a topological groupoid endowed with a cocycle $c : H \to \mathbb{Z}$ (a cocycle is a map satisfying $c(gh) = c(g) + c(h)$ whenever $s(g) = r(h)$) which is continuous. Let G be another topological groupoid, and suppose that $\alpha : G \to G$ is an automorphism of G. In this section,

we describe how to construct a *twisted product groupoid* $H \times_{c,\alpha} G$. Our construction makes sense for any H and G, but we will later be interested primarily in the situation where H is the groupoid associated to $\mathcal{O}_\infty$ (see (3.1)).

Lemma 3.1. *Let G and H be groupoids. Suppose that $c : H \to \mathbb{Z}$ is a cocycle, and that $\alpha : G \to G$ is an algebraic automorphism. Let*

$$H \times_{c,\alpha} G := H \times G,$$

and $(H \times_{c,\alpha} G)^{(0)} := H^{(0)} \times G^{(0)}$. Define $r, s : H \times_{c,\alpha} G \to (H \times_{c,\alpha} G)^{(0)}$ by $r(h, g) := \big(r(h), r(g)\big)$ and $s(h, g) := \big(s(h), \alpha^{c(h)}(s(g))\big)$. For $(h_1, g_1), (h_2, g_2) \in H \times_{c,\alpha} G$ with $s(h_1, g_1) = r(h_2, g_2)$, define

$$(h_1, g_1)(h_2, g_2) := (h_1 h_2, g_1 \alpha^{-c(h_1)}(g_2)) \quad and \quad (h, g)^{-1} := (h^{-1}, \alpha^{c(h)}(g^{-1})).$$

Under these structure maps, the set $H \times_{c,\alpha} G$ is a groupoid.

Proof. Clearly $r((h_1, g_1)(h_2, g_2)) = r(h_1, g_1)$. Since $\mathbb{Z}$ is abelian and α is an automorphism, we have $\alpha^{c(h_1 h_2)}(s(\alpha^{-c(h_1)}(g_2))) = \alpha^{c(h_2)} \circ \alpha^{c(h_1)}(\alpha^{-c(h_1)} s(g_2)) = \alpha^{c(h_2)}(s(g_2))$, and so

$$s((h_1, g_1)(h_2, g_2)) = \big(s(h_1 h_2), \alpha^{c(h_1 h_2)}(s(\alpha^{-c(h_1)}(g_2)))\big) = \big(s(h_1 h_2), \alpha^{c(h_2)}(s(g_2))\big) = s(h_2, g_2).$$

Now if $(h_1, g_1), (h_2, g_2)$ and (h_3, g_3) are composable, then

$$\big((h_1, g_1)(h_2, g_2)\big)(h_3, g_3) = (h_1 h_2, g_1 \alpha^{-c(h_1)}(g_2))(h_3, g_3) = (h_1 h_2 h_3, g_1 \alpha^{-c(h_1)}(g_2)\alpha^{-c(h_1 h_2)}(g_3))$$
$$= (h_1, g_1)\big(h_2 h_3, g_1 \alpha^{-c(h_1)}(g_2 \alpha^{-c(h_2)}(g_3))\big) = (h_1, g_1)\big((h_2, g_2)(h_3, g_3)\big).$$

So multiplication preserves ranges and sources and is associative. We have

$$(h, g)s(h, g) = (h, g)(s(h), \alpha^{c(h)}(s(g))) = (hs(h), g\alpha^{-c(h)}(\alpha^{c(h)}(s(g)))) = (h, g),$$

and an even simpler calculation gives $r(h, g)(h, g) = (h, g)$. We have

$$(h, g)^{-1}(h, g) = (h^{-1} h, \alpha^{c(h)}(g^{-1})\alpha^{-c(h^{-1})}(g)) = (s(g), \alpha^{c(h)}(s(g))) = s(h, g),$$

and a similar calculation gives $(h, g)(h^{-1}, \alpha^{c(h)}(g^{-1})) = r(h, g)$. Finally we have

$$s((h, g)^{-1}) = s(h^{-1}, \alpha^{c(h)}(g^{-1})) = \big(s(h^{-1}), \alpha^{c(h^{-1})}\big(s(\alpha^{c(h)}(g^{-1}))\big)\big) = (s(h^{-1}), s(g^{-1})) = r(h, g),$$

completing the proof. $\square$

We now show that if H and G are both locally compact, Hausdorff, étale groupoids, then so is $H \times_{c,\alpha} G$.

Lemma 3.2. *Let H and G be locally compact, Hausdorff, étale groupoids, $c : H \to \mathbb{Z}$ be a continuous cocycle and $\alpha : G \to G$ an automorphism. Then the groupoid $H \times_{c,\alpha} G$ is locally compact, Hausdorff and étale when endowed with the product topology. If G and H are second countable, then so is $H \times_{c,\alpha} G$. If G and H are ample, then so is $H \times_{c,\alpha} G$.*

Proof. Since H and G are locally compact and Hausdorff, so is the product $H \times G$. Since $H \times_{c,\alpha} G$ is $H \times G$ as a topological space, we deduce that $H \times_{c,\alpha} G$ is locally compact and Hausdorff. Composition and inversion in $H \times_{c,\alpha} G$ are continuous because each α^n is continuous and composition and inversion in each of H and G are continuous. Since the range map on $H \times_{c,\alpha} G$ coincides with that on the étale groupoid $H \times G$, it is a local homeomorphism, and so $H \times_{c,\alpha} G$ is étale too.

The statement about second countability is clear. For the final statement, observe that if $\mathcal{U}$ and $\mathcal{V}$ are bases of compact open bisections for H and G respectively, then $\mathcal{U} \times \mathcal{V}$ is a base of compact open bisections for $H \times_{c,\alpha} G$. $\square$

3.1. **The twisted product groupoid G_α^∞.** From now on, we restrict our attention to the situation where G is locally compact, Hausdorff and étale and H is the groupoid associated to the Cuntz algebra $\mathcal{O}_\infty$, which we make precise below.

Let E be the graph with one vertex v and infinitely many edges $\{e_i : i \in \mathbb{N}\}$. Thus, the graph C^*-algebra $C^*(E)$ is canonically isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ (see for example [22, page 42]). We denote by H_∞ the groupoid G_E. Since $C^*(H_\infty) \cong \mathcal{O}_\infty$, we call H_∞ the *groupoid associated to the Cuntz algebra $\mathcal{O}_\infty$*. As discussed in Section 2, writing $P := E^* \cup E^\infty$, the groupoid H_∞ consists of ordered triples

(3.1) $$H_\infty = \{(\alpha x, |\alpha| - |\beta|, \beta x) : x \in P, \alpha, \beta \in E^* r(x)\} \subseteq P \times \mathbb{Z} \times P,$$

with operations $(x, m, y)(y, n, z) = (x, m + n, z)$ and $(x, m, y)^{-1} = (y, -m, x)$. We identify $H_\infty^{(0)}$ with P via $(x, 0, x) \mapsto x$. The topology on H_∞ described in Section 2.2 is generated by the sets

$$Z((\alpha, \beta) \setminus F) = \{(\alpha x, |\alpha| - |\beta|, \beta x) : x \in P, x_1 \notin F\}$$

indexed by pairs $\alpha, \beta \in E^*$ and finite subsets $F \subseteq \{e_i : i \in \mathbb{N}\}$. We define $Z(\alpha, \beta) := Z((\alpha, \beta) \setminus \emptyset)$. For $\alpha \in P$ we have

$$\alpha P = Z(\alpha) = Z(\alpha, \alpha) = \{\alpha x : x \in P\}.$$

With this structure H_∞ is a second-countable, amenable, locally compact, Hausdorff, ample groupoid; the sets $Z((\alpha, \beta) \setminus F)$ are compact open bisections.

Lemma 3.3. *For every nonempty open set $W \subseteq H_\infty^{(0)}$ there exists $\lambda \in E^*$ such that $Z(\lambda) \subseteq W$.*

Proof. Fix any $u \in W$. If $u \in E^\infty$ then the sets $\{Z(u(0, n)) : n \in \mathbb{N}\}$ are a neighbourhood base at u, so we can take $\lambda = u(0, n)$ for large n. If $u \in E^*$, then the sets $Z(u \setminus F)$, where $F \subseteq s(u)E^1$ is finite, form a neighbourhood base at u. Hence there exists finite F such that $Z(u \setminus F) \subseteq W$. Let $n = \max\{j : e_j \in F\} + 1$. Then $\lambda = ue_n$ satisfies $Z(\lambda) \subseteq Z(u \setminus F) \subseteq W$.   □

Notation 3.4. Let G be any locally compact, Hausdorff, étale groupoid and let α be an automorphism of G. We let G_α^∞ denote the twisted product of G and H_∞ with its canonical cocycle, i.e.,

(3.2) $$G_\alpha^\infty := H_\infty \times_{c,\alpha} G,$$

as in Lemma 3.1, where $c : H_\infty \to \mathbb{Z}$ is given by $c(x, m, y) = m$.

Lemma 3.5. *Let G be a locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Then G_α^∞ is amenable if and only if G is amenable; and then $C^*(G_\alpha^\infty) \cong C_r^*(G_\alpha^\infty)$ is nuclear and in the UCT class.*

Proof. First suppose that G is amenable. Define a map $\tilde{c} : G_\alpha^\infty \to \mathbb{Z}$ by $\tilde{c}(h, g) = c(h)$. Then $\tilde{c}$ is a cocycle on G_α^∞. Using [31, Proposition 9.3] (see also [26, Corollary 4.5]), we show that G_α^∞ is amenable by showing that $\tilde{c}^{-1}(0)$ is amenable. The formula for multiplication in G restricts to coordinatewise multiplication on the subgroupoid $\tilde{c}^{-1}(0)$, and so $\tilde{c}^{-1}(0) \cong c^{-1}(0) \times G$ as topological groupoids. Since products of amenable groupoids are again amenable, and since G is amenable by assumption, it therefore suffices to show that $c^{-1}(0) \subseteq H_\infty$ is amenable. As above, let E be the graph with one vertex and infinitely many loops, so that H_∞ is the graph groupoid of E. For each $n \in \mathbb{N}$, Paterson shows [19, Proposition 4.1] that the subgroupoid $R_n := \bigcup_{i=1}^n \{(\alpha x, 0, \beta x) : \alpha, \beta \in E^i, x \in E^* \cup E^\infty\}$ of H_∞ is amenable. He then shows (see the two paragraphs following [19, Proposition 4.1]) that $R := \bigcup_{n=1}^\infty R_n$ is also amenable. This R is precisely $c^{-1}(0)$, so $c^{-1}(0)$, and hence $\tilde{c}^{-1}(0)$ is amenable.

Now suppose that G_α^∞ is amenable. Fix $x \in H_\infty^{(0)}$. Then the subgroupoid $\{x\} \times G$ of G_α^∞ is closed, and therefore locally closed, so [2, Proposition 5.1.1] shows that $\{x\} \times G$ is amenable. Since G is canonically isomorphic to $\{x\} \times G$, it follows that G is amenable.

By Lemma 3.2, G_α^∞ is locally compact, Hausdorff and étale and so $C_r^*(G_\alpha^\infty)$ is defined. Suppose that G_α^∞ is amenable. Then $C^*(G_\alpha^\infty) \cong C_r^*(G_\alpha^\infty)$ by [2, Proposition 6.1.18]. It is nuclear

by [2, Corollary 6.2.14(i)], and belongs to the UCT class by [34, Lemma 3.3 and Proposition 10.7]. $\qquad\square$

4. REALISING $C^*(G_\alpha^\infty)$ AS A TOEPLITZ ALGEBRA

Throughout this section we assume that G is amenable. Under this hypothesis we will show that $C^*(G_\alpha^\infty)$ can be realised as the Toeplitz algebra of a correspondence over $C^*(G)$. Provided that $C^*(G)$ is separable, it will then follow from [21, Theorem 4.4] that $C^*(G_\alpha^\infty)$ is KK-equivalent to $C^*(G)$.

We continue to write H_∞ for the groupoid (3.1) associated to the Cuntz algebra $\mathcal{O}_\infty$. For $\phi \in C_c(H_\infty)$ and $f \in C_c(G)$ we write $\phi \times f$ for the function in $C_c(G_\alpha^\infty)$ defined by

$$\phi \times f(h, g) = \phi(h)f(g).$$

Lemma 4.1. *Let G be an amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Then the map $\iota_G : f \mapsto 1_{H_\infty^{(0)}} \times f$ from $C_c(G)$ to $C_c(G_\alpha^\infty)$ extends to an embedding $\iota_G : C^*(G) \to C^*(G_\alpha^\infty)$.*

Proof. Since $H_\infty^{(0)}$ is compact and open in H_∞, the characteristic function $1_{H_\infty^{(0)}}$ belongs to $C_c(H_\infty)$, and so ι makes sense. Since c is zero on $H_\infty^{(0)}$, the operations on $H_\infty^{(0)} \times G$ in G_α^∞ agree with those in $H_\infty \times G$. Since $f \mapsto 1_{H_\infty^{(0)}} \times f$ is multiplicative and star-preserving from $C_c(G)$ to $C_c(H_\infty \times G)$, it follows that ι is a homomorphism.

Fix $(u, x) \in H_\infty^{(0)} \times G^{(0)}$ and $h \in (H_\infty)_u$. Define $y := \alpha^{-c(h)}(x) \in G^{(0)}$ and let P_h denote the projection of $\ell^2((G_\alpha^\infty)_{(u,x)})$ onto the subspace $\ell^2(\{h\} \times G_y)$. Define $V_h : \ell^2(\{h\} \times G_y) \to \ell^2(G_y)$ by $V_h\delta_{(h,\gamma)} = \delta_\gamma$. Fix $(s, \lambda) \in (G_\alpha^\infty)_{(u,x)}$. We have $V_h\delta_{(s,\lambda)} = V_hP_h\delta_{(s,\lambda)}$, which is equal to δ_λ if $s = h$ and zero otherwise. So (2.1) gives

$$V_h^* R_y(f)V_h\delta_{(s,\lambda)} = 1_{\{h\}}(s)V_h^* R_y(f)\delta_\lambda = 1_{\{h\}}(s)V_h^* \sum_{\mu \in G_{r(\lambda)}} f(\mu)\delta_{\mu\lambda} = 1_{\{h\}}(s) \sum_{\mu \in G_{r(\lambda)}} f(\mu)\delta_{(h,\mu\lambda)}.$$

Moreover, by (2.1) again, $R_{(u,x)}(\iota_G(f))\delta_{(s,\lambda)} = \sum_{(g,\mu) \in (G_\alpha^\infty)_{r(s,\lambda)}} \iota_G(f)(g,\mu)\delta_{(g,\mu)(s,\lambda)}$ with nonzero terms in the sum only when $g = r(s)$, in which case $\iota_G(f)(g,\mu)\delta_{(g,\mu)(r,\lambda)} = f(\mu)\delta_{(r,\mu\lambda)}$. We get

$$P_h R_{(u,x)}(\iota_G(f))\delta_{(s,\lambda)} = P_h \sum_{(r(s),\mu) \in (G_\alpha^\infty)_{r(s,\lambda)}} f(\mu)\delta_{(s,\mu\lambda)} = 1_{\{h\}}(s) \sum_{\mu \in G_{r(\lambda)}} f(\mu)\delta_{(h,\mu\lambda)}.$$

Summing over all $h \in (H_\infty)_u$, we have $R_{(u,x)}(\iota_G(f)) = \bigoplus_h V_h^* R_{\alpha^{-c(h)}(x)}(f)V_h$. That is, each regular representation of G_α^∞ is equivalent to a direct sum of regular representations of $C_c(G)$, which gives $\|R_{(u,x)}(\iota_G(f))\| \le \|f\|_{C_r^*(G)}$ for all f. Also, each regular representation R_x of G is equivalent to a direct summand in a regular representation of $C_c(G_\alpha^\infty)$, giving $\|R_x(f)\| \le \|\iota_G(f)\|_{C_r^*(G_\alpha^\infty)}$. So ι_G is isometric for the reduced norms on $C_c(G)$ and $C_c(G_\alpha^\infty)$. $\qquad\square$

Recall that H_∞ is the groupoid of the graph with one vertex v and infinitely many edges $\{e_i : i \in \mathbb{N}\}$. For $i \in \mathbb{N}$, let $x_i := 1_{Z(e_i,v)} \in C_c(H_\infty)$. Define

$$X := \overline{\text{span}}\{x_i \times f : f \in C_c(G), i \in \mathbb{N}\} \subseteq C^*(G_\alpha^\infty).$$

We will show that X is a C^*-correspondence over $C^*(G)$ and then apply [11, Theorem 3.1] to show that $C^*(G_\alpha^\infty) \cong \mathcal{T}_X$.

Lemma 4.2. *Let G be an amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). With*

$$a \cdot x := \iota_G(a)x, \qquad x \cdot a := x\iota_G(a), \qquad and \qquad \langle x, y\rangle_{C^*(G)} := \iota_G^{-1}(x^*y)$$

for $x, y \in X$ and $a \in C^*(G)$, the space X is a C^*-correspondence over $C^*(G)$.

Proof. We first show that for $x, y \in X$, we have $x^* y \in \iota_G(C^*(G))$. Since convolution and involution are continuous and linear, it suffices to consider $x = x_i \times f$ and $y = x_j \times f'$. We have

$$(4.1) \quad (x_i \times f)^* * (x_j \times f')(h, g) = \sum_{(h_1, g_1)(h_2, g_2) = (h, g)} \overline{(x_i \times f)((h_1, g_1)^{-1})} (x_j \times f')(h_2, g_2)$$

$$= \sum_{(h_1, g_1)(h_2, g_2) = (h, g)} \overline{(x_i \times f)(h_1^{-1}, \alpha^{c(h_1)}(g_1^{-1}))} (x_j \times f')(h_2, g_2)$$

$$= \sum_{(h_1, g_1)(h_2, g_2) = (h, g)} 1_{Z(v, e_i)}(h_1) \overline{(f \circ \alpha^{c(h_1)})(g_1^{-1})} 1_{Z(e_j, v)}(h_2) f'(g_2)$$

$$= \sum_{((r(h), -1, e_i r(h)), g_1)((e_j s(h), 1, s(h)), g_2) = (h, g)} (f \circ \alpha^{-1})^*(g_1) f'(g_2)$$

$$= \delta_{i,j} 1_{H_\infty^{(0)}}(h) \sum_{g_1 \alpha(g_2) = g} (f \circ \alpha^{-1})^*(g_1) f'(g_2)$$

$$= \delta_{i,j} 1_{H_\infty^{(0)}}(h) \sum_{g_1 g_2' = g} (f \circ \alpha^{-1})^*(g_1) (f' \circ \alpha^{-1})(g_2')$$

$$= \delta_{i,j} \iota_G \big((f \circ \alpha^{-1})^* * (f' \circ \alpha^{-1}) \big)(h, g),$$

which belongs to $\iota_G(C^*(G))$ as claimed. In particular the inner product is well defined. Since ι_G is isometric by Lemma 4.1, The inner-product norm on X agrees with the C^*-norm on $C^*(G_\alpha^\infty)$. So X is complete.

Now we check that $\iota_G(C^*(G))X$ and $X\iota_G(C^*(G))$ are contained in X. For $f, f' \in C_c(G)$ and $i \in \mathbb{N}$ we have

$$(4.2) \quad (f' \cdot (x_i \times f))(h, g) = \big(\iota_G(f') * (x_i \times f) \big)(h, g)$$

$$= \sum_{(h_1, g_1)(h_2, g_2) = (h, g)} (1_{H_\infty^{(0)}} \times f')(h_1, g_1)(1_{Z(e_i, v)} \times f)(h_2, g_2)$$

$$= \sum_{g_1 \alpha^{-c(r(h))}(g_2) = g} f'(g_1) 1_{Z(e_i, v)}(h) f(g_2)$$

$$= 1_{Z(e_i, v)}(h) \sum_{g_1 g_2 = g} f'(g_1) f(g_2) = \big(x_i \times (f' * f) \big)(h, g).$$

Thus by continuity of multiplication in $C^*(G_\alpha^\infty)$, we have $\iota_G(C^*(G))X \subseteq X$. Similarly,

$$((x_i \times f) \cdot f')(h, g) = \sum_{(h_1, g_1)(h_2, g_2) = (h, g)} (1_{Z(e_i, v)} \times f)(h_1, g_1)(1_{H_\infty^{(0)}} \times f')(h_2, g_2)$$

$$= \sum_{g_1 \alpha^{-1}(g_2) = g} 1_{Z(e_i, v)}(h) f(g_1) f'(g_2)$$

$$= 1_{Z(e_i, v)}(h) \sum_{g_1 g_2' = g} f(g_1) f'(\alpha(g_2'))$$

$$= \big(x_i \times (f * (f' \circ \alpha)) \big)(h, g).$$

Since $\alpha \in \mathrm{Aut}(G)$ we have $f' \circ \alpha \in C_c(G)$. So continuity gives $X\iota_G(C^*(G)) \subseteq X$.

The C^*-algebra $C^*(G_\alpha^\infty)$ is a correspondence over itself with actions given by multiplication and inner-product $(a, b) \mapsto a^* b$, and we have just showed that $X \subseteq C^*(G_\alpha^\infty)$ is an inner-product bimodule over $C^*(G) \subseteq C^*(G_\alpha^\infty)$ under the inherited operations. Thus X is a correspondence over $C^*(G)$ as required. □

To fix notation, recall that $\mathcal{T}_X$ is generated by a universal representation $(i_X, i_{C^*(G)})$ of X. Moreover, every Toeplitz representation (ψ, π) of X in $B(\mathcal{H})$ induces a homomorphism $\psi \times \pi : \mathcal{T}_X \to B(\mathcal{H})$ satisfying $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_{C^*(G)} = \pi$ ([11, Proposition 1.3]).

Lemma 4.3. *Let X be a C^*-correspondence over a C^*-algebra A. Suppose that $X = \bigoplus_{i \in \mathbb{N}} X_i$ and that for each nonzero $a \in A$ the set $\{i \in \mathbb{N} : a \cdot X_i \neq \{0\}\}$ is infinite. Suppose that (ψ, π) is a representation of X in $B(\mathcal{H})$ such that π is injective. Then $\psi \times \pi$ is injective.*

Proof. Fix a finite set $F \subseteq \mathbb{N}$. By [11, Theorem 3.1], $\psi \times \pi$ is injective provided that the compression of π to $(\psi(\oplus_{j \in F} X_j)\mathcal{H})^\perp$ is faithful. Fix $a \in A \setminus \{0\}$. Take $i \notin F$ and $x \in X_i$ such that $a \cdot x \neq 0$. Since π is injective if follows from [11, Remark 1.1] that ψ is isometric. We can therefore find $h \in \mathcal{H}$ satisfying $\psi(a \cdot x)h \neq 0$. For each $j \in F$, $y \in X_j$ and $k \in \mathcal{H}$ we have $\big(\psi(x)h \mid \psi(y)k\big) = \big(h \mid \psi(x)^*\psi(y)k\big) = \big(h \mid \pi(\langle x, y \rangle_{C^*(G)})k\big) = 0$, so

$$0 \neq \psi(a \cdot x)h = \pi(a)\psi(x)h \in \pi(a)(\psi(\oplus_{i \in F} X_i)\mathcal{H})^\perp.$$

We conclude that the compression of π to $(\psi(\oplus_{j \in F} X_j)\mathcal{H})^\perp$ is faithful. $\square$

Define $X_i := \overline{\text{span}}\{x_i \times f : f \in C_c(G)\}$ for each $i \in \mathbb{N}$, with the module structure and left action inherited from X.

Lemma 4.4. *Let G be an amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). For each $i \in \mathbb{N}$, the space X_i is C^*-correspondence over $C^*(G)$ and the module X of Lemma 4.2 is isomorphic to $\bigoplus_i X_i$.*

Proof. That each X_i is a correspondence follows from the argument of Lemma 4.2. By (4.1) it follows that $\langle x, y \rangle_{C^*(G)} = 0$ for $x \in X_i$, $y \in X_j$ and $i \neq j$. Using this, it is routine to check that the map from the algebraic direct sum of the X_i to X that carries $(\xi_i)_{i=1}^\infty$ to $\sum_{i=1}^\infty \xi_i$ preserves the inner product and so is isometric. Since the module actions in both X and $\bigoplus X_i$ are implemented by multiplication in $C^*(G_\alpha^\infty)$, the map $(x_i) \mapsto \sum x_i$ is also a bimodule map, completing the proof. $\square$

Lemma 4.5. *Let G be an amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Suppose X and X_i are the C^*-correspondences of Lemma 4.4. For each $a \in C^*(G)$ there exist $f \in C_0(G^{(0)})$ such that*

$$\|a \cdot (x_i \times f)\| \geq \frac{\|a\|}{3}$$

for all $i \in \mathbb{N}$. In particular for $a \neq 0$, the set $\{i \in \mathbb{N} : a \cdot X_i \neq \{0\}\}$ is infinite.

Proof. Fix $a \in C^*(G)$. Choose $a_0 \in C_c(G)$ satisfying $\|a - a_0\| \leq \|a\|/3$. Then the set

$$s(\text{supp}(a_0)) \subseteq G^{(0)}$$

is compact, so we can choose $f \in C_c(G^{(0)})$ such that $\|f\| = 1$ and $f|_{s(\text{supp}(a_0))} \equiv 1$, and hence $a_0 * f = a_0$. The computations (4.1) and (4.2) show that

$$\|a_0 \cdot (x_i \times f)\| = \|x_i \times (a_0 * f)\| = \|(a_0 * f) \circ \alpha^{-1}\| = \|a_0 \circ \alpha^{-1}\| = \|a_0\|$$

for every $i \in \mathbb{N}$. Since $\|a - a_0\| \leq \|a\|/3$, we have $\|a_0\| \geq 2\|a\|/3$, and so for each i we obtain $\|a_0 \cdot (x_i \times f)\| \geq 2\|a\|/3$. So for $i \in \mathbb{N}$, we have

$$(4.3) \qquad \|a \cdot (x_i \times f)\| \geq \|a_0(x_i \times f)\| - \|(a - a_0)(x_i \times f)\| \geq 2\|a\|/3 - \|(a - a_0)(x_i \times f)\|.$$

Since $\|f\| = 1$, each $\|x_i \times f\| = \|f \circ \alpha^{-1}\| = 1$, and since $\|a - a_0\| \leq \|a\|/3$, we deduce that each $\|(a - a_0)(x_i \times f)\| \leq \|a\|/3$. This and (4.3) give $\|a \cdot (x_i \times f)\| \geq 2\|a\|/3 - \|a\|/3 = \|a\|/3$ for all i. So, for $a \neq 0$, $\{i \in \mathbb{N} : a \cdot X_i \neq \{0\}\} = \mathbb{N}$, which is certainly infinite. $\square$

We can now prove that $\mathcal{T}_X \cong C^*(G_\alpha^\infty)$. First observe that if $\phi : C^*(G_\alpha^\infty) \to B(\mathcal{H})$ is a representation, then we obtain a Toeplitz representation (ψ, π) of X in $\mathcal{B}(\mathcal{H})$ by setting

$$(4.4) \qquad\qquad \psi(x) = \phi(x) \qquad \text{and} \qquad \pi(a) = \phi(\iota_G(a)).$$

Proposition 4.6. *Let G be an amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Suppose X is the C^*-correspondence of Lemma 4.2. Then there is an isomorphism $\rho : C^*(G_\alpha^\infty) \to \mathcal{T}_X$ such that $\rho \circ \iota_G = i_{C^*(G)}$.*

Proof. Choose a faithful representation ϕ of $C^*(G_\alpha^\infty)$, and let (ψ, π) be the representation of (4.4). Lemmas 4.3 and 4.5 show that the induced representation $\psi \times \pi$ of $\mathcal{T}_X$ is faithful. So it suffices to show that the image of $\psi \times \pi$ is equal to the image of ϕ. Clearly we have

$$(\psi \times \pi)(\mathcal{T}_X) = C^*\big(i_X(X) \cup i_{C^*(G)}(C^*(G))\big) = C^*(\phi(X \cup C^*(G))) \subseteq \phi(C^*(G_\alpha^\infty)).$$

To see that $\operatorname{im} \psi \times \pi = \operatorname{im} \phi$, it suffices to show that the *-subalgebra $\mathcal{A}$ of $C_c(G_\alpha^\infty)$ generated by X is dense in $C^*(G_\alpha^\infty)$. Let

$$\mathcal{C} := \{Z(\mu, \nu) \times U \mid \mu, \nu \in E^* \text{ and } U \text{ is an open bisection in } G\},$$

which is a collection of open bisections of G_α^∞. Note that $\mathcal{A}$ contains all the functions (with support in $\mathcal{C}$) of the form $1_{Z(\mu,\nu)} \times f$ where $\operatorname{supp} f \subseteq U$ for some open bisection $U \subseteq G$.

Proposition 3.14 of [10] says that each element of $C_c(G_\alpha^\infty)$ can be written as a finite sum of the form $\sum_n f_n$ where each f_n is supported on some $B_n \in \mathcal{C}$. Thus it suffices to show that for $B \in \mathcal{C}$, any element of $C_c(G_\alpha^\infty)$ supported on B can be approximated by functions in $\mathcal{A}$. We have $\|f\|_\infty \leq \|f\|$ for all $f \in C_c(G_\alpha^\infty)$ (see, for example, [4, Lemma 2.1(1)]), and Proposition 3.14 of [10] gives the reverse inequality for f supported on $B \in \mathcal{C}$. So the result follows from the Stone–Weierstrass theorem.

Finally with $\rho := (\psi \times \pi)^{-1} \circ \phi$, we use $(\psi \times \pi) \circ i_{C^*(G)} = \pi$ and $\pi = \phi \circ \iota_G$ to deduce that $\rho \circ \iota_G = (\psi \times \pi)^{-1} \circ \pi = i_{C^*(G)}$. $\qquad\square$

Corollary 4.7. *Let G be a second-countable, amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Then the KK-class of the map $\iota_G : C^*(G) \to C^*(G_\alpha^\infty)$ of Lemma 4.1 is a KK-equivalence. In particular, $K_*(\iota_G) : K_*(C^*(G)) \to K_*(C^*(G_\alpha^\infty))$ is an isomorphism.*

Proof. Let X be the C^*-correspondence of Lemma 4.2. Then Proposition 4.6 gives $C^*(G_\alpha^\infty) \cong \mathcal{T}_X$. Since G is second countable, $C^*(G)$ is separable, and so [21, Theorem 4.4] implies that the KK-class of the inclusion $i_{C^*(G)} : C^*(G) \to \mathcal{T}_X$ is a KK-equivalence. In particular, ι_G induces an isomorphism in K-theory (see [28, page 38]). $\qquad\square$

Although we do not need it in the current paper, we digress now to prove that $\mathcal{T}_X$, and so $C^*(G_\alpha^\infty)$, coincides with $\mathcal{O}_X$. Let $\varphi_X : C^*(G) \to \mathcal{L}(X)$ denote the homomorphism implementing the left action on X.

Proposition 4.8. *Let G be a second-countable, amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Let X be the C^*-correspondence of Lemma 4.2. Then*

(1) *we have $\varphi_X(C^*(G)) \cap \mathcal{K}(X) = \{0\}$; and*
(2) *we have $C^*(G_\alpha^\infty) \cong \mathcal{O}_X$.*

Proof. For (1), first identify $X = \bigoplus_i X_i$ using Lemma 4.4. Since $\bigoplus_i X_i = \overline{\bigcup_n \bigoplus_{i=1}^n X_i}$, it follows that $\mathcal{K}(X) = \bigcup_{n=1}^\infty \mathcal{K}(\bigoplus_{i=1}^n X_i)$. For $T \in \mathcal{K}(\bigoplus_{i=1}^n X_i)$, we have $T(x_j \times f) = 0$ for $j > n$ and $f \in C_0(G^{(0)})$. It then follows from continuity that

$$(4.5) \qquad \lim_{j \to \infty} \|T(x_j \times f)\| = 0 \qquad \text{for all } T \in \mathcal{K}(X) \text{ and } f \in C_0(G^{(0)}).$$

Fix $T \in \varphi_X(C^*(G)) \cap \mathcal{K}(X)$, say $T = \varphi_X(a)$. By Lemma 4.5 there exist $f \in C_0(G^{(0)})$ such that $\|a \cdot (x_i \times f)\| \geq \|a\|/3$ for each $i \in \mathbb{N}$. Using (4.5), we see that

$$0 = \lim_{j \to \infty} \|T(x_j \times f)\| = \lim_{j \to \infty} \|a \cdot (x_j \times f)\| \geq \|a\|/3,$$

This forces $a = 0$ and hence $0 = \varphi_X(a) = T$, giving (1). For (2), recall that $C^*(G_\alpha^\infty) \cong \mathcal{T}_X$ by Proposition 4.6, and so the result follows from (1) and the definition of $\mathcal{O}_X$ (see [21, Corollary 3.14]). $\qquad\square$

5. Structural properties of G_α^∞

Recall that a Kirchberg algebra is a nuclear, separable, simple, purely infinite C^*-algebra. Our goal in this section is to identify conditions on G under which G_α^∞ is principal and $C^*(G_\alpha^\infty)$ is a Kirchberg algebra in the UCT class with the same K-theory as $C^*(G)$. We have established that G being amenable forces $C^*(G_\alpha^\infty)$ to be nuclear and in the UCT class in Lemma 3.5. If G is also second countable, then $C^*(G_\alpha^\infty)$ is separable, and has the same K-theory as $C^*(G)$. Thus, once we find conditions on G under which G_α^∞ is principal and minimal, all that will remain is to consider pure infiniteness.

Proposition 5.1. *Let G be a locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). The following are equivalent*

(1) G_α^∞ *is principal,*
(2) G *is principal and α induces a free action of $\mathbb{Z}$ on the orbit-space of G in the sense that*

(5.1) $$\text{if } x \in G^{(0)} \text{ and } l \in \mathbb{Z} \text{ satisfy } [x] = [\alpha^l(x)] \text{ then } l = 0.$$

Proof. (1) $\implies$ (2): Suppose that G_α^∞ is principal. Then for each unit x of H_∞, the subgroupoid $\{x\} \times G$ of G_α^∞ is also principal. Since $(x, g) \mapsto g$ is an isomorphism of $\{x\} \times G$ onto G, it follows that G is principal. To establish (5.1), fix $x \in G^{(0)}$ and suppose that $[x] = [\alpha^l(x)]$ for some $l \in \mathbb{Z}$. By assumption, there exists $g_0 \in G$ such that

$$s(g_0) = \alpha^l(x) \quad \text{and} \quad r(g_0) = x.$$

Recall that $H_\infty^{(0)}$ is the space of all paths (finite and infinite) of the graph with one vertex v and infinitely many edges e_i. Let $y_0 = e_0 e_0 e_0 \cdots \in H_\infty^{(0)}$, and let $h_0 := (y_0, -l, y_0) \in H_\infty$. Let $\gamma_0 = (h_0, g_0)$. Then

$$s(\gamma_0) = (y_0, \alpha^{-l}(s(\alpha^l(x)))) = (y_0, x) = r(\gamma_0).$$

Since G_α^∞ is principal we then have $\gamma_0 \in (G_\alpha^\infty)^{(0)}$, so $h_0 \in H_\infty^{(0)}$ forcing $l = 0$.

(2) $\implies$ (1): Suppose that (2) holds. Fix $\gamma = (h, g) \in G_\alpha^\infty$ such that $s(\gamma) = r(\gamma)$. Then $s(h) = r(h)$ and $s(\alpha^{c(h)}(g)) = r(g)$. Hence

$$[s(g)] = [r(g)] = [s(\alpha^{c(h)}(g))] = [\alpha^{c(h)}(s(g))].$$

Using (5.1) we see that $c(h) = 0$, and hence $h = (r(h), 0, s(h)) \in H_\infty^{(0)}$. Since G is principal, we also have $g \in G^{(0)}$, and so $\gamma \in H_\infty^{(0)} \times G^{(0)} = (G_\alpha^\infty)^{(0)}$. $\square$

Remark 5.2. We had to work a little to ensure that G_α^∞ is principal in Proposition 5.1; but it is automatically topologically principal whenever G is. To see this, observe that if $G_u^u = \{u\}$ and $(H_\infty)_x^x = \{x\}$, then $(h, g) \in (G_\alpha^\infty)_{(x,u)}^{(x,u)}$ forces $h \in (H_\infty)_x^x = \{x\}$, and then

$$g \in G_{\alpha^{c(h)}(u)}^u = G_u^u = \{u\}; \text{ so } (h, g) = (x, u) \in (G_\alpha^\infty)^{(0)}.$$

Since the product of dense subsets of $H_\infty^{(0)}$ and $G^{(0)}$ is dense in $H_\infty^{(0)} \times G^{(0)} = (G_\alpha^\infty)^{(0)}$, and since H_∞ is topologically principal, it follows that if G is topologically principal, then so is G_α^∞.

We next describe a necessary and sufficient condition for G_α^∞ to be minimal.

Proposition 5.3. *Let G be a locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). The groupoid G_α^∞ is minimal if and only if for each $y \in G^{(0)}$ the set $\bigcup_{n \le 0} \alpha^n([y])$ is dense in $G^{(0)}$. So if G is minimal, so is G_α^∞.*

Proof. First suppose G_α^∞ is minimal and choose $x, y \in G^{(0)}$. Choose an open neighbourhood V of x in $G^{(0)}$. Since $[y] = s(G^y)$, it suffices to find $\gamma \in G^y$ and $n \le 0$ such that $\alpha^n(s(\gamma)) \in V$. Recall that H_∞ is constructed from the graph with a single vertex v. Since G_α^∞ is minimal there exists $(h, \gamma) \in G_\alpha^\infty$ such that $(r(h), r(\gamma)) = r(h, \gamma) = (v, y)$ and $(s(h), \alpha^{c(h)}(s(\gamma))) = s(h, \gamma) \in H_\infty^{(0)} \times V$. So $\gamma \in G^y$, and $\alpha^{c(h)}(s(\gamma)) \in V$. Since $r(h) = v \in E^0$, we have $h = (v, -|\delta|, \delta)$ for some $\delta \in E^*$. So $n := c(h) = -|\delta| \le 0$ satisfies $\alpha^n(s(\gamma)) \in V$ as required.

Now suppose that for all $y \in G^{(0)}$, the set $\bigcup_{n \leq 0} \alpha^n([y])$ is dense in $G^{(0)}$. Observe that by applying this with $y = \alpha^p(z)$ we deduce that

$$(5.2) \qquad \bigcup_{n \leq p} \alpha^n([z]) \text{ is dense in } G^{(0)} \text{ for every } p \in \mathbb{Z} \text{ and } z \in G^{(0)}.$$

Choose $(u, w), (x, y) \in H_\infty^{(0)} \times G^{(0)}$ and let $U \times V$ be a basic open set containing (u, w). We must find $\xi \in G_\alpha^\infty$ such that $r(\xi) = (x, y)$ and $s(\xi) \in U \times V$. By Lemma 3.3, there exists $\lambda \in E^*$ such that $Z(\lambda) \subseteq U$. Using (5.2), we can choose $n \leq -|\lambda|$ and $\gamma \in G$ such that $r(\gamma) = y$ and $\alpha^n(s(\gamma)) \in V$. Choose $\lambda' \in E^*$ such that $|\lambda\lambda'| = -n$. Let $h := (x, n, \lambda\lambda'x) \in H_\infty$. Then $r(h) = x$ and $s(h) \in Z(\lambda) \subseteq U$. So $\xi := (h, \gamma)$ satisfies $r(\xi) = (x, y)$ and $s(\xi) = (s(h), \alpha^n(s(\gamma))) \in U \times V$ as required.

The final statement follows immediately: if G is minimal, then $[y] \subseteq \bigcup_{n \leq 0} \alpha^n([y])$ is dense in $G^{(0)}$ for every y. $\qquad\square$

The idea of our construction is that the C^*-algebra $C^*(G_\alpha^\infty)$ is something like a twisted tensor product of $C^*(G)$ with $\mathcal{O}_\infty$, and so we can expect it frequently to be purely infinite even when $C^*(G)$ is not. While there is not as yet a completely satisfactory characterisation of the groupoids G whose C^*-algebras are purely infinite (but see [4]), there is a very useful sufficient condition due to Anantharaman-Delaroche [1, Proposition 2.4]. Our next result provides conditions on G and α that ensure G_α^∞ satisfies this condition. Our proof requires the additional assumption that G is ample (this is our main source of examples in any case); but we expect that a similar result should hold more generally.

Recall that an ample groupoid L is *locally contracting* if for every nonempty open set $W \subseteq L^{(0)}$, there exists a compact open bisection B such that

$$r(B) \subsetneq s(B) \subseteq W.$$

The groupoid H_∞ is certainly locally contracting: by Lemma 3.3, each nonempty open set in $W \subseteq H_\infty^{(0)}$ contains a set of the form $Z(\lambda)$, and then $B = Z(\lambda e_1, \lambda)$ satisfies $r(B) = Z(\lambda e_1) \subsetneq Z(\lambda) = s(B) \subseteq W$.

Lemma 5.4. *Let G be a locally compact, Hausdorff, étale groupoid, α an automorphism of G, and G_α^∞ the groupoid (3.2). Suppose that G is ample and $\mathcal{B}$ is a basis for the topology on $G^{(0)}$ consisting of compact open bisections such that*

$$(5.3) \qquad \text{for every } V \in \mathcal{B} \text{ there exists } l \geq 1 \text{ such that } \alpha^{-l}(V) \subseteq V.$$

Then G_α^∞ is locally contracting.

Proof. Let W be a nonempty open set in $(G_\alpha^\infty)^{(0)} = H_\infty^{(0)} \times G^{(0)}$. Then there exist nonempty clopen subsets $V_H \subseteq H_\infty^{(0)}$ and $V_G \subseteq G^{(0)}$ such that $V_G \in \mathcal{B}$ and $V_H \times V_G \subseteq W$. By Lemma 3.3, there exists $\lambda \in E^*$ such that $Z(\lambda) \subseteq V_H$. Since each $Z(\lambda e_i) \subseteq Z(\lambda)$, we can assume that $|\lambda| \geq 1$. Define $N := |\lambda|$.

Fix $l \geq 1$ such that $\alpha^{-l}(V_G) \subseteq V_G$. Then $\alpha^{-Nl}(V_G) \subseteq V_G$. Let λ^l denote the finite path $\lambda\lambda\cdots\lambda$ in which λ is repeated l times. Let $U := Z(\lambda^{2l}, \lambda^l)$. Then

$$r(U) = Z(\lambda^{2l}) \text{ and } s(U) = Z(\lambda^l),$$

and we also have $U \subseteq c^{-1}(Nl)$.

Let $B := U \times \alpha^{-Nl}(V_G) \subseteq G_\alpha^\infty$. Then B is a compact open bisection, and

$$r(B) = r(U) \times \alpha^{-Nl}(V_G) \subsetneq Z(\lambda^l) \times V_G = Z(\lambda^l) \times \alpha^{Nl}(\alpha^{-Nl}(V_G)) = s(B) \subseteq W.$$

Hence G_α^∞ is locally contracting. $\qquad\square$

Putting all this together, we obtain our main result.

Theorem 5.5. *Let G be a second-countable, amenable, locally compact, Hausdorff, étale groupoid, α an automorphism of G satisfying conditions (5.1) and (5.3), and G_α^∞ the groupoid (3.2). Then G_α^∞ is also a second-countable, amenable, locally compact, Hausdorff, étale groupoid. Furthermore*

(1) the inclusion $\iota_G : C^*(G) \to C^*(G_\alpha^\infty)$ of Lemma 4.1 induces a KK-equivalence;

(2) if G principal, then so is G_α^∞;

(3) if G ample, then G_α^∞ is ample and locally contracting; and

(4) if G minimal, so is G_α^∞.

In particular, if G is principal, ample and minimal, then $C^*(G_\alpha^\infty)$ is a Kirchberg algebra in the UCT class with the same K-theory as $C^*(G)$.

Proof. The first statement and (3) follow from Remark 2.1 and Lemma 3.2, Lemma 3.5 and Lemma 5.4. Item (1) follows from Corollary 4.7, item (2) follows from Lemma 5.1, and item (4) follows from the final statement of Proposition 5.3.

For the final statement observe that $C^*(G_\alpha^\infty)$ is nuclear and in the UCT class by Lemma 3.5. It is separable because G_α^∞ is second countable by Lemma 3.2. Items (2) and (4) combine with [25, Corollary 4.6] (see also [3, Theorem 5.1]) to show that $C^*(G_\alpha^\infty)$ is simple, and Lemma 5.4 combines with [1, Proposition 2.4] to show that it is purely infinite. Item (1) (or Corollary 4.7) says that $C^*(G_\alpha^\infty)$ has the same K-theory as $C^*(G)$, completing the proof. $\qquad\square$

It will be useful to be able to stabilise the C^*-algebras of the groupoids G appearing in Theorem 5.5. The following result allows us to do so. We define K to be the complete equivalence relation on $\mathbb{Z}$ regarded as a discrete groupoid. So $C^*(K)$ is canonically isomorphic to $\mathcal{K}(\ell^2(\mathbb{Z}))$ (see for example Theorem 3.1 of [17]).

Lemma 5.6. *Suppose that G is a second-countable, amenable, locally compact, Hausdorff, étale groupoid, and let α be an automorphism of G that satisfies conditions (5.1) and (5.3). Then the cartesian product $G \times K$ and the automorphism $\alpha \times \mathrm{id} : G \times K \to G \times K$ given by*

$$(\alpha \times \mathrm{id})(g, (m, n)) = (\alpha(g), (m, n))$$

also have all these properties. Moreover,

$$C^*(G \times K) \cong C^*(G) \otimes \mathcal{K}(\ell^2(\mathbb{Z})),$$

and $G \times K$ is principal if G is principal, minimal if G is minimal, and ample if G is ample.

Proof. The cartesian product of second-countable locally compact Hausdorff spaces is itself second-countable, locally compact and Hausdorff. Since $r : G \to G^{(0)}$ and $r' : K \to K^{(0)}$ are local homeomorphisms the product map $r \times r' : G \times K \to (G \times K)^{(0)}$ is too, and so $G \times K$ is étale. Since G is amenable, $C^*(G) = C_r^*(G)$ is nuclear. By definition of K, the associated C^*-algebra $C^*(K) = C_r^*(K)$ is nuclear and simple. Using [5, Proposition 10.1.5], we see that $C_r^*(G \times K) \cong C_r^*(G) \otimes C_r^*(K)$ is nuclear, and then $G \times K$ is amenable by [2, Corollary 6.2.14(ii)]. Since $C^*(K)$ is canonically isomorphic to $\mathcal{K}(\ell^2(\mathbb{Z}))$, the C^*-algebra $C^*(G \times K)$ is canonically isomorphic to $C^*(G) \otimes C^*(K) = C^*(G) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. For the final statement, note that since K is principal, minimal and ample, taking cartesian products with K preserves these properties. $\quad\square$

6. Examples from Bratteli diagrams

The hypotheses of Theorem 5.5 may appear restrictive, but we show in this and the subsequent section that there are many examples. By the Kirchberg–Phillips classification theorem, for each simple dimension group $D \neq \mathbb{Z}$ and each $d \in D^+$ there is a unique Kirchberg algebra A_D with $(K_0(A_D), [1_{A_D}], K_1(A_D)) \cong (D, d, \{0\})$. In this section we show that A_D can be realised as the C^*-algebra of an amenable, principal, ample groupoid with compact unit space.

Theorem 6.1. *Let D be a simple dimension group other than $\mathbb{Z}$. Let A be a UCT Kirchberg algebra with trivial K_1-group and suppose that there is an isomorphism $\pi : K_0(A) \to D$. If A is unital, suppose further that $\pi([1_A]) \in D^+$. Then there is an amenable, principal, ample groupoid $\mathcal{G}$ such that $A \cong C^*(\mathcal{G})$. If A is unital, then $\mathcal{G}^{(0)}$ is compact.*

The remainder of this section will be spent proving Theorem 6.1. Our convention is that a *Bratteli diagram* is a row-finite directed graph whose vertex set E^0 is partitioned into finite sets

V_n, $n = 0, 1, 2, \ldots$ and whose edge set is partitioned as $E^1 = \bigcup_{n=0}^{\infty} V_n E^1 V_{n+1}$. We will further assume that $vE^1 \neq \emptyset$ for all $v \in E^0$ and that $E^1 v \neq \emptyset$ for all $v \in E^0 \setminus V_0$.

Lemma 6.2. *Let D be a simple dimension group other than $\mathbb{Z}$. Then there exists a Bratteli diagram E such that $K_0(C^*(E)) \cong D$, $K_1(C^*(E)) \cong \{0\}$ and $k_{vw} := |vEw| > n$ for each $n \in \mathbb{N}$, $v \in V_n$ and $w \in V_{n+1}$. Let G denote the graph groupoid (2.2) of E. Then G is a second-countable, amenable, locally compact, Hausdorff, ample groupoid that is minimal and principal.*

Proof. Let B be an AF algebra whose K_0-group is isomorphic to D. Then $K_1(B) = \{0\}$ because B is AF. By [6, Corollary IV.5.2], B is simple. Let E be a Bratteli diagram for B. By [35, Theorem 1] (see also [7, Theorem 1]), B is Morita equivalent to $C^*(E)$. By telescoping E, we can assume that for every $n \in \mathbb{N}$, every $v \in V_n$ and every $w \in V_{n+1}$, we have $k_{vw} = |vEw| > n$: this follows, for example, from Lemmas A4.3 and A4.4 of [8]—see the proof of Theorem 6.2(2) on page 163 of [18].

Since G is the path groupoid (2.2) associated to E, it is a second-countable, amenable, locally compact, Hausdorff, ample groupoid, and we have $C^*(E) \cong C^*(G)$ (see Remark 2.1). Since simplicity is preserved by Morita equivalence and B is simple, $C^*(G)$ is simple and hence G is minimal (see, for example, [3, Theorem 5.1]). Since E has no cycles, G is also principal by [12, Proposition 8.1]. $\square$

With E, k_{vw} and G as in Lemma 6.2, we label the edges in vEw as $\{(vw)_1, \ldots, (vw)_{k_{vw}}\}$. To define an automorphism α of G, first consider the graph automorphism, also called α, of E such that

$$\alpha((vw)_i) = (vw)_{((i+1) \bmod k_{vw})} \quad \text{for all } v \in V_n, w \in V_{n+1} \text{ and } i < k_{vw}.$$

Then α restricts to a permutation of the set vE^1w for $v \in V_n$ and $w \in V_{n+1}$, and α pointwise fixes E^0.

Lemma 6.3. *With G as in Lemma 6.2, and α as above, there is an automorphism α of G satisfying (5.1) and (5.3) such that $\alpha(x, m, y) = (\alpha(x), m, \alpha(y))$ for $(x, m, y) \in G$.*

Proof. Since α is an automorphism of E, it determines a bijection of $P = P_E$ by $\alpha(x)_i = \alpha(x_i)$ for $i \leq |x|$. This map clearly commutes with σ^n for all n, and so determines an algebraic automorphism α of G as described. Since this automorphism carries each basic open set $Z((\mu, \nu) \setminus F)$ bijectively onto $Z((\alpha(\mu), \alpha(\nu)) \setminus \alpha(F))$, it is a homeomorphism, and hence an automorphism of the topological groupoid G. In particular, we have $\alpha^l(Z(\mu)) = Z(\alpha^l(\mu))$. Since the orbit of any $\mu \in E^*$ under α is contained in the finite set $r(\mu)E^{|\mu|}s(\mu)$, it is finite, and so there exists $l > 0$ such that $\alpha^{-l}(\mu) = \mu$, giving $\alpha^{-l}(Z(\mu)) = Z(\mu)$. Since E is row-finite, $P_E = E^\infty$, so G and α satisfy (5.3) with $\mathcal{B} = \{Z(\mu) : \mu \in E^*\}$.

It remains to show α satisfies condition (5.1). Fix $x \in G^{(0)} = E^\infty$ and suppose $[x] = [\alpha^l(x)]$ for some $l \in \mathbb{Z}$. Then there exist m, n such that

$$\sigma^m(x) = \sigma^n(\alpha^l(x)).$$

Thus, $r(\sigma^m(x)) = r(\sigma^n(\alpha^l(x)))$. We have $r(x) \in V_j$ for some j, and since α fixes vertices of the graph E, we then have $r(\alpha^l(x)) = r(x) \in V_j$ as well. So $V_{j+m} \ni r(\sigma^m(x)) = r(\sigma^n(\alpha^l(x))) \in V_{j+n}$. Since the V_i are mutually disjoint, we obtain $m = n$. By replacing x with $\sigma^m(x)$, we may assume that $x = \alpha^l(x)$. For each $p \in \mathbb{N}$, we have $x(p, p+1) = \alpha^l((x)(p, p+1))$. For each p, let $v_p = r(x(p, p+1))$ and $w_p = s(x(p, p+1))$. Then $l = 0 \mod k_{v_p w_p}$ for all p. Lemma 6.2 shows that $k_{v_p w_p} \to \infty$, and it follows that $l = 0$. $\square$

Proof of Theorem 6.1. Let E be the Bratteli diagram of Lemma 6.2 and let α be the automorphism of the path groupoid G of E described in Lemma 6.3. By Lemmas 6.2 and 6.3, we have $K_*(A) \cong K_*(C^*(G))$, the groupoid G is second-countable, amenable, locally compact, Hausdorff, ample, minimal and principal, and α satisfies conditions (5.1) and (5.3).

Now Lemma 5.6 implies that $\widetilde{G} = G \times K$ and $\widetilde{\alpha} = \alpha \times \mathrm{id}$ have all the properties of G and α discussed in the preceding paragraph. It also implies that $C^*(\widetilde{G})$ is stable. Applying Theorem 5.5, we obtain a second-countable, amenable, locally compact, Hausdorff, ample groupoid $\widetilde{G}_\alpha^\infty$ that is minimal and principal and an inclusion $\iota_{\widetilde{G}} : C^*(\widetilde{G}) \to C^*(\widetilde{G}_\alpha^\infty)$ such that $K_*(\iota_{\widetilde{G}})$ is an isomorphism and $C^*(\widetilde{G}_\alpha^\infty)$ is a Kirchberg algebra in the UCT class.

First suppose that A is nonunital. Since $\mathcal{G} := \widetilde{G}_\alpha^\infty$ has noncompact unit space, $C^*(\widetilde{G}_\alpha^\infty)$ is also nonunital. So A and $C^*(\mathcal{G})$ are nonunital UCT Kirchberg algebras with the same K-theory because

$$K_*(A) \cong K_*(C^*(E)) \cong K_*(C^*(G)) \cong K_*(C^*(\widetilde{G})) \cong K_*(C^*(\widetilde{G}_\alpha^\infty)) = K_*(C^*(\mathcal{G})).$$

Hence $A \cong C^*(\mathcal{G})$ by the Kirchberg–Phillips classification theorem [14, 20].

Now suppose that A is unital. There is an order isomorphism $\phi : K_0(C^*(E)) \cong \varinjlim(\mathbb{Z}V_n, B_n)$ where $B_n \in M_{V_{n+1}, V_n}(\mathbb{N})$ is given by $B_n(v, w) = |vE^1w|$ (see, for example, [7]). So $D = \varinjlim(\mathbb{Z}V_n, B_n)$, and D^+ is the union of the images of the subsemigroups $\mathbb{N}V_n \subseteq \mathbb{Z}V_n$. Writing $B_{n,\infty} : \mathbb{Z}V_n \to \varinjlim(\mathbb{Z}V_n, B_n)$ for the canonical inclusion, we have $\phi([p_v]) = B_{n,\infty}(\delta_v)$ for $v \in V_n$. Since the isomorphism $C^*(G) \cong C^*(E)$ carries $1_{Z(v)}$ to p_v [16], we obtain an isomorphism $\tilde{\phi} : K_0(C^*(G)) \cong D$ such that $\tilde{\phi}(1_{Z(v)}) = B_{n,\infty}(\delta_v)$ for $v \in V_n$.

By assumption, $\pi([1_A]) \in D^+$, so there exist $n \geq 1$ and $a \in \mathbb{N}V_n$ such that $\pi([1_A]) = B_{n,\infty}(a)$. Define $V \subseteq \widetilde{G}^{(0)} = G^{(0)} \times \mathbb{Z}$ by $V := \bigcup_{v \in V_n} \bigcup_{1 \leq j \leq a(v)} Z(v) \times \{j\}$; observe that V is clopen. The canonical isomorphism $K_0(C^*(\widetilde{G})) \cong K_0(C^*(G))$ induced by the isomorphism $C^*(\widetilde{G}) \cong C^*(G) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ carries $[1_V]$ to $\sum_{v \in V_n} a(v)[1_{Z(v)}]$. So composing this isomorphism with $\tilde{\phi}$ gives an isomorphism $K_0(C^*(\widetilde{G})) \cong D$ that takes $[1_V]$ to $\pi([1_A])$.

Since $K_*(\iota_{\widetilde{G}}) : K_*(C^*(\widetilde{G})) \to K_*(C^*(\widetilde{G}_\alpha^\infty))$ is an isomorphism, it follows that

$$(K_0(A), 1_A) \cong (K_0(C^*(\widetilde{G}_\alpha^\infty)), [1_W]),$$

where $1_W = \iota_{\widetilde{G}}(1_V)$. Let $\mathcal{G}$ be the restriction $\widetilde{G}_\alpha^\infty|_W = \{g \in \widetilde{G}_\alpha^\infty : r(g), s(g) \in W\}$ of $\widetilde{G}_\alpha^\infty$ to the compact open subset W of its unit space. Then $C^*(\mathcal{G}) \cong 1_W C^*(\widetilde{G}_\alpha^\infty) 1_W$ is a corner of $C^*(\widetilde{G}_\alpha^\infty)$, which is full since $C^*(\widetilde{G}_\alpha^\infty)$ is simple. We therefore have

$$(K_0(C^*(\mathcal{G})), 1_{C^*(\mathcal{G})}) \cong (K_0(A), 1_A).$$

Since $\widetilde{G}_\alpha^\infty$ is amenable, principal and ample, so is $\mathcal{G}$. Since simplicity, nuclearity, separability, pure infiniteness and membership of the UCT class pass to full corners, $C^*(\mathcal{G})$ is a UCT Kirchberg algebra. So the Kirchberg–Phillips theorem [14, 20] gives $C^*(\mathcal{G}) \cong A$ as required. □

7. Examples from rank-2 Bratteli diagrams

In this section, we show that Theorem 5.5 can also be applied to groupoids associated to the rank-2 Bratteli diagrams of [18]. Recall that a matrix A with nonnegative entries is *proper* if each row and each column of A has at least one nonzero entry.

Theorem 7.1. *Let $\{c_n : n \in \mathbb{N}\}$ be positive integers. For each n, let $A_n, B_n \in M_{c_{n+1}, c_n}(\mathbb{N})$ be proper matrices, and let $T_n \in M_{c_n}(\mathbb{N})$ be a proper diagonal matrix. Suppose that $A_n T_n = T_{n+1} B_n$ for all n and $\varinjlim(\mathbb{Z}^{c_n}, A_n)$ is a simple dimension group not isomorphic to $\mathbb{Z}$. Let A be a Kirchberg algebra such that $K_0(A) \cong \varinjlim(\mathbb{Z}^{c_n}, A_n)$ and $K_1(A) \cong \varinjlim(\mathbb{Z}^{c_n}, B_n)$. Further suppose, if A is unital, that the isomorphism $K_0(A) \cong \varinjlim(\mathbb{Z}^{c_n}, A_n)$ carries $[1_A]$ to an element of the canonical positive cone of $\varinjlim(\mathbb{Z}^{c_n}, A_n)$. Then there exists an amenable, principal, ample groupoid $\mathcal{G}$ such that $C^*(\mathcal{G}) \cong A$. If A is unital, then $G^{(0)}$ is compact.*

The main ingredients in the proof of Theorem 7.1 are Lemmas 7.2–7.4. We briefly recall the notions involved, starting with rank-2 Bratteli diagrams. For a row-finite 2-graph Λ we will identify the *blue graph* $f_1^* \Lambda := \{(\lambda, n) : n \in \mathbb{N}, \lambda \in \Lambda^{ne_1}\}$ discussed in [18] with the subgraph $\Lambda^{\mathbb{N}e_1} \subseteq \Lambda$, and similarly we identify the *red graph* $f_2^* \Lambda$ with $\Lambda^{\mathbb{N}e_2} \subseteq \Lambda$. Following [18], a *rank-2*

Bratteli diagram (of infinite depth) is a row-finite 2-graph Λ such that Λ^0 is a disjoint union of nonempty finite sets or *levels* $(V_n)_{n=0}^\infty$ which satisfy:

(1) for every blue edge $e \in \Lambda^{e_1}$, there exists n such that $e \in V_n E^1 V_{n+1}$;
(2) the blue graph has no sources and all sinks belong to V_0; and
(3) every vertex of Λ lies on an *isolated cycle*[1] of the red graph, and $\Lambda^{e_2} = \bigcup_n V_n \Lambda^{e_2} V_n$.

The C^*-algebra $C^*(\Lambda)$ of a rank-2 Bratteli diagram Λ is an A$\mathbb{T}$ algebra. Its K-groups can be computed as follows: Recall that every V_n is a disjoint union of sets $\bigcup_{i=1}^{c_n} V_{n,i}$ where each $V_{n,i}$ consists of the vertices on an isolated red cycle. For each $0 \le n$, $1 \le j \le c_n$, and $1 \le i \le c_{n+1}$ define

$$A_n(i,j) := |\{v\Lambda^{e_1}V_{n+1,i}\}| \quad \text{and} \quad B_n(i,j) := |\{V_{n,j}\Lambda^{e_1}w\}|,$$

for any choice of $v \in V_{n,j}$, $w \in V_{n+1,i}$ (the result is independent of these choices). By [18, Theorem 4.3], there is an isomorphism $K_0(C^*(\Lambda)) \cong \lim(\mathbb{Z}^{c_n}, A_n)$ that carries s_v to $\delta_j \in \mathbb{Z}^{c_n}$ if $v \in V_{n,j} \subseteq V_n$, and $K_1(C^*(\Lambda)) \cong \lim(\mathbb{Z}^{c_n}, B_n)$.

For each n, define a diagonal matrix $T_n \in M_{c_n}(\mathbb{N})$ by $T_n(j,j) := |V_{n,j}|$. Then $A_n T_n = T_{n+1} B_n$ for all n (see [18, Lemma 4.2]).

In a rank-2 Bratteli diagram Λ the factorisation property induces a permutation $\mathcal{F}$ of the edges of the blue graph: For each $e \in \Lambda^{e_1}$, let f be the unique element of $\Lambda^{e_2}r(e)$, and define $\mathcal{F}(e)$ to be the unique element of Λ^{e_1} such that $fe = \mathcal{F}(e)f'$ for some red edge f'. The *order* $o(e)$ of e is then defined to be the the smallest $k > 0$ such that $\mathcal{F}^k(e) = e$. Observe that if $e \in V_n$, then $o(e) \le |V_n\Lambda^{e_1}|$ is finite since Λ is row-finite and V_n is finite. For each $n \in \mathbb{N}$, define

$$O_n := \text{lcm}\{o(e) \mid e \in V_n\Lambda^{e_1}\},$$

and m_n (inductively) by $m_0 := 0$ and $m_{n+1} := m_n + nO_n$ for $n \ge 0$ (see Figure 1 for an illustrative example).

Lemma 7.2. *Let $\{c_n : n \in \mathbb{N}\}$ be positive integers. For each n, let $A_n, B_n \in M_{c_{n+1},c_n}(\mathbb{N})$ be proper matrices, and let $T_n \in M_{c_n}(\mathbb{N})$ be a proper diagonal matrix. Suppose that $A_n T_n = T_{n+1} B_n$ for all n and that $\lim(\mathbb{Z}^{c_n}, A_n)$ is a simple dimension group not isomorphic to $\mathbb{Z}$. Then there exists a rank-2 Bratteli diagram Λ with $\Lambda^0 = \bigcup_{n=0}^\infty V_n$ such that $K_0(C^*(\Lambda)) \cong \lim(\mathbb{Z}^{c_n}, A_n)$, $K_1(C^*(\Lambda)) \cong \lim(\mathbb{Z}^{c_n}, B_n)$ and*

$$o(e) > n \cdot m_n,$$

for every $n \in \mathbb{N}$ and $e \in V_n\Lambda^{e_1}$.

Proof. We follow the proof of [18, Theorem 6.2(2)] modulo a small adjustment of the construction of the subsequence $(l(n))_{n=1}^\infty$ which we define as follows:

For $n \ge m$ define $A_{n,m} := A_{n-1}A_{n-2}\ldots A_m$. As in [18] we can find a subsequence $(l'(i))_{i=1}^\infty$ of $\mathbb{N}$ with the property that every entry of $A_{l'(n+1),l'(n)}$ is at least n, and hence

(7.1) $$\lim_{k\to\infty} \min_{p,q}\{A_{l'(i+k),l'(i)}(p,q)\} = \infty \qquad \text{for all } i.$$

Let $l(0) := l'(1)$. Using induction we construct sequences M_n and $l(n+1)$, for all $n \in \mathbb{N}$. Step 0: Put $M_0 := 0$ and $l(0+1) := l'(2)$. Step 1: Put $M_1 := 0$ and $l(1+1) := l'(3)$. For the induction step $n+1$, put $M_{n+1} := M_n + n \cdot \prod_{i,j} A_{l(n+1),l(n)}(i,j)T_{l(n)}(j,j)$ and use (7.1) to find $l(n+2) > l(n+1)$ such that every entry of $A_{l(n+2),l(n+1)}$ is strictly grater than $(n+1)M_{n+1}$.

Proceeding as in the proof of [18, Theorem 6.2(2)] we let Λ be the rank-2 Bratteli diagram obtained by applying [18, Proposition 6.4] to the data

$$c_n' := c_{l(n)}, \quad A_n' := A_{l(n+1),l(n)}, \quad B_n' := B_{l(n+1),l(n)} \quad \text{and} \quad T_n' := T_{l(n)}$$

for $n \in \mathbb{N}$. As in [18] it follows that $K_0(C^*(\Lambda)) \cong \lim(\mathbb{Z}^{c_n}, A_n)$ and $K_1(C^*(\Lambda)) \cong \lim(\mathbb{Z}^{c_n}, B_n)$.

---

[1]A *cycle* in a k-graph Λ is an element $\lambda \in \Lambda$ with the property that $d(\lambda) \ne 0$, $r(\lambda) = s(\lambda)$ and if $\lambda = \lambda'\lambda''$ with $0 < d(\lambda') < d(\lambda)$, then $s(\lambda') \ne s(\lambda)$. The cycle λ is *isolated* if, whenever $\lambda = \lambda'\lambda''$ with $d(\lambda') \ne 0$, the sets

$$r(\lambda)\Lambda^{d(\lambda')} \setminus \{\lambda'\} \qquad \text{and} \qquad \Lambda^{d(\lambda'')}s(\lambda) \setminus \{\lambda''\}$$

are both empty. See [18, p. 140].

We now prove that $o(e) > n \cdot M_n \geq n \cdot m_n$ for every $n \in \mathbb{N}$ and $e \in V_n \Lambda^{e_1}$ using induction. The statement is trivial when $n = 0$ and 1 because each $o(e) \geq 1$ and $M_1 = m_1 = 0$. For the induction step, we first consider $e \in V_n \Lambda^{e_1}$, say $e \in V_{n,j} \Lambda V_{n+1,i}$. By [18, Proposition 6.4],

$$(7.2) \qquad o(e) = A'_n(i,j)|V_{n,j}| = A'_n(i,j)T'_n(j,j) = A_{l(n+1),l(n)}(i,j)T_{l(n)}(j,j).$$

Hence

$$m_{n+1} = m_n + n \cdot \mathrm{lcm}\{o(e) \mid e \in V_n \Lambda^{e_1}\} \leq M_n + n \cdot \prod_{i,j} A_{l(n+1),l(n)}(i,j)T_{l(n)}(j,j) = M_{n+1},$$

where we have used that $m_n \leq M_n$.

Secondly we consider any $e \in V_{n+1}\Lambda^{e_1}$. Since $T_{l(n)}(j,j)$ is positive, we know from (7.2) that $o(e) \geq A_{l(n+2),l(n+1)}(i,j)$ for some (in fact for any) i, j. By construction every entry of $A_{l(n+2),l(n+1)}$ is strictly grater than $(n+1)M_{n+1}$, so

$$o(e) > (n+1)M_{n+1} \geq (n+1) \cdot m_{n+1}. \qquad \square$$

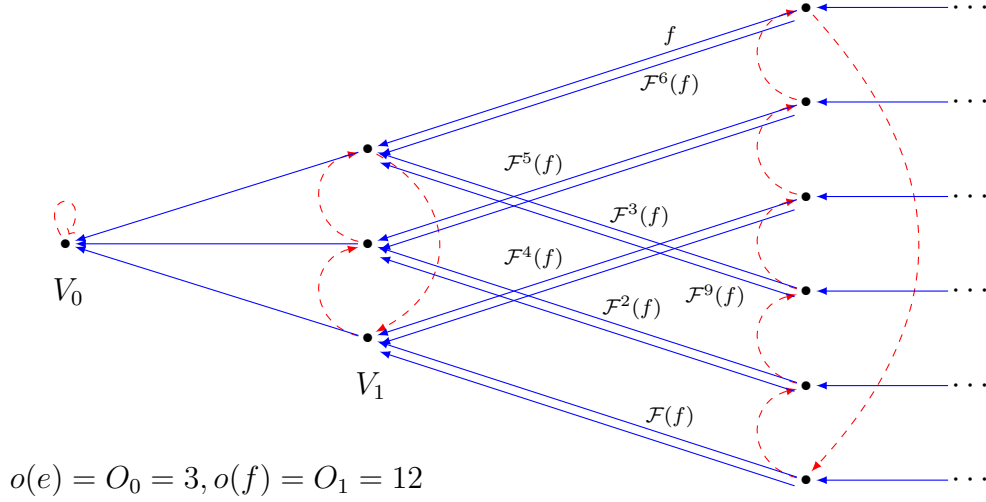

$$o(e) = O_0 = 3, o(f) = O_1 = 12$$

FIGURE 1. Since $f \in V_1 \Lambda^{e_1}$, we have $r(\mathcal{F}^{lO_1}(f)) = r(f)$ for any $l \in \mathbb{N}$. Indeed, $r(\mathcal{F}^{lO_0}(f)) = r(f)$ for any $l \in \mathbb{N}$.

We now recall the definition of the groupoid G associated to a rank-2 Bratteli diagram Λ as defined in [15]. Denote by Ω_k the k-graph with vertices $\Omega_k^0 := \mathbb{N}^k$, paths $\Omega_k^m := \{(n, n+m), n \in \mathbb{N}^k\}$ for $m \in \mathbb{N}^k$, $r((n, n+m)) = n$ and $s((n, n+m)) = n + m$. The *infinite paths* in a k-graph Λ with no sources are degree-preserving functors $x \colon \Omega_k \to \Lambda$. The collection of all infinite paths of Λ is denoted Λ^∞, and we write $r(x)$ for the vertex $x(0)$ and call it the the range of x. For $p \in \mathbb{N}^k$ and $x \in \Lambda^\infty$, $\sigma^p(x) \in \Lambda^\infty$ is defined by $\sigma^p(x)(m,n) := x(m+p, n+p)$ [15, Definition 2.1]. For $\lambda \in \Lambda$ and $x \in \Lambda^\infty$ with $s(\lambda) = r(x)$ we write λx for the unique element $y \in \Lambda^\infty$ such that $\lambda = y(0, d(\lambda))$ and $x = \sigma^{d(\lambda)} y$. Then, as a set,

$$G = \{(x, n, y) \in \Lambda^\infty \times \mathbb{Z}^2 \times \Lambda^\infty \colon \sigma^l(x) = \sigma^m(y), n = l - m\}.$$

Composition and inverse are given by $(x, n, y)(y, l, z) = (x, n+l, z)$ and $(x, n, y)^{-1} = (y, -n, x)$, implying $r(x, n, y) = x$, $s(x, n, y) = y$. The topology on the groupoid G has basic open sets $Z(\lambda, \mu) := \{(\lambda z, d(\lambda) - d(\mu), \mu z) \colon z \in s(\lambda)\Lambda^\infty\}$, indexed by pairs $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$. We call the topological groupoid G the *path groupoid* of Λ. It is second countable, amenable, locally compact, Hausdorff and ample.

Lemma 7.3. *Let Λ be a rank-2 Bratteli diagram. There is a unique automorphism α of Λ such that $\alpha(e) = \mathcal{F}^{m_n}(e)$ for all $e \in V_n \Lambda^{e_1}$. This α induces a homeomorphism, also denoted α, of*

Λ^∞ by $\alpha(x) := \alpha \circ x$, which in turn induces an automorphism α of G such that

$$\alpha((x, m, y)) = (\alpha(x), m, \alpha(y)).$$

Proof. There is at most one automorphism with the desired property because each $v\Lambda^{e_2}$ is a singleton and Λ is generated as a category by $\Lambda^{e_1} \cup \Lambda^{e_2}$. To show that there exist such an $\alpha \colon \Lambda \to \Lambda$, first we consider $e, f \in \Lambda^{e_1}$ with $s(e) = r(f)$. We show $s(\alpha(e)) = r(\alpha(f))$. Let n denote the level of $r(e)$. Then $r(f)$ is on level $n + 1$. Since $\mathcal{F}$ is a morphism and $\mathcal{F}^{O_n}(e) = e$ we get

$$r(\alpha(f)) = r(\mathcal{F}^{m_{n+1}}(f)) = s(\mathcal{F}^{m_{n+1}}(e)) = s(\mathcal{F}^{m_n+nO_n}(e)) = s(\mathcal{F}^{m_n}(e)) = s(\alpha(e)).$$

Thus we can extend α edgewise so that it is defined on any blue path. Because Λ is a rank-2 Bratelli diagram, α extends to an automorphism of Λ as follows: for $\lambda \in \Lambda$, factor λ so that $\lambda = \lambda_1 \lambda_2$ where λ_1 is blue and λ_2 is red. Then $\alpha(\lambda_1)$ makes sense and we define $\alpha(\lambda_2)$ to be the unique red path with degree $d(\lambda_2)$ and range equal to $s(\alpha(\lambda_1))$. Now it is routine to check that the formula $\alpha(\lambda) := \alpha(\lambda_1)\alpha(\lambda_2)$ is bijective and preserves composition and hence defines an automorphism of Λ.

Now α induces a homeomorphism $\alpha \colon \Lambda^\infty \to \Lambda^\infty$ by $\alpha(x) := \alpha \circ x$. One checks that $\sigma^m \circ \alpha = \alpha \circ \sigma^m$ for all $m \in \mathbb{N}^k$, and so there is an algebraic automorphism α of G given by

$$\alpha((x, m, y)) = (\alpha(x), m, \alpha(y)).$$

Since $\alpha \colon \Lambda^\infty \to \Lambda^\infty$ carries each $Z(\lambda, \mu)$ bijectively onto $Z(\alpha(\lambda), \alpha(\mu))$, α is a homeomorphism, and hence an automorphism of the topological groupoid G. □

Lemma 7.4. *Let Λ be a rank-2 Bratteli diagram. Suppose that $o(e) > n \cdot m_n$ for every $n \in \mathbb{N}$ and $e \in V_n \Lambda^{e_1}$. Then the automorphism α of G given by Lemma 7.3 satisfies conditions (5.1) and (5.3), and G is principal and minimal.*

Proof. To establish (5.1) and prove that G is principal, fix $x \in \Lambda^\infty$ and $l \in \mathbb{Z}$ such that $[x] = [\alpha^l(x)]$. We claim that $l = 0$. The shift-tail equivalence class of x is given by

$$[x] := \{\lambda\sigma^p(x) \colon p \in \mathbb{N}^2, \lambda \in \Lambda x(p)\}.$$

In particular $\sigma^p(x) = \sigma^q(\alpha^l(x))$ for some $p, q \in \mathbb{N}^2$. Let n denote the level of $r(x)$. The map σ^p increases the level of the range of an infinite path by p_1, so $r(\sigma^p(x)) \in V_{n+p_1}$. On the other hand α does not change the level, so $r(\sigma^q(\alpha^l(x))) \in V_{n+q_1}$, ensuring $p_1 = q_1$. Without loss of generality we may assume that $q_2 \geq p_2$ (if not, replace l by $-l$ and interchange p, q). Define $y = \sigma^p(x)$ and $s = q_2 - p_2$. Then $y = \sigma^p(x) = \sigma^{q-p+p}(\alpha^l(x)) = \sigma^{(0,s)}\alpha^l(y)$. Now observe that for $r \in \mathbb{N}$, $y(re_1, re_1 + e_1 + e_2) = ef = f'e'$ for some $e, e' \in \Lambda^{e_1}$, $f, f' \in \Lambda^{e_2}$. It follows that $\sigma^{e_2}y(re_1, re_1 + e_1) = e'$, while $\mathcal{F}(e') = y(re_1, re_1 + e_1)$. So for $t \geq n + p_1$, putting $r_t := (t - n - p_1)e_1$ and $f_t := y(r_t, r_t + e_1)$, we have

$$f_t = y(r_t, r_t + e_1) = (\sigma^{(0,s)}\alpha^l(y))(r_t, r_t + e_1) = \alpha^l(y(r_t, r_t + e_1 + se_2)).$$

Since $y(r_t) \in V_t$, and by definition of α, we have $\alpha^l(y(r_t, r_t+e_1+se_2)) = \mathcal{F}^{lm_t}(y(r_t, r_t+e_1+se_2))$, and this in turn is equal to $\mathcal{F}^{lm_t-s}(y(r_t, r_t+e_1))$ because $\mathcal{F}^{-s}(y(r_t, r_t+e_1)) = \sigma^{(0,s)}y(r_t, r_t+e_1)$. Putting all this together, we obtain $f_t = \mathcal{F}^{lm_t-s}(f_t)$.

Hence, for each $t > n + p_1$ we have $lm_t - s = 0 \bmod o(f_t)$. By assumption, we have $t \cdot m_t < o(f_t)$ for all t, and hence $|lm_t - s| < o(f_t)$ for large t. This forces $lm_t - s = 0$ for large t. Since $m_t \to \infty$, this forces $l = 0$ establishing (5.1). It also forces $s = 0$ and hence $p_2 = q_2$. So if $\sigma^p(x) = \sigma^q(x)$ then $p = q$, and this implies that $(x, m, x) \in G$ only for $m = 0$, so G is principal.

To establish (5.3), for $\lambda \in \Lambda$ define $Z(\lambda) := \{\lambda z \colon z \in s(\lambda)\Lambda^\infty\}$. Recall that Λ^∞ is endowed with the topology generated by the collection $\mathcal{B} := \{Z(\lambda) \colon \lambda \in \Lambda\}$. Take any $\lambda \in \Lambda$. We claim that $\alpha^{-l}(Z(\lambda)) \subseteq Z(\lambda)$ for some $l \in \mathbb{N} \setminus \{0\}$. To see this, let n, i be the pair with $r(\lambda) \in V_{n,i}$. Since $\mathcal{F}$ permutes the elements of each $V_{n,j}$, so does α and hence each α^l is a permutation of

the finite set $V_{n,i}\Lambda^{d(\lambda)}$. In particular $\alpha^l(\lambda) = \lambda$ for some $l > 0$. Hence for each $x \in Z(\lambda)$, with $x = \lambda z$, we have $\alpha^{-l}(x) = \alpha^{-l}(\lambda z) = \alpha^{-l}(\lambda)\alpha^{-l}(z) = \lambda\alpha^{-l}(z) \in Z(\lambda)$.

It remains to check that G is minimal, which follows from [3, Theorem 5.1] because $C^*(G)$ is simple by [18, Theorem 5.1]. $\qquad\square$

Proof of Theorem 7.1. Using Lemma 7.2, we construct a rank-2 Bratteli diagram Λ with $\Lambda^0 = \bigcup_{n=0}^\infty V_n$ such that $K_0(C^*(\Lambda)) \cong \varinjlim(\mathbb{Z}^{c_n}, A_n)$, $K_1(C^*(\Lambda)) \cong \varinjlim(\mathbb{Z}^{c_n}, B_n)$ and $o(e) > n \cdot m_n$ for every $n \in \mathbb{N}$ and $e \in V_n\Lambda^{e_1}$.

Let G denote the path groupoid of Λ. By construction G is a second countable, amenable, locally compact, Hausdorff, ample groupoid and $C^*(\Lambda) \cong C^*(G)$ [15]. By Lemmas 7.3 and 7.4, there is an automorphism α of G that satisfies conditions (5.1) and (5.3), and G is principal and minimal, $K_*(\iota_{\widetilde{G}})$ is an isomorphism, and $C^*(\widetilde{G}_\alpha^\infty)$ is a Kirchberg algebra in the UCT class.

We now proceed as in the proof of Theorem 6.1. Lemma 5.6 shows that $\widetilde{G} := G \times K$ and $\widetilde{\alpha} := \alpha \times \mathrm{id}$ have all the properties established for G and α in the preceding paragraphs. We apply Theorem 5.5 to see that $\widetilde{G}_\alpha^\infty$ is second-countable, amenable, locally compact, Hausdorff, ample groupoid that is principal and minimal. Moreover, $C^*(\widetilde{G})$ is nonunital because $\widetilde{G}^{(0)} = G^{(0)} \times \mathbb{Z}$ is noncompact.

First suppose that A is nonunital. Then A and $C^*(\widetilde{G}_\alpha^\infty)$ are nonunital Kirchberg algebras with the same K-theory:

$$K_0(A) \cong \varinjlim(\mathbb{Z}^{c_n}, A_n) \cong K_0(C^*(\Lambda)) \cong K_0(C^*(G)) \cong K_0(C^*(\widetilde{G})) \cong K_0(C^*(\widetilde{G}_\alpha^\infty)),$$

and a similar calculation gives $K_1(A) \cong K_1(C^*(\widetilde{G}_\alpha^\infty))$. Hence $A \cong C^*(\widetilde{G}_\alpha^\infty)$ by the Kirchberg–Phillips theorem [14, 20].

Now suppose that A is unital. Then by assumption, the isomorphism $K_0(A) \cong \varinjlim(\mathbb{Z}^{c_n}, A_n)$ carries $[1_A]$ to an element of the positive cone, and hence the class of some $a \in \overrightarrow{\mathbb{N}^{c_n}} \subseteq \mathbb{Z}^{c_n}$. For each $j \leq c_n$ chose $v_j \in V_{n,j}$. As discussed on page 16 above, Theorem 6.2(2) of [18] shows that the isomorphism $\varinjlim(\mathbb{Z}^{c_n}, A_n) \cong K_0(C^*(G))$ takes a to $\sum_{j \leq c_n} a(j)[1_{Z(v_j)}]$. So the isomorphism $\varinjlim(\mathbb{Z}^{c_n}, A_n) \cong K_0(C^*(\widetilde{G}))$ carries a to $\sum_{j \leq c_n}\sum_{k \leq a(j)}[1_{Z(v_j)\times\{k\}}]$, which is the class of a characteristic function 1_V of a clopen subset V of $\widetilde{G}^{(0)}$. So the argument of the final few paragraphs of the proof of Theorem 6.1 shows that $\mathcal{G} := W\widetilde{G}_\alpha^\infty W$ with $W := H_\infty^{(0)} \times V$ has the desired properties. $\qquad\square$

References

[1] C. Anantharaman-Delaroche, *Purely infinite C^*-algebras arising from dynamical systems*, Bull. Soc. Math. France **125** (1997), 199–225.

[2] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, With a foreword by Georges Skandalis and Appendix B by E. Germain, L'Enseignement Mathématique, Geneva, 2000, 196pp.

[3] J.H. Brown, L.O. Clark, C. Farthing and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.

[4] J.H. Brown, L.O. Clark and A. Sierakowski, *Purely infinite C^*-algebras associated to étale groupoids*, Ergodic Theory Dynam. Systems **35** (2015), 2397–2411.

[5] N. Brown and N. Ozawa, C^*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

[6] K.R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996. xiv+309 pp.

[7] D. Drinen, *Viewing AF-algebras as graph algebras*, Proc. Amer. Math. Soc. **128** (2000), 1991–2000.

[8] E.G. Effros, Dimensions and C^*-algebras, Conference Board of the Mathematical Sciences, Washington, D.C., 1981, v+74pp.

[9] G.A. Elliott and A. Sierakowski, *The K-theory of certain purely infinite crossed products*, preprint 2011 (arXiv:1110.6614 [math.OA]).

[10] R. Exel, *Inverse semigroups and combinatorial C^*-algebras*, Bull. Braz. Math. Soc. **39** (2008), 191–313.

[11] N. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J. **48** (1999), 155–181.

[12] R. Hazelwood and A. an Huef, *Strength of convergence in the orbit space of a groupoid*, J. Math. Anal. Appl. **383** 2011, 1–24.

[13] T. Katsura, *A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. IV. Pure infiniteness*, J. Funct. Anal. **254** (2008), 1161–1187.

[14] E. Kirchberg, *The classification of purely infinite simple C^*-algebras using Kasparov's theory*, unpublished, 1994 (3rd draft).

[15] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.

[16] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids and Cuntz–Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.

[17] P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987), 3–22.

[18] D. Pask, I. Raeburn, M. Rørdam and A. Sims, *Rank-two graphs whose C^*-algebras are direct limits of circle algebras*, J. Funct. Anal. **239** (2006), 137–178.

[19] A. Paterson, *Graph inverse semigroups, groupoids and their C^*-algebras*, J. Operator Theory **48** (2002), 645–662.

[20] N.C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Documenta Math. **5** (2000), 49–114.

[21] M.V. Pimsner, A class of C^*-algebras generalizing both Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$, Free probability theory, 189–212, Fields Inst. Commun., **12**, Amer. Math. Soc., Providence, RI, 1997.

[22] I. Raeburn and W. Szymański, *Cuntz–Krieger algebras of infinite graphs and matrices*, Trans. Amer. Math. Soc. **356** (2003), 39–59.

[23] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Math. Surveys and Monographs, vol. 60, Amer. Math. Soc., Providence, 1998.

[24] J. Renault, A groupoid approach to C^*-algebras. Lecture Notes in Mathematics, **793**. Springer, Berlin, 1980. ii+160 pp.

[25] J. Renault, *The ideal structure of groupoid crossed product C^*-algebras*, J. Operator Theory **25** (1991), 3–36.

[26] J. Renault and D.P. Williams, *Amenability of groupoids arising from partial semigroup actions and topological higher rank graphs*, Trans. Amer. Math. Soc., to appear (arXiv:1501.03027 [math.OA]).

[27] A. Rennie, D. Robertson and A. Sims, *Groupoid algebras as Cuntz–Pimsner algebras*, Math. Scand., to appear (arXiv:1402.7126 [math.OA]).

[28] M. Rørdam, *Classification of nuclear, simple C^*-algebras.* Classification of nuclear C^*-algebras. Entropy in operator algebras, 1–145, Encyclopaedia Math. Sci., **126**, Springer, Berlin, 2002.

[29] M. Rørdam and A. Sierakowski, *Purely infinite C^*-algebras arising from crossed products*, Ergodic Theory Dynam. Systems **32** (2012), 273–293.

[30] E. Ruiz, A. Sims and A.P.W. Sørensen, *UCT Kirchberg algebras have nuclear dimension 1*, Adv. Math. **279** (2015), 1–28.

[31] J. Spielberg, *Groupoids and C^*-algebras for categories of paths*, Trans. Amer. Math. Soc. **366** (2014), 5771–5819.

[32] Y. Suzuki, *Amenable minimal Cantor systems of free groups arising from diagonal actions*, J. reine angew. Math. (2014), 31 pages. DOI:10.1515/crelle-2014-0071

[33] W. Szymański, *The range of K-invariants for C^*-algebras of infinite graphs*, Indiana Univ. Math. J. **51** (2002), 239–249.

[34] J.-L. Tu, *La conjecture de Baum-Connes pour les feuilletages moyennables*, K-Theory **17** (1999), 215–264.

[35] J. Tyler, *Every AF-algebra is Morita equivalent to a graph algebra*, Bull. Austral. Math. Soc. **69** (2004) 237–240.

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