IDEAL STRUCTURE OF C^*-ALGEBRAS OF COMMUTING LOCAL HOMEOMORPHISMS

KEVIN AGUYAR BRIX, TOKE MEIER CARLSEN, AND AIDAN SIMS

Abstract. We determine the primitive ideal space and hence the ideal lattice of a large class of separable groupoid C^*-algebras that includes all 2-graph C^*-algebras. A key ingredient is the notion of harmonious families of bisections in étale groupoids associated to finite families of commuting local homeomorphisms. Our results unify and recover all known results on ideal structure for crossed products of commutative C^*-algebras by free abelian groups, for graph C^*-algebras, and for Katsura's topological graph C^*-algebras.

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1. Introduction

Background. The lattice of ideals of a C^*-algebra is a fundamental structural feature that is notoriously difficult to compute. In the case of commutative C^*-algebras, type I C^*-algebras (e.g. continuous trace C^*-algebras), and other continuous fields of simple C^*-algebras, or just-infinite C^*-algebras [GMR18] the primitive ideal space is a key piece of data for classifying the C^*-algebras [GN43, DD63, GMR18]. However, in most cases where C^*-algebras are built from dynamical or combinatorial data such as shifts of finite type [CK80], local homeomorphisms [Ren80, Dea95, A-D97, ER07], or directed graphs

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(and their higher-rank analogues) [Rae05], work has focussed on conditions that ensure simplicity [KPR98, KP00], or that reduce the complexity of the ideal structure of the C^*-algebra of a dynamical system [Ren91, BHRS02, WvW21]; perhaps due to the Elliott classification program (see, for example, [Ell76, Ell93, Phi00, Kir95, TWW17, Win17]) whereby simple C^*-algebras can be classified by K-theory and traces.

Cuntz [Cu81] determined the ideal structure of non-simple Cuntz–Krieger algebras assuming condition (II) in terms of the irreducible components of the underlying shift of finite type. Cuntz' condition (II) ensures that all ideals are gauge-invariant (such ideals are called *dynamical* in our companion paper [BCS22]). The later groundbreaking work of an Huef and Raeburn [aHR97] further developed this technique and classified all gauge-invariant ideals of non-simple Cuntz–Krieger algebras. This was shortly followed by complete results for graph algebras [HS04], and very recently for topological graphs (via actions of $\mathbb{N}$ by local homeomorphisms) [Kat21]. The key idea of an Huef and Raeburn underpins both analyses: the primitive ideals are indexed by a quotient of $X \times \mathbb{T}$, where X is the space of infinite paths; and the hull-kernel topology is computed using a *sandwiching lemma*: each primitive ideal is sandwiched between a pair of gauge-invariant ideals for which the subquotient is Morita equivalent to a crossed product of the form $C_0(U) \rtimes \mathbb{Z}$.

As a result of a significant program dating back to work of Mackey [Mac51], Rieffel [Rie74], and Green [Gre80], the primitive-ideal spaces of such crossed products are well understood (see, for example, [Wi07]): if X is a second-countable locally compact Hausdorff space and G is a second-countable locally compact abelian group acting on X, then there is an equivalence relation on $X \times \widehat{G}$ such that $(x, \chi) \sim (y, \rho)$ if x and y have the same orbit closure, and $\chi\overline{\rho}$ annihilates the stabiliser of x. Each (x, χ) determines an irreducible representation $\pi_{x,\chi}$ of $C_0(X) \rtimes G$ on $L^2(G \cdot x)$, and the kernels of $\pi_{x,\chi}$ and $\pi_{y,\rho}$ coincide precisely if $(x, \chi) \sim (y, \rho)$. So $\pi\colon (x, \chi) \to \ker(\pi_{x,\chi})$ descends to a bijection from $(X \times \widehat{G})/{\sim}$ onto $\operatorname{Prim}(C_0(X) \rtimes G)$, and it transpires that this map is in fact a homeomorphism. Few other general results on ideal structure of C^*-algebras of dynamical systems like groupoids are available in the literature, beyond those such as Bönicke and Li's results [BL20] on strongly effective étale groupoids for which every ideal is dynamical, and those of van Wyk and Williams [WvW21] in which continuity conditions are imposed on the isotropy groups. Katsura's results [Kat21] are the furthest reaching results that do not impose regularity conditions on isotropy groups.

Our results. In this paper, we make substantial new progress on the problem of ideal structure in separable C^*-algebras of étale groupoids. Our main result Corollary 7.6 (see also Theorem 7.1) describes a base for the topology of the primitive ideal space, for a large class of separable C^*-algebras. The formal statement of the result is complicated, but we show by example that in many cases of interest our description is genuinely computable (see, for example, Section 11). In particular, our results are the first of their kind to cover large classes of higher-rank graph C^*-algebras including all row-finite 2-graphs with no sources.

The specific class of C^*-algebras that we study are those arising from actions $T\colon \mathbb{N}^k \curvearrowright X$ by local homeomorphisms of second-countable locally compact Hausdorff spaces X. All the cases mentioned above (actions by free abelian groups, shifts of finite type, directed graphs, and Katsura's topological graphs) provide examples of such actions. The associated topological groupoid G_T (sometimes referred to as the *Deaconu–Renault groupoid*) is well behaved in the sense that it is second-countable locally compact Hausdorff, amenable, and étale. Conceptually, G_T can be regarded as a proxy for the orbit space of T.

For such actions, Sims and Williams [SW16, Theorem 3.2] discover a surjective map

$$\pi\colon X \times \mathbb{T}^k \to \operatorname{Prim}(C^*(G_T))$$

and an equivalence relation on $X \times \mathbb{T}^k$ (akin to the equivalence relation in the case of crossed products mentioned above) such that $\pi(x, z) = \pi(x', z')$ precisely when (x, z) and (x', z') are equivalent. Importantly, the map π is a parameterisation and so does not *a priori* say anything about the hull-kernel topology on the primitive ideal space. The sandwiching lemma for general étale groupoids in our companion paper [BCS22, Lemma 3.6] applies to this setting, so every ideal is optimally sandwiched between dynamical ideals. However, the resulting subquotients do not admit natural descriptions as crossed products, and the approach of [aHR97, HS04, Kat21] does not naturally extend to this setting; moreover, the isotropy-group bundle is typically badly discontinuous, so the results of [WvW21] do not apply either. Without access to the powerful Mackey–Green–Rieffel machine, that has underpinned previous analyses, we needed a new approach.

We describe the ideals in $C^*(G_T)$ by directly analysing which subsets of $X \times \mathbb{T}^k$ are preimages under π of open sets in $\mathrm{Prim}(C^*(G_T))$. To do this, we introduce the *essential isotropy* (Definition 2.1) of G_T that will play a key role; in particular, it determines certain isotropy subgroups $\mathcal{J}_x$ over a unit x. We first describe a necessary condition for a subset A of $X \times \mathbb{T}^k$ to be the preimage of an open set in terms of a family $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_x}$ of homogeneous open bisections of G_T indexed by the isotropy subgroups $\mathcal{J}_x$ where x is a unit in the the projection of A onto X. Given any such family $(B_\alpha)_{\alpha \in \mathcal{J}_x}$ of bisections, we write $\mathcal{B}^{\mathrm{ess}}$ for its intersection with the essential isotropy. We identify a system of $\mathcal{B}$-*saturated* subsets $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp$ of $X \times \mathbb{T}^k$, indexed by pairs (U, V) consisting of an open neighbourood U of x and an open subset V of $\mathbb{T}^k$, with the property that if $A \subseteq X \times \mathbb{T}^k$ is the preimage of an open set of primitive ideals and contains a point $(x, z) \in X \times \mathbb{T}^k$, then there is a pair (U, V) such that $(x, z) \in U \times V$, and $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp$ is contained in A (Theorem 4.2). This, our first main result, applies to any G_T and gives useful information about ideal structure; in particular, it follows that π is continuous (Corollary 4.4).

Theorem. *Let X be a locally compact Hausdorff space and suppose that $T \colon \mathbb{N}^2 \curvearrowright X$ is an action by local homeomorphisms. The surjection $\pi \colon X \times \mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ described above (cf.* [SW16]*) is continuous.*

Next, we aim for a complete description of the ideal structure. The key idea is the notion of *harmonious families of bisections* (Definition 6.1). These are families of bisections $\mathcal{B}$ as above, whose intersections with the essential isotropy of G_T satisfy additional consistency conditions. These conditions allow us to employ harmonic analysis on $\mathbb{T}^k = \widehat{\mathbb{Z}}^k$ to prove a kind of noncommutative Urysohn lemma (Proposition 7.2): given a harmonious family of bisections $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_x}$, a point $z \in \mathbb{T}^k$ and open neighbourhoods U of x and V of z, we construct an element of $C^*(G_T)$ that does *not* belong to the ideal $\pi(x, z)$ but does belong to $\pi(y, w)$ for all (y, w) in the complement of $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp$. We believe this result is of independent interest, but here it is the engine room of our proof that the sets $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp$ are preimages of open sets (Theorem 7.1). Our main result (Corollary 7.6) uses this to describe a base for the topology on $\mathrm{Prim}(C^*(G_T))$:

Theorem. *Let X be a second-countable locally compact Hausdorff space and suppose that $T \colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits harmonious families of bisections. A base for the hull-kernel topology of $\mathrm{Prim}(C^*(G_T))$ is given by sets of the form*

$$\pi((U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp)$$

where $\mathcal{B}$ is a harmonious family of bisections at a unit x, U is an open neighbourhood of x, and V is open in $\mathbb{T}^k$.

In particular, we recover known results about effective groupoids: if G_T is minimal and effective then the primitive ideal space is a singleton (so $C^*(G_T)$ is simple); and if G_T is strongly effective, then the primitive ideal space is homeomorphic to the quasi-orbit space of G_T.

Using our main theorem, we provide an explicit description of the lattice of ideals of $C^*(G_T)$ in terms of subsets of $X \times \mathbb{T}^k$ (Proposition 8.2), and a characterisation of convergence of sequences in $\mathrm{Prim}(C^*(G_T))$ (Theorem 9.5).

At present, we do not know whether every action by commuting local homeomorphisms admits harmonious families of bisections, but we confirm this for actions by free abelian groups, (topological) graphs, and more importantly a large class of higher-rank graphs including all row-finite 2-graphs with no sources. These are the first general results for irreversible dynamical systems of rank greater than 1.

We conclude the paper with a detailed analysis of the case of higher-rank graph C^*-algebras [KP00], which were a significant motivator and source of examples for our work. Given a row-finite higher-rank graph Λ with no sources and with infinite-path space Λ^∞, the associated groupoid G_Λ is the Deaconu–Renault groupoid for the action $T\colon \mathbb{N}^k \curvearrowright \Lambda^\infty$ by shift maps. Our main theorem identifies the collection $\mathcal{A}_\Lambda$ of subsets of $\Lambda^\infty \times \mathbb{T}^k$ that are the preimages of open subsets of $\mathrm{Prim}(C^*(\Lambda))$ with an appropriate collection $\mathcal{D}_\Lambda$ of subsets of $\Lambda^0 \times \mathbb{T}^k$ (Corollary 11.7), so that such subsets index the ideals of $C^*(\Lambda)$. This result is in the spirit of [CS16, Theorem 5.1]. We finish by working through two concrete examples, completely determining the ideal structure of two 2-graph C^*-algebras that are not accessible to any pre-existing computations of ideal structure.

The ideas and techniques we develop here are flexible, and we suspect they can be applied to significantly larger classes of groupoid C^*-algebras, particularly when combined with the sandwiching lemma of [BCS22] as we do for the case of single local homeomorphisms in Section 10.4. Although we restrict our attention here to Deaconu–Renault groupoids, the notions of essential isotropy and of harmonious families of bisections make sense, and are potentially useful, for arbitrary étale groupoids.

Outline. We start in Section 2 by outlining necessary background and notation for topological groupoids with examples from dynamics and graphs, the ideal structure of separable C^*-algebras, the results of [SW16] on a parametrisation of primitive ideals in Deaconu–Renault-groupoid C^*-algebras, and harmonic analysis on $\mathbb{T}^k$. In Section 3 we describe and analyse a family of representations of Deaconu–Renault groupoids that interpolate between the regular representations and the orbit-space representations. These are a key ingredient in the proof of our main theorem. In Section 4, we establish our first main result (Theorem 4.2): a necessary condition for a subset of $X \times \mathbb{T}^k$ to be the preimage of an open subset of $\mathrm{Prim}(C^*(G_T))$. It follows that $\pi\colon X \times \mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ is continuous (Corollary 4.4). In Section 5, we clarify how the nested open invariant sets of our general sandwiching lemma [BCS22] relate to subsets of $X \times \mathbb{T}^k$, and we show that Deaconu–Renault groupoids admit obstruction ideals in the sense of [AL18, BCS22]. In Section 6 we introduce and study harmonious families of bisections, which are the main new tool we apply to study ideal structure. We identify two sufficient conditions to generate a harmonious family of bisections: one of these (Lemma 6.8) is particularly useful in ample groupoids such as those of (higher-rank) graphs; the other (Lemma 6.6) is applicable to a wider class of groupoids but requires more stringent hypotheses. In Section 7, we prove our main result (Theorem 7.1) using harmonious families of bisections: we determine the preimages in $X \times \mathbb{T}^k$ of open subsets of $\mathrm{Prim}(C^*(G_T))$ for Deaconu–Renault groupoids that admit harmonious families of bisections. We use this to describe the ideal lattice of the C^*-algebras of such groupoids in Section 8. We characterise convergence of sequences in $\mathrm{Prim}(C^*(G_T))$ in Section 9. Section 10 details a number of examples including all actions by commuting homeomorphisms, all graph C^*-algebras, and actions by a local homeomorphism on a second-countable locally compact Hausdorff space. Moreover, we show that many higher-rank graphs (including all 2-graphs) admit harmonious families of bisections. Finally, in Section 11 we use our main theorem to completely describe the ideal structure of the C^*-algebras of higher-rank graphs whose groupoids admit harmonious families of bisections.

2. Background material

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the integers and the nonnegative integers, respectively.

2.1. **Isotropy in étale groupoids.**

We use the notation and conventions for groupoids of [Sim20].

A *groupoid* is a small category in which every morphism is invertible. A groupoid G is a Hausdorff *étale* groupoid if it carries a locally compact Hausdorff topology with respect to which the range and source maps r and s are local homeomorphisms onto the unit space $G^{(0)} = \{r(\alpha) : \alpha \in G\}$, the inversion map $\alpha \mapsto \alpha^{-1}$ is continuous, and composition is continuous in the subspace topology on the space $G^{(2)}$ of composable pairs. If G is étale then the open subsets of G on which both r and s restrict to homeomorphisms form a basis for the topology; we call such sets *open bisections*. The unit space $G^{(0)}$ in a Hausdorff étale groupoid G is both closed and open in G.

The *orbit* $[x]$ of a unit $x \in G^{(0)}$ is the set $\{r(\gamma) : \gamma \in G, s(\gamma) = x\} \subseteq G^{(0)}$, and the *orbit closure* $\overline{[x]}$ is the closure of $[x]$ in $G^{(0)}$. The restriction $G|_{\overline{[x]}} = r^{-1}\big(\overline{[x]}\big) \cap s^{-1}\big(\overline{[x]}\big)$ is itself a Hausdorff étale groupoid when G is. The groupoid is *minimal* if every orbit is dense.

Let G be an étale groupoid and take $x \in G^{(0)}$. The *range fibre* of x is $G^x = r^{-1}(x)$ and similarly the source fibre is $G_x = s^{-1}(x)$; they are both discrete subsets of G. An element $\gamma \in G$ is *isotropy* at x if $r(\gamma) = x = s(\gamma)$, and the *isotropy group* at $x \in G^{(0)}$ is the discrete group $\mathcal{I}(G)_x := \{\gamma \in G : r(\gamma) = x = s(\gamma)\} = G^x \cap G_x$. The *isotropy subgroupoid* $\mathcal{I}(G)$ is the group bundle $\bigsqcup_{x\in G^{(0)}} \mathcal{I}_x(G)$. This is an algebraic subgroupoid of G. The topological interior of the isotropy is an open subgroupoid of G denoted $\mathcal{I}^\circ(G)$. We write $\mathcal{I}^\circ(G)_x = \mathcal{I}^\circ(G) \cap \mathcal{I}(G)_x$. The groupoid is *effective* if $\mathcal{I}^\circ(G) = G^{(0)}$.

The notion of essential isotropy will be important for our main results.

Definition 2.1. Let G be an étale groupoid. The *essential isotropy at* $x \in G^{(0)}$ is $\mathcal{I}^{\mathrm{ess}}_x(G) := \mathcal{I}^\circ(G_{\overline{[x]}})_x \subseteq G_{\overline{[x]}}$, and the *essential isotropy of* G is the bundle of discrete groups

$$\mathcal{I}^{\mathrm{ess}}(G) := \bigsqcup_{x\in G^{(0)}} \mathcal{I}^{\mathrm{ess}}_x(G);$$

that is, $\mathcal{I}^{\mathrm{ess}}(G)$ is the collection of all points that are interior to the isotropy in the restriction of G to the orbit-closure of their source.

By a *normal subgroupoid* of the isotropy of a groupoid G we mean a subset $H \subseteq \mathcal{I}(G)$ that is closed under inversion and composition and has the property that if $\alpha \in H$ and $\beta \in G_{r(\alpha)}$ then $\beta\alpha\beta^{-1} \in H$.

Lemma 2.2. *Let G be an étale groupoid. Then $\mathcal{I}^\circ(G)$ and $\mathcal{I}^{\mathrm{ess}}(G)$ are both normal subgroupoids of G.*

Proof. If $(\alpha, \beta) \in \mathcal{I}^\circ(G) \cap G^{(2)}$, then there are open bisections $A \ni \alpha$ and $B \ni \beta$ consisting of isotropy, and then AB is an open subset of the isotropy containing $\alpha\beta$, and A^{-1} is an open subset of the isotropy containing α^{-1}. So $\mathcal{I}^\circ(G)$ is closed under composition and inversion.

To see that it is normal, suppose $\gamma \in \mathcal{I}^\circ(G)$ and $\eta \in G_{r(\gamma)}$. Choose a bisection $U \subseteq \mathcal{I}(G)$ with $\gamma \in U$, and a bisection V containing η. Then VUV^{-1} is an open subset of $\mathcal{I}(G)$ containing $\eta\gamma\eta^{-1}$, so $\eta\gamma\eta^{-1} \in \mathcal{I}^\circ(G)$.

To see that $\mathcal{I}^{\mathrm{ess}}(G)$ is a normal subgroupoid, observe that if $\alpha, \beta \in \mathcal{I}^{\mathrm{ess}}(G) \cap G^{(2)}$, and if $\eta \in G_{r(\alpha)}$, then the units $r(\eta)$, $r(\alpha)$, $s(\alpha) = r(\beta)$ and $s(\beta)$ all have the same orbit closure K. Since $\alpha, \beta \in \mathcal{I}^\circ(G|_K)$, the first statement of the lemma shows that $\alpha\beta$, α^{-1} and $\eta\alpha\eta^{-1}$ all belong to $\mathcal{I}^\circ(G|_K)$, and hence to $\mathcal{I}^{\mathrm{ess}}(G)$. □

Remark 2.3. The essential isotropy of a groupoid is an *algebraic* subgroupoid. It is in general not an open subgroupoid as demonstrated by Example 2.7 below. We do not know whether the essential isotropy is always closed, but we suspect not.

Notation 2.4. We define $\mathcal{J} = \mathcal{J}(G) := \overline{\mathcal{I}^{\mathrm{ess}}(G)}$, the smallest closed subgroupoid of G that contains the essential isotropy. For $x \in G^{(0)}$, we let $\mathcal{J}_x := \mathcal{J} \cap \mathcal{I}_x$. Since $\mathcal{I}$ is closed, we have $\mathcal{J} \subseteq \mathcal{I}$.

2.2. **Deaconu–Renault groupoids.** As above, we follow the notational conventions of [Sim20] for Deaconu–Renault groupoids.

If X is a locally compact Hausdorff space, then an *action* of $\mathbb{N}^k$ on X by local homeomorphisms is a monoid homomorphism $T\colon n \mapsto T^n$ from $\mathbb{N}^k$ to the monoid of local homeomorphisms of X. We use the shorthand $T\colon \mathbb{N}^k \curvearrowright X$ to mean that T is an action of $\mathbb{N}^k$ on X by local homeomorphisms. The orbit of a point x under T is the set

$$[x]_T = \bigcup_{n,m \in \mathbb{N}^k} T^{-m}(T^n x),$$

and T is *irreducible* if there exists $x \in X$ such that $\overline{[x]_T} = X$; we say that T is *minimal* if $\overline{[x]_T} = X$ for all $x \in X$. If T is clear from context, we simply write $[x]$ for $[x]_T$.

Suppose that X is a locally compact Hausdorff space and that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. We write G_T for the set

$$\{(x, m, y) \in X \times \mathbb{Z}^k \times X : \text{there exist } p, q \in \mathbb{N}^k \text{ such that } T^p(x) = T^q(y) \text{ and } p - q = m\}.$$

This set is a groupoid, called the *Deaconu–Renault groupoid* of T with composable pairs $\big\{\big((x, m, y), (y', n, z)\big) \in G_T \times G_T : y = y'\big\}$ and multiplication map

$$(x, m, y)(y, n, z) = (x, m + n, z).$$

The inversion operation is $(x, m, y) = (y, -m, x)$. The unit space of G_T is

$$G_T^{(0)} = \{(x, 0, x) : x \in X\},$$

and we silently identify it with X. With this identification the orbit $[x]_T$ of $x \in X$ under T as defined above agrees with the orbit $[x]$ of x in G_T as defined in Section 2.1.

For open sets $U, V \subseteq X$ and elements $p, q \in \mathbb{N}^k$, we define $Z(U, p, q, V) \subseteq G_T$ by

$$Z(U, p, q, V) := \{(x, p - q, y) : x \in U, y \in V \text{ and } T^p(x) = T^q(y)\}.$$

The collection of all such sets is a basis for a locally compact Hausdorff topology on G_T under which it becomes an étale groupoid. If $(x, p - q, y) \in Z(U, p, q, V)$, then using that T^p and T^q are local homeomorphisms, we can choose precompact open neighbourhoods U' of x and V' of y such that $T^p|_{U'}$ and $T^q|_{V'}$ are homeomorphisms onto their ranges. Putting $W = T^p(U') \cap T^q(V')$ and then setting $U'' = U' \cap (T^p)^{-1}(W)$ and $V'' = V' \cap (T^q)^{-1}(W)$, we obtain a basic open set $Z(U'', p, q, V'') \subseteq Z(U, p - q, V)$ containing $(x, p - q, y)$ with the property that $T^p|_{U''}$ and $T^q_{V''}$ are homeomorphisms onto the same precompact open subset W of X. So the collection of all such sets is a basis of precompact open bisections for the same topology on G_T.

There is a canonical 1-cocycle $c_T \colon G_T \to \mathbb{Z}$ (that is, a group-valued groupoid homomorphism) on G_T given by $c_T(x, m, y) = m$ for all $(x, m, y) \in G_T$. If T is clear from context, we just write c for c_T. A subset B of G_T is *c_T-homogeneous* (or simply *homogeneous*) if $c_T(B)$ is a singleton subset of $\mathbb{Z}^k$.

Suppose that T is irreducible. Writing G for G_T, Proposition 3.1 of [SW16] says that there is an open set $Y \subseteq X$ such that in the reduction $G|_Y = \{\gamma \in G : r(\gamma), s(\gamma) \in Y\}$ of G to Y, the interior of the isotropy is closed. We shall need to know that in fact the interior of the isotropy in G itself is closed.

Lemma 2.5. *Let X be a second-countable locally compact Hausdorff space and suppose that $T \colon \mathbb{N}^k \curvearrowright X$ be an action by local homeomorphisms. If T is irreducible, then $\mathcal{I}^\circ(G_T)$ is closed in G_T.*

Proof. By [SW16, Proposition 3.10] there is an open set $Y \subseteq X$ such that $T^p Y \subseteq Y$ for all $p \in \mathbb{N}^k$ and $\mathcal{I}^\circ(G_T|_Y)$ is closed in $G_T|_Y$. Since T is irreducible and Y is open, we have $X = \bigcup_{p \in \mathbb{N}^k} T^{-p}(Y)$.

Suppose that $(\gamma_n)_{n=1}^\infty$ is a sequence in $\mathcal{I}^\circ(G_T)$ that converges to $\gamma \in G_T$. We must show that $\gamma \in \mathcal{I}^\circ(G_T)$. It clearly belongs to $\mathcal{I}(G_T)$. For each n let $x_n = r(\gamma_n)$ and let $r(x) = \gamma$. So each $\gamma_n = (x_n, m_n, x_n)$ and $\gamma = (x, m, x)$ for some m_n and m in $\mathbb{Z}^k$. By discarding finitely many terms, we can assume without loss of generality that $m_n = m$ for all n.

Fix p such that $r(\gamma) \in T^{-p}(Y)$. Since Y and hence $T^{-p}(Y)$ is open, by discarding finitely many terms again, we can assume that $r(\gamma_n) \in T^{-p}(Y)$ for all n as well.

Write $m = a - b$ with $a, b \in \mathbb{N}^k$ satisfying $T^a(x) = T^b(x)$ and such that $a, b \geq p$ (we can arrange this by replacing a with $a + p$ and b with $b + p$ for example). Choose a neighbourhood $U \subseteq T^{-p}(Y)$ of x such that T^a and T^b restrict to homeomorphisms of U. Then $T^a U, T^b U \subseteq Y$ because $a, b \geq p$. Since $Z(U, a, b, U)$ is an open neighbourhood of $\gamma = (x, p, x)$, we can assume without loss of generality that $\gamma_n = (x_n, p, x_n)$ belongs to $Z(U, a, b, U)$ for all n.

Let $B := \{(z, p, T^p z) : z \in U\}$. Then $\gamma \mapsto B\gamma B^{-1}$ is a homeomorphism of $Z(U, a, b, U)$ onto $BZ(U, a, b, U)B^{-1} \subseteq G_T|_Y$. Hence the sequence $B\gamma_n B^{-1}$ converges in $G_T|_Y$ to $B\gamma B^{-1}$. Since each $\gamma_n \in \mathcal{I}^\circ(G_T)$, which is normal in G_T by Lemma 2.2, each $B\gamma_n B^{-1} \in \mathcal{I}^\circ(G_T)$ too. Since the interior of the isotropy in $G_T|_Y$ is closed by [SW16, Proposition 3.10], it follows that $B\gamma B \in \mathcal{I}^\circ((G_T)|_Y)$. Since $G_T|_Y \subseteq G_T$ is open, this gives $B\gamma B \in \mathcal{I}^\circ(G_T)$. Using Lemma 2.2 again, we see that $\gamma \in \mathcal{I}^\circ(G_T)$. □

Example 2.6.
(1) If G_T is minimal, then the essential isotropy of G_T coincides with interior of the isotropy, and by the above result the interior of the isotropy is closed, so $\mathcal{J}(G_T) = \mathcal{I}^{\mathrm{ess}}(G_T) = \mathcal{I}^\circ(G_T)$.
(2) If G_T is strongly effective (every reduction to a closed invariant set is effective), then the essential isotropy is trivial and $\mathcal{J}(G_T)$ may be identified with the unit space X.
(3) If G_T is minimal and effective, then $\mathcal{J}(G_T) = X$.

2.3. **Graph groupoids.** We will frequently use graph groupoids to describe examples. Let E be a row-finite directed graph with no sources (see [Rae05] for definitions and conventions). The path space E^* of E consists of finite strings $\mu = \mu_1 \cdots \mu_n$ of edges of E such that $s(\mu_i) = r(\mu_{i+1})$ for all $i < n$; we write $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$. The infinite-path space E^∞ of E consists of strings $x = x_1 x_2 \cdots$ all of whose initial segments $x_1 \cdots x_n$ are paths; we write $r(x)$ for $r(x_1)$. The space E^∞ is a totally disconnected locally compact Hausdorff space under the topology generated by the cylinder sets $Z(\mu) = \{\mu x : s(\mu) = r(x)\}$ of finite paths μ, and the shift map $\sigma \colon x \mapsto x_2 x_3 \ldots$ is a local homeomorphism (it restricts to a homeomorphism on $Z(\mu)$ whenever μ has length at least 1). The resulting Deaconu–Renault groupoid $G_E = G_\sigma$ is called the *graph groupoid* of E.

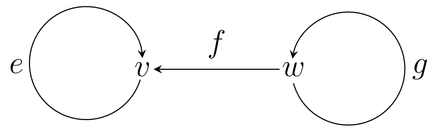
For finite paths $\mu, \nu \in E^*$ such that $s(\mu) = s(\nu)$, both $\sigma^{|\mu|}|_{Z(\mu)}$ and $\sigma^{|\nu|}|_{Z(\nu)}$ are homeomorphisms onto $Z(s(\mu)) \subseteq G_\sigma^{(0)} \cong E^\infty$, and the open sets

$$Z(\mu, \nu) := Z(Z(\mu), |\mu|, |\nu|, Z(\nu)) = \{(\mu x, |\mu| - |\nu|, \nu x) : x \in Z(s(\mu))\}$$

in G_σ indexed by such pairs constitute a basis of compact open sets for G_σ.

We will refer to the following elementary but illustrative example a number of times throughout the paper.

Example 2.7 (The Dumbbell graph)**.** Consider the directed graph E depicted below.



We call this the *dumbbell graph.* The infinite-path space of E is

$$E^\infty = \{e^\infty\} \cup \{e^n f g^\infty : n \in \mathbb{N}\} \cup \{g^\infty\}.$$

A straightforward argument shows that in the topology on E^∞ the subset $\{e^n f g^\infty : n \in \mathbb{N}\} \cup \{g^\infty\}$ is a discrete open subset, and E^∞ is homeomorphic to the one-point compactification of this subset with e^∞ as the point at infinity. The Deaconu–Renault groupoid is

$$\begin{aligned} G_E = \{(e^\infty, n, e^\infty) : n \in \mathbb{Z}\} &\cup \{(\alpha g^\infty, |\alpha| - |\beta|, \beta g^\infty) : \alpha, \beta \in \{e^n f : n \in \mathbb{N}\}\} \\ &\cup \{(g^\infty, n, g^\infty) : n \in \mathbb{Z}\}. \end{aligned}$$

An important point is that the topology of cylinder sets is finer than the topology inherited from $E^\infty \times \mathbb{Z} \times E^\infty$. To see this, note that although $(e^n f g^\infty, 0, e^n f g^\infty) \to (e^\infty, 0, e^\infty)$ as $n \to \infty$, for $m \in \mathbb{Z} \setminus \{0\}$ the sequence $(e^n f g^\infty, m, e^n f g^\infty)_{n \in \mathbb{N}}$ belongs to G_σ but has no convergent subsequence, and in particular does not converge to (e^∞, m, e^∞). To see this note that for any p, q with $p - q = m$, when $n \geq \max\{p, q\}$ we have $\sigma^p(e^n f g^\infty) = e^{n-p} f g^\infty \neq e^{n-q} f g^\infty = \sigma^q(e^n f g^\infty)$, and so $(e^n f g^\infty, m, e^n f g^\infty)$ does not belong to the open neighbourhood $Z(e^p, e^q)$ of (e^∞, m, e^∞).

We claim that $\mathcal{I}_{G_\sigma}^{\text{ess}} = \mathcal{I}(G_\sigma) = \{(x, m, x) : x \in E^\infty, m \in \mathbb{Z}\}$. Clearly $\mathcal{I}_{G_\sigma}^{\text{ess}} \subseteq \{(x, m, x) : x \in E^\infty, m \in \mathbb{Z}\}$. Since $E^\infty \setminus \{e^\infty\}$ is an open discrete subset of E^∞, the set $\{(x, m, x) : x \in E^\infty \setminus \{e^\infty\}, m \in \mathbb{Z}\}$ is an open discrete subset of G_σ and hence contained in $\mathcal{I}_{G_\sigma}^{\text{ess}}$. So it suffices to show that $\{(e^\infty, m, e^\infty) : m \in \mathbb{Z}\} \subseteq \mathcal{I}_{G_\sigma}^{\text{ess}}$. For this, note that the orbit of e^∞ is the singleton $\{e^\infty\}$, so $\overline{[e^\infty]} = \{e^\infty\}$, and $(G_\sigma)|_{\overline{[x]}} = \{(e^\infty, m, e^\infty) : m \in \mathbb{Z}\}$ is a discrete group isomorphic to $\mathbb{Z}$. In particular, the interior of the isotropy in this groupoid is the whole groupoid, and we obtain $\{(e^\infty, m, e^\infty) : m \in \mathbb{Z}\} \subseteq \mathcal{I}_{G_\sigma}^{\text{ess}}$ as claimed.

Since, as discussed above, $(e^{m+n} f g^\infty, m, e^n f g^\infty)_{n \geq |m|}$, the space $\mathcal{I}_{G_\sigma}^{\text{ess}}$ is not open.

It is instructive to describe the relative topology on $\mathcal{I}_{G_\sigma}^{\text{ess}}$. The complement $\mathcal{I}_{G_\sigma}^{\text{ess}} \setminus E^\infty$ of the unit space is a discrete clopen subset, while the unit space is homeomorphic to the one-point compactification of $\{e^n f g^\infty : n \in \mathbb{N}\} \cup \{g^\infty\}$ as described above.

We find it helpful to picture $\mathcal{I}_{G_\sigma}^{\text{ess}}$ as a subset of $\mathbb{R}^3$ as follows: for $x \in E^\infty$, we let $\theta(x) := 1/\min\{n : x_n = g\}$ with the convention that $\theta(e^\infty) = 0$; we then define points in $\mathcal{I}_{G_\sigma}^{\text{ess}}$ with points in $\mathbb{R}^3$ by

$$(x, m, x) \mapsto \begin{cases} (\theta(x), m, 0) & \text{if } x \neq e^\infty \\ (\theta(x), 0, m) & \text{if } x = e^\infty. \end{cases}$$

Example 2.8 (Essential isotropy of graph groupoids)**.** For a general graph groupoid, we can describe the essential isotropy relatively cleanly. It suffices to discuss row-finite graphs with no sources, because up to groupoid equivalence all graph groupoids can be realised by such graphs. Let E be a row-finite with no sources (in particular, the unit space of the graph groupoid is identified with the infinite-path space). A *maximal tail* of E is a set $T \subseteq E^0$ of vertices with the property that:

(1) $s(\alpha) \in T$ implies $r(\alpha) \in T$;
(2) if $v \in T$ then there exists $e \in vE^1$ such that $s(e) \in T$; and
(3) T is cofinal in the sense that whenever $u, v \in T$ there exists $w \in T$ such that uE^*w and vE^*w are both nonempty.

The orbit closures in E^∞ are the sets $V_T := \{x \in E^\infty : s(x_i) \in T \text{ for all } i\}$ indexed by maximal tails T of E: if x is an infinite path, then $T_x := \bigcup_n \{v \in E^0 : vE^*s(x_n) \neq \varnothing\}$ is a maximal tail, and $\overline{[x]} = V_{T_x}$. By [HS04, Lemma 2.1], if T is a maximal tail, then T can contain (up to cyclic permutation of edges and vertices) at most one cycle μ with no entrance in T; if there is such a μ, then $T = T_{\mu^\infty} = \{v \in E^0 : vE^*r(\mu) \neq \varnothing\}$. It is routine to check using the arguments of Example 2.7 that for a maximal tail T, the interior of the isotropy in $G_{ET} := G|_{V_T}$ is trivial if T contains no cycle with an entrance, and is equal to

$$\{(\alpha\mu^\infty, n|\mu|, \alpha\mu^\infty) : \alpha \in E^*r(\mu), \alpha \notin E^*\mu, n \in \mathbb{Z}\}$$

if $T = T_{\mu^\infty}$ is a maximal tail containing a cycle μ with no entrance in T. It follows that

$$\mathcal{I}^{\mathrm{ess}}(G_E) = G_E^{(0)} \cup \{(\alpha\mu^\infty, n|\mu|, \alpha\mu^\infty) : r(\mu) = s(\mu),\ \mu \text{ has no entrance in } T_{\mu^\infty},\\ \alpha \in E^*r(\mu), \alpha \notin E^*\mu, n \in \mathbb{Z}\}.$$

To describe the topology on $\mathcal{I}^{\mathrm{ess}}$, first note that $G_E^{(0)}$ is a clopen subset of $\mathcal{I}^{\mathrm{ess}}$. We claim that the complement of $G_E^{(0)}$ in $\mathcal{I}^{\mathrm{ess}}$ is discrete. For this, fix a cycle μ of nonzero length with no entrance in $T = T_\mu$, a path α with $s(\alpha) = r(\mu)$ and an integer n. We must show that given any sequence ν_i of cycles each having no entrance in $T_{\nu_i^\infty}$, any sequence β_i of paths with $s(\beta_i) = r(\nu_i)$ and $\beta_i \notin E^*\nu_i$, and any sequence $p_i \in \mathbb{Z}$, if $(\beta_i\nu_i^\infty, p_i|\beta_i|, \beta_i\nu_i^\infty) \to (\alpha\mu^\infty, n|\mu|, \alpha\mu^\infty)\}$, then $(\beta_i\nu_i^\infty, p_i|\beta_i|, \beta_i\nu_i^\infty) = (\alpha\mu^\infty, n|\mu|, \alpha\mu^\infty)$ for large i. We will argue the case when $n > 0$; the case $n < 0$ follows by taking inverses. Observe that since the range and source maps and the cocycle $c : (z, m, y) \mapsto m$ are continuous, we have $\beta_i\nu_i^\infty \to \alpha\mu^\infty$ and $p_i|\nu_i| = n|\mu|$ for large i; we may as well assume that $p_i|\nu_i| = n|\mu|$ for all i. Fix I such that $(\beta_i\nu_i^\infty, p_i|\beta_i|, \beta_i\nu_i^\infty) \in Z(\alpha\mu^{2n}, n|\mu|, \alpha\mu^n)$ for $i \geq I$. Fix $i \geq I$. We claim that $\beta_i\nu_i^\infty \in Z(\alpha\mu^{(2+k)n})$ for all $k \geq 0$. We prove this by induction. The base case is trivial since $(\beta_i\nu_i^\infty, p_i|\beta_i|, \beta_i\nu_i^\infty) \in Z(\alpha\mu^{2n}, n|\mu|, \alpha\mu^n)$ implies $\beta_i\nu_i^\infty \in Z(\alpha\mu^{2n})$. So suppose inductively that $\beta_i\nu_i^\infty \in Z(\alpha\mu^{(2+k)n})$. Then $\beta_i\nu_i^\infty = \alpha\mu^{(2+k)n}y$ for some y. Since $(\beta_i\nu_i^\infty, p_i|\beta_i|, \beta_i\nu_i^\infty) \in Z(\alpha\mu^{2n}, n|\mu|, \alpha\mu^n)$, we have

$$\sigma^{|\alpha|+2n|\mu|}(\beta_i\nu_i^\infty) = \sigma^{|\alpha|+n|\mu|}(\beta_i\nu_i^\infty) = \sigma^{|\alpha|+n|\mu|}(\alpha\mu^{(2+k)n}y) = \mu^{(1+k)n}y.$$

Since $\beta_i\nu_i^\infty \in Z(\alpha\mu^{2n})$ by the base case,

$$\beta_i\nu_i^\infty = \alpha\mu^{2n}\sigma^{|\alpha|+2n|\mu|}(\beta_i\nu_i^\infty) = \alpha\mu^{2n}\mu^{(1+k)n}y \in Z(\alpha\mu^{(2+(k+1))n}).$$

So $\beta_i\nu_i^\infty \in Z(\alpha\mu^{(2+k)n})$ for all $k \geq 0$ by induction, and hence $\beta_i\nu_i^\infty = \alpha\mu^\infty$.

2.4. **Ideals in C^*-algebras.** Here we use the exposition from [RW98, Appendix A2] to which the reader is also referred for details.

Let A be a C^*-algebra. By an *ideal* in A, we will always mean a closed and two-sided ideal. The ideals in A are therefore exactly the kernels of *-homomorphisms defined on A. An ideal I in A is *primitive* if it is the kernel of an irreducible representation of A, and the collection of primitive ideals, $\operatorname{Prim} A$, is endowed with the *hull-kernel* topology (or *Jacobson* topology) which is specified by the closure operation: the closure of a subset $F \subseteq \operatorname{Prim} A$ is given by

$$\bar{F} = \{P \in \operatorname{Prim} A : \bigcap_{I \in F} I \subseteq P\}.$$

Let I be an ideal in A. The *hull* of I is the closed set of primitive ideals

$$h(I) = \{P \in \operatorname{Prim} A : I \subseteq P\}.$$

Conversely, if F is a closed set of primitive ideals, then the *kernel* of F is

$$k(F) = \bigcap_{J \in F} J.$$

The two operations are inverses of each other and therefore define a bijection between the ideals in A and the closed subset of $\operatorname{Prim} A$. Similarly, ideals in A correspond bijectively to open sets of primitive ideals as in $I \mapsto \{P : I \not\subseteq P\}$.

2.5. **Primitive ideals in C^*-algebras of Deaconu–Renault groupoids.** Recall that if G is an étale groupoid then $C^*(G)$ is the universal C^*-algebra generated by a *-representation of the convolution algebra $C_c(G)$ (see, for example, [Sim20]). Since $G^{(0)}$ is a clopen subset of G, the completion of $C_c(G^{(0)})$ in $C^*(G)$ is a subalgebra isomorphic to $C_0(G^{(0)})$. We identify the two and regard $C_0(G^{(0)})$ as a subset of $C^*(G)$.

If $G = G_T$ is the Deaconu–Renault groupoid for an action $T\colon \mathbb{N}^k \curvearrowright X$, then for $z \in \mathbb{T}$ and for each $f \in C_c(G_T)$ the function $\gamma_z(f)\colon (x, n, y) \mapsto z^n f(x, n, y)$ also belongs to $C_c(G_T)$. The map γ_z is a *-homomorphism, so the universal property of $C^*(G_T)$ implies that γ_z extends to an endomorphism of $C^*(G_T)$. A routine $\varepsilon/3$-argument shows that $z \mapsto \gamma_z(a)$ is continuous for $a \in C^*(G)$. Since $\gamma_z \circ \gamma_w = \gamma_{zw}$ on $C_c(G_T)$ and since γ_1 is the identity on $C_c(G_T)$, this γ is an action of $\mathbb{T}^k$ on $C^*(G_T)$ called the *gauge action*.

Let G_T be the Deaconu–Renault groupoid of an action $T\colon \mathbb{N}^k \curvearrowright X$. For each $(x, z) \in X \times \mathbb{T}^k$, there is an irreducible representation $\pi_{x,z} : C^*(G_T) \to \mathcal{B}(\ell^2([x]))$ such that

$$\pi_{x,z}(f)e_y = \sum_{\gamma \in (G_T)_y} z^{c(\gamma)} f(\gamma) e_{r(\gamma)}$$

for every $f \in C_c(G_T)$ and $y \in [x]$. By [SW16, Theorem 3.2], there is a surjection $\pi\colon X \times \mathbb{T}^k \to \operatorname{Prim}(C^*(G_T))$ such that $\pi(x, z) = \ker(\pi_{x,z})$ for all $(x, z) \in X \times \mathbb{T}^k$. Moreover, $\pi_{x,z}$ and $\pi_{x',z'}$ have the same kernel precisely if $\overline{[x]} = \overline{[x']}$ and z and z' induce the same character on the group

$$H(x) := \bigcup_{\substack{\varnothing \neq U \subseteq \overline{[x]} \\ U \text{ relatively open}}} \{m - n : T^m y = T^n y \text{ for all } y \in U\} \subseteq \mathbb{Z}^k$$

described immediately before [SW16, Theorem 3.2]. Observe that $H(x) = H(y)$ whenever $\overline{[x]} = \overline{[y]}$.

We claim that $H(x) = c(\mathcal{I}^\circ((G_T)|_{\overline{[x]}}))$: the containment $\supseteq$ follows from the definition of $H(x)$, and the reverse containment follows from [SW16, Lemma 3.9] since $(G_T)|_{\overline{[x]}}$ is irreducible by definition. We claim further that

$$H(x) = c(\mathcal{I}_x^{\mathrm{ess}}).$$

Indeed, if $n \in H(x)$, then there exists $\gamma \in \mathcal{I}^\circ((G_T)|_{\overline{[x]}})$ with $c(\gamma) = n$. So there is an open bisection B in $\mathcal{I}^\circ((G_T)|_{\overline{[x]}})$ containing γ, and since c is locally constant, we can assume that $B \subseteq c^{-1}(n)$. Since $[x]$ is dense in $\overline{[x]}$, there exists $\eta \in (G_T)_x$ with $r(\eta) \in s(B)$. Lemma 2.2 shows that $\eta^{-1}\gamma\eta \in \mathcal{I}^\circ((G_T)|_{\overline{[x]}})_x$, and so $n \in c(\mathcal{I}_x^{\mathrm{ess}})$. The reverse containment is clear because $\mathcal{I}_x^{\mathrm{ess}} \subseteq \mathcal{I}^\circ((G_T)|_{\overline{[x]}})$.

2.6. **Harmonic analysis.** For the proof of our main result we will need a bit of harmonic analysis. We use notation and results from [Fol99, Chapter 8].

Let $K \subseteq \mathbb{Z}^k$ be a subgroup. Its annihilator is the compact subgroup $K^\perp := \{z \in \mathbb{T}^k : z^h = 1, \text{ for all } h \in K\} \leq \mathbb{T}^k$. The annihilator acts on $\mathbb{T}^k$ by translation. The Pontryagin dual of K (defined as the group of continuous homomorphisms from H into $\mathbb{T}$) is isomorphic to the quotient group $\mathbb{T}^k/K^\perp \cong \widehat{K}$. Let $\psi \in C^\infty(\widehat{K})$. The Fourier coefficients of ψ are

$$\widehat{\psi}(h) := \int_{\widehat{K}} \psi(\eta)\overline{\eta(h)}\, d\eta,$$

where the integration is with respect to normalised Haar measure on $\widehat{K}$. When ψ is smooth (that is, all partial derivatives exist and are continuous), its Fourier coefficients are absolutely summable, cf. [Fol99, p. 257].

Given $h_0 \in K$, we may *perturb* ψ by h_0 to obtain $\psi_{h_0} \in C^\infty(\widehat{K})$ given by

$$\psi_{h_0}(\eta) = \eta(h_0)\psi(\eta) \tag{2.1}$$

for $\eta \in \widehat{K}$. The Fourier coefficients of ψ_{h_0} are $\widehat{\psi}_{h_0}(h) = \widehat{\psi}(h - h_0)$, for $h \in K$. If ψ is supported on an open subset $V \subseteq \widehat{K}$, then ψ_{h_0} is also supported on V.

If $H \leq K$ is a subgroup, then $K^\perp \subseteq H^\perp$, and there is a canonical quotient map $\widehat{q}_{H,K} \colon \widehat{K} \to \widehat{H}$. We define an *averaging map* $\Phi_{H,K} \colon C(\widehat{K}) \to C(\widehat{H})$ by

$$\Phi_{H,K}(\psi)(\widehat{q}_{H,K}(\eta)) := \int_{H^\perp} \psi(\chi\eta)\, d\chi \tag{2.2}$$

for $\psi \in C(\widehat{K})$ and $\eta \in \widehat{K}$. Again we integrate with respect to normalised Haar measure on $H^\perp$. If $\psi \in C(\widehat{K})$ is smooth, then it is the $\|\cdot\|_\infty$-limit of its Fourier series:

$$\psi(\eta) = \sum_{h \in K} \eta(h)\widehat{\psi}(h).$$

In this case,

$$\Phi_{H,K}(\psi)(q_{H,K}(\eta)) = \sum_{h \in K} \eta(h)\widehat{\psi}(h) \int_{H^\perp} \chi(h)\, d\chi = \sum_{h \in H} \eta(h)\widehat{\psi}(h). \tag{2.3}$$

In particular, $\big(\Phi_{H,K}(\psi)\widehat{\ }\big)(h) = \widehat{\psi}(h)$ for all $h \in H$.

If $\psi \in C(\widehat{K})$ is supported on $\widehat{V} \subseteq \widehat{K}$, then $\Phi_{H,K}(\psi)$ is supported on $\widehat{q}_{H,K}(\widehat{V})$.

Lemma 2.9. *Suppose that $z \in \mathbb{T}^k$ and $\psi \in C^\infty(\mathbb{T}^k)$ satisfy $\psi(z) \neq 0$. For $h_0 \in \mathbb{Z}^k$ let ψ_{h_0} be as in (2.1). For any subgroup $H \subseteq \mathbb{Z}^k$ there exists $h_0 \in \mathbb{Z}^k$ such that*

$$\sum_{h \in H} z^h \widehat{\psi}_{h_0}(h) \neq 0.$$

Proof. Since ψ is smooth, it is the norm limit of its Fourier series, so

$$0 \neq \psi(z) = \sum_{h \in \mathbb{Z}^k} z^h \widehat{\psi}(h).$$

Choose a section σ for the quotient map $\mathbb{Z}^k \to \mathbb{Z}^k/H$. For each $y \in \mathbb{Z}^k/H$, consider the $\sigma(y)$-pertubation of ψ,

$$\psi_{\sigma(y)}(\eta) = \eta(\sigma(y))\psi(\eta). \tag{2.4}$$

As discussed above, for $h \in \mathbb{Z}^k$ the corresponding Fourier coefficient of $\psi_{\sigma(y)}$ is $\widehat{\psi}_{\sigma(y)}(h) = \widehat{\psi}\big(h - \sigma(y)\big)$. We have

$$\Phi_{H,\mathbb{Z}^k}(\psi_{\sigma(y)})\big(\widehat{q}(\chi)\big) = \sum_{h \in H} z^h \widehat{\psi}\big(h - \sigma(y)\big). \tag{2.5}$$

Let $q : \mathbb{T}^k \to \widehat{H}$ be the quotient map so that $q(z)(h) = z^h$ for $h \in H$ and $z \in \mathbb{T}^k$. Since the Fourier coefficients of ψ are absolutely summable, we can rearrange the summation (2.5) to see that

$$0 \neq \sum_{h\in\mathbb{Z}^k} z^h\widehat{\psi}(h) = \sum_{y\in\mathbb{Z}^k/H} \Big(\sum_{h\in H} z^h\widehat{\psi}\big(h-\sigma(y)\big)\Big) = \sum_{y\in\mathbb{Z}^k/H} \Phi_{H,\mathbb{Z}^k}(\psi_{\sigma(y)})(q(z)).$$

In particular, there exists $y \in \mathbb{Z}^k/H$ such that $h_0 = \sigma(y)$ satisfies

$$0 \neq \Phi_{H,\mathbb{Z}^k}(\psi_{h_0})(z) = \sum_{h\in\mathbb{Z}^k} z^h\widehat{\psi}(h). \qquad \square$$

3. A family of representations

Our analysis of ideals depends upon the existence and behaviour of representations of the C^*-algebra of a Deaconu–Renault groupoid that interpolate between the regular representation on $\ell^2((G_T)_x)$ and the representation on $\ell^2([x])$ induced by the action of G_T on $[x]$. We establish these technical results in this section.

Let X be a second-countable locally compact Hausdorff space and let $T\colon \mathbb{N}^k \curvearrowright X$ be an action by local homeomorphisms. Suppose that H is a subgroup of $\mathbb{Z}^k$. We let $\sim_H$ be the equivalence relation on G_T given by

$$(x_1,h_1,y_1) \sim_H (x_2,h_2,y_2) \iff x_1 = x_2,\ y_1 = y_2,\ \text{and } h_1 - h_2 \in H.$$

This is an equivalence relation, and for $\xi \in G_T$, we let $[\xi]_H$ denote the equivalence class of ξ with respect to $\sim_H$.

Proposition 3.1. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For each $(x,z) \in X\times\mathbb{T}$ there is a representation $\pi^H_{(x,z)}\colon C^*(G_T) \to B(\ell^2((G_T)_x/{\sim_H}))$ such that for each $f \in C_c(G_T)$ and $\xi_1,\xi_2 \in (G_T)_x$,*

$$\langle e_{[\xi_1]_H}, \pi^H_{(x,z)}(f)e_{[\xi_2]_H}\rangle = \sum_{(x_1,h,x_2)\in[\xi_1\xi_2^{-1}]_H} z^h f(x_1,h,x_2). \tag{3.1}$$

If H_1 and H_2 are subgroups of $\mathbb{Z}^k$ and $H_1 \subseteq H_2$, then $\ker(\pi^{H_1}_{(x,z)}) \subseteq \ker(\pi^{H_2}_{(x,z)})$.

Proof. The canonical cocycle $c_T\colon G_T \to \mathbb{Z}^k$ defined by $c_T((y,n,x)) = n$ for every $(y,n,x) \in G_T$ determines a coaction (dual to the gauge action) $\delta_T\colon C^*(G_T) \to C^*(G_T)\otimes C^*(\mathbb{Z}^k)$ that satisfies $\delta_T(f) = f\otimes u_n$ whenever $f \in C_c(c_T^{-1}(n)) \subseteq C_c(G_T)$.

Fix $x \in X$ and consider the representation $\varepsilon_x\colon C^*(G_T) \to B(\ell^2([x]))$ of [BCFS14, Proposition 5.2] satisfying $\varepsilon_x(f)e_y = \sum_{\gamma\in(G_T)_y} f(\gamma)e_{r(\gamma)}$ for all $f \in C_c(G_T)$ and $y \in [x]$. Fix a subgroup $H \le \mathbb{Z}^k$ and $z \in \mathbb{T}^k$. Let $\{u_n : n \in \mathbb{Z}^k\} \subseteq C^*(\mathbb{Z}^k)$ be the canonical generators. Similarly, let $\{u_{n+H} : n+H \in \mathbb{Z}^k/H\} \subseteq C^*(\mathbb{Z}^k/H)$ be the canonical generators. Since $n+H \mapsto z^n u_{n+H}$ is a unitary representation of $\mathbb{Z}^k$, there is a homomorphism $\rho_{z,H}\colon C^*(\mathbb{Z}^k) \to C^*(\mathbb{Z}^k/H)$ such that $\rho_{z,H}(u_n) = z^n u_{n+H}$. Finally, let $\lambda^{\mathbb{Z}^k/H}$ be the left regular representation of $C^*(\mathbb{Z}^k/H)$ on $\ell^2(\mathbb{Z}^k/H)$.

Consider the representation

$$\psi^H_{(x,z)} := (1\otimes\lambda^{\mathbb{Z}^k/H})\circ(\varepsilon_x\otimes\rho_{z,H})\circ\delta_T\colon C^*(G_T) \to B\big(\ell^2([x])\otimes\ell^2(\mathbb{Z}^k/H)\big). \tag{3.2}$$

Observe that $c(\mathcal{I}(G_T)_x)$ is a subgroup of $\mathbb{Z}^k$, so $c(\mathcal{I}(G_T)_x)+H$ is also a subgroup. Let $\Sigma \subseteq \mathbb{Z}^k$ be a complete set of coset representatives for the cosets of $c(\mathcal{I}(G_T)_x)+H$ in $\mathbb{Z}^k$ such that $0 \in \Sigma$ is the representative of the coset $c(\mathcal{I}(G_T)_x)+H$ itself.

Our strategy is as follows. We decompose $\ell^2([x]) \otimes \ell^2(\mathbb{Z}^k/H)$ into a direct sum of subspaces $\mathcal{H}_t$, indexed by elements t in Σ, that are invariant for $\psi^H_{(x,z)}$. We then identify a unitary isomorphism $W_j\colon \ell^2((G_T)_x/{\sim_H}) \to \mathcal{H}_0$ such that $\pi^H_{(x,z)} := \operatorname{Ad}(W_j^*) \circ \psi^H_{(x,z)}|_{\mathcal{H}_0}$ satisfies (3.1), establishing the first statement. We then show that the restrictions $\psi^H_{(x,z)}|_{\mathcal{H}_t}$ are all unitarily equivalent to one another, and conclude that $\ker(\pi^H_{(x,z)}) = \ker(\psi^H_{(x,z)})$. Finally we will show that if $H_1 \le H_2 \le \mathbb{Z}^k$, then $\ker(\psi^{H_1}_{(x,z)}) \subseteq \ker(\psi^{H_2}_{(x,z)})$, giving the final statement.

We first claim that

$$[x] \times \mathbb{Z}^k/H = \bigsqcup_{t\in\Sigma} \{(y, n+t+H) : (y,n,x) \in G_T\}. \tag{3.3}$$

It is clear that the left-hand side contains the union of the sets on the right, so we have to show that the union on the right contains the left-hand side, and that the sets on the right are disjoint. For the first assertion, fix $y \in [x]$ and $n \in \mathbb{Z}^k$ so that $(y, n+H)$ is a typical element of $[x] \times \mathbb{Z}^k/H$. Since $y \in [x]$ there exists n_y such that $(y, n_y, x) \in G_T$. Let $t \in \Sigma$ be the unique element such that $t \in n - n_y + H + c(\mathcal{I}(G_T)_x)$. So $n = n_y + t + h + m$ for some $h \in H$ and $m \in c(\mathcal{I}(G_T)_x)$. Since $m \in c(\mathcal{I}(G_T)_x)$ we have $(x,m,x) \in G_T$, and so $(y, n_y+m, x) = (y, n_y, x)(x,m,x) \in G_T$. We then have $(y, n+H) = (y, n-h+H) = (y, (n_y+m)+t+H)$, which belongs to the right-hand side of (3.3). We must now show that $\{(y, n+t+H) : (y,n,x) \in G_T\}_{t\in\Sigma}$ are mutually disjoint. So suppose that $s, t \in \Sigma$ and that $(y, n+t+H) = (y, n'+s+H)$ for some $(y,n,x), (y,n',x) \in G_T$. So $s - t \in n - n' + H$. We have $(x, n-n', x) = (x,n,y)(x,n',y)^{-1} \in G_T$ so that $n - n' \in c(\mathcal{I}(G_T)_x)$. Hence $s - t \in c(\mathcal{I}(G_T)_x) + H$. Therefore, $s \in \Sigma \cap (t + c(\mathcal{I}(G_T)_x) + H) = \{t\}$, so $s = t$ as required.

For $t \in \Sigma$, we define

$$\mathcal{H}_t := \overline{\operatorname{span}}\{e_y \otimes e_{n+t+H} : (y,n,x) \in G_T\}.$$

Then (3.3) shows that $\ell^2([x]) \otimes \ell^2(\mathbb{Z}^k/H) \cong \bigoplus_{t\in\Sigma} \mathcal{H}_t$. To show that each $\mathcal{H}_t$ is invariant for $\psi^H_{(x,z)}$, we fix $t \in \Sigma$. Since $C^*(G_T)$ is the closed linear span of functions supported on basic open sets, it suffices to fix open sets $U, V \subseteq X$ and $p, q \in \mathbb{N}^k$ such that $T^p|_U$ and $T^q|_V$ are homeomorphisms onto the same open set $W \subseteq X$, a function $f \in C_c(Z(U,p,q,V))$ and an element $(y,n,x) \in G_T$ so that $e_y \otimes e_{n+t+H}$ is a typical basis element of $\mathcal{H}_t$, and show that $\psi^H_{(x,z)}(f)(e_y \otimes e_{n+t+H}) \in \mathcal{H}_t$. This is a straightforward calculation:

$$\begin{aligned}\psi^H_{(x,z)}(f)(e_y \otimes e_{n+t+H}) &= (1 \otimes \lambda^{\mathbb{Z}^k/H}) \circ (\varepsilon_x(f) \otimes z^{p-q} u_{p-q+H})(e_y \otimes e_{n+t+H}) \\ &= \sum_{\gamma \in (G_T)_y} z^{p-q} f(\gamma)(e_{r(\gamma)} \otimes e_{n+p-q+t+H}).\end{aligned}$$

If $y \notin V$, then $f(\gamma) = 0$ for all $\gamma \in (G_T)_y$, and then $\psi^H_{(x,z)}(f)(e_y \otimes e_{n+t+H}) = 0$. If $y \in V$, then there is a unique $u \in U$ such that $T^p(u) = T^q(y)$, and then $\gamma = (u, p-q, y)$ is the unique element of $(G_T)_y$ such that $f(\gamma) \neq 0$, so the calculation above gives

$$\psi^H_{(x,z)}(f)(e_y \otimes e_{n+t+H}) = z^{p-q} f(u, p-q, y)(e_u \otimes e_{n+p-q+t+H}). \tag{3.4}$$

Since $(u, n+p-q, x) = (u, p-q, y)(y, n, x) \in G_T$, we have $e_u \otimes e_{n+p-q+t+H} \in \mathcal{H}_t$, so $\mathcal{H}_t$ is invariant for $\psi^H_{(x,z)}$ as claimed.

In particular, the subspace $\mathcal{H}_0 = \overline{\operatorname{span}}\{e_y \otimes e_{n+H} : (y,n,x) \in G_T\}$ is invariant for $\psi^H_{(x,z)}$. We show that $\mathcal{H}_0$ is isomorphic to $\ell^2((G_T)_x/{\sim_H})$. To see this, observe that there is a map $\widetilde{j}\colon (G_T)_x \to [x] \times \mathbb{Z}^k/H$ satisfying $\widetilde{j}(y,n,x) = (y, n+H)$, and we have $(y,n,x) \sim_H (y', n', x)$ if and only if $y = y'$ and $n - n' \in H$, and hence if and only if $\widetilde{j}(y,n,x) =$

$\widetilde{j}(y', n', x)$. This means that there is an injective map $j\colon (G_T)_x/{\sim_H} \to [x] \times \mathbb{Z}^k/H$ satisfying $j([y,n,x]) = (y, n+H)$. Since j induces a bijection $e_{[y,n,x]} \mapsto e_y \otimes e_{n+H}$ between orthonormal bases for $\ell^2((G_T)_x/{\sim_H})$ and $\mathcal{H}_0$, it induces a unitary $W_j\colon \ell^2((G_T)_x/{\sim_H}) \to \mathcal{H}_0$.

Since $\mathcal{H}_0$ is invariant for $\psi^H_{(x,z)}$ we obtain a representation

$$\pi^H_{(x,z)} := \operatorname{Ad}(W_j^*) \circ \psi^H_{(x,z)} \colon C^*(G_T) \to B(\ell^2((G_T)_x/{\sim_H})).$$

We claim that this representation satisfies (3.1). Once again it suffices to establish (3.1) for $f \in C_c(Z(U, p-q, V))$ where $T^p|_U$ and $T^q|_V$ are homeomorphisms onto the same open set W. Fix $[\xi_1] = [w, m, x]$ and $[\xi_2] = [y, n, x]$ in $(G_T)_x/{\sim_H}$. As in the paragraph including (3.4), we have

$$\langle e_{[\xi_1]_H}, \pi^H_{(x,z)}(f) e_{[\xi_2]_H}\rangle = \begin{cases} z^{p-q} f(w, p-q, y) & \text{if } y \in V,\ w \in U,\ T^p(z) = T^q(y), \\ & \quad \text{and } n+p-q+H = m+H, \\ 0 & \text{otherwise.}\end{cases}$$

Note that $[\xi_1\xi_2^{-1}] = \{(w, m-n+h, y) : h \in H\} \cap G_T$ and that f is nonzero at at most one point in this set, which occurs if $y \in V$, $w \in U$, $T^p(w) = T^q(y)$ and $m-n \in p-q+H$. So

$$\sum_{(x_1,h,x_2)\in[\xi_1\xi_2^{-1}]_H} z^h f(x_1, h, x_2) = \begin{cases} z^{p-q} f(w, p-q, y) & \text{if } y \in V,\ w \in U,\ T^p(z) = T^q(y), \\ & \quad \text{and } p-q+H = m-n+H, \\ 0 & \text{otherwise.}\end{cases}$$

Comparing the last two displayed equations we see that $\pi^H_{(x,z)}$ satisfies (3.1) as claimed. This completes the proof of the first statement.

Next we claim that $\ker(\pi^H_{(x,z)}) = \ker(\psi^H_{(x,z)}) = \ker((\varepsilon_x \otimes \rho_{z,H}) \circ \delta_T)$. First note that for each $t \in \Sigma$ the map $e_y \otimes e_{n+t+H} \mapsto e_y \otimes e_{n+H}$ is a bijection between orthonormal bases for $\mathcal{H}_t$ and $\mathcal{H}_0$ and so induces a unitary $V_t\colon \mathcal{H}_t \to \mathcal{H}_0$. The definition of V_t and the paragraph including (3.4) show that U_t commutes with $\psi^H_{(x,z)}(f)$ for $f \in C_c(Z(U,p,q,V)$. Hence $\operatorname{Ad}(V_t) \circ \psi^H_{(x,z)}|_{\mathcal{H}_0} = \psi^H_{(x,z)}|_{\mathcal{H}_t}$. It follows that

$$\psi^H_{(x,z)} = \bigoplus_{t\in\Sigma} \operatorname{Ad}(V_t) \circ \psi^H_{(x,z)}|_{\mathcal{H}_0} = \bigoplus_{t\in\Sigma} \operatorname{Ad}(V_t W_j) \circ \pi^H_{(x,z)},$$

so $\ker(\pi^H_{(x,z)}) = \ker(\psi^H_{(x,z)})$. Since $\mathbb{Z}^k/H$ is amenable, the regular representation $\lambda^{\mathbb{Z}^k/H}$ is faithful, and $C^*(\mathbb{Z}^k/H)$ is nuclear so $1 \otimes \lambda^{\mathbb{Z}^k/H}$ is faithful on $B(\ell^2([x])) \otimes C^*(\mathbb{Z}^k/H)$. Therefore, $\ker(\psi^H_{(x,z)}) = \ker((\varepsilon_x \otimes \rho_{z,H}) \circ \delta_T)$.

It remains to verify that if $H_1 \leq H_2 \leq \mathbb{Z}^k$ are subgroups then $\ker(\pi^{H_1}_{(x,z)}) \subseteq \ker(\pi^{H_2}(x,z))$ for all $(x,z) \in X \times \mathbb{T}^k$. From the above observations, we just need to show that

$$\ker(\varepsilon_x \otimes \rho_{z,H_1}) \subseteq \ker(\varepsilon_x \otimes \rho_{z,H_2}). \tag{3.5}$$

The quotient map $n + H_1 \mapsto n + H_2$ from $\mathbb{Z}^k/H_1$ to $\mathbb{Z}^k/H_2$ induces a homomorphism $q\colon C^*(\mathbb{Z}^k/H_1) \to C^*(\mathbb{Z}^k/H_2)$ satisfying $\rho_{(z,H_2)} = q \circ \rho_{(z,H_1)}$. Therefore $(1 \otimes q) \circ (\varepsilon_x \otimes \rho_{z,H_1}) = \varepsilon_x \otimes (q \circ \rho_{z,H_1}) = \varepsilon_x \otimes \rho_{z,H_2}$, and this proves (3.5). □

Remark 3.2. When $H = \mathbb{Z}^k$, the equivalence relation $\sim_{\mathbb{Z}^k}$ is given by $(x,m,y) \sim_{\mathbb{Z}^k} (w,n,z)$ if and only if $x = w, y = z$ and $m - n \in \mathbb{Z}^k$; that is, if and only if $x = w$ and $y = z$. Hence there is a bijection $(G_T)_x/{\sim_{\mathbb{Z}^k}} \to [x]$ given by $[y,n,x] \mapsto y$. Using this bijection to induce a unitary $\ell^2((G_T)_x/{\sim_{\mathbb{Z}^k}}) \to \ell^2([x])$, we see that $\pi^{\mathbb{Z}^k}_{(x,z)}$ is unitarily equivalent to the representation $\pi_{x,z}$ of $C^*(G_T)$ on $\ell^2([x])$ appearing in [SW16, Theorem 3.2] (cf. Section 2.5).

It will be important later that the nesting of kernels described in the final statement of Proposition 3.1 is equality if H_1 is the image under c_T of the essential isotropy at x, and H_2 is the whole of $\mathbb{Z}^k$.

Lemma 3.3. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Let $c_T : G_T \to \mathbb{Z}^k$ be the canonical cocycle. Fix $(x, z) \in X \times \mathbb{T}^k$, and let $K := c_T(\mathcal{I}^{\mathrm{ess}}_x)$. Then $\ker(\pi^K_{(x,z)}) = \ker(\pi_{(x,z)})$.*

Proof. Let $Y := \overline{[x]} \subseteq X$, the orbit-closure of x. Then Y is a closed invariant subset of X for T, so T restricts to an action $S : \mathbb{N}^k \curvearrowright Y$. The resulting Deaconu–Renault groupoid G_S is precisely the reduction of (G_T) to the closed invariant subspace Y of its unit space. Let $q_Y : C^*(G_T) \to C^*(G_S)$ be the surjective homomorphism extending restriction of functions $C_c(G_T) \to C_c(G_S)$ [Sim20, Proposition 10.3.2].

By definition, $\pi^K_{(x,z)}$ and $\pi_{(x,z)}$ annihilate $\ker(q_Y)$. Consequently, $\pi^K_{(x,z)}$ and $\pi_{(x,z)}$ factor through the corresponding representations $\widetilde{\pi}^K_{(x,z)}$ and $\widetilde{\pi}_{(x,z)}$ of $C^*(G_S)$ respectively. So it suffices to show that $\ker(\widetilde{\pi}^K_{(x,z)}) = \ker(\widetilde{\pi}_{(x,z)})$.

Since $G^{(0)}_S = Y = \overline{[x]}$, the groupoid G_S is irreducible. Hence Lemma 2.5 implies that $\mathcal{I}^\circ(G_S)$ is closed in G_S, and so [SW16, Proposition 2.5] implies that $G' := G_S/\mathcal{I}^\circ(G_S)$ is an amenable effective locally compact Hausdorff étale groupoid. By definition, $\mathcal{I}^\circ(G_S) = \mathcal{I}^{\mathrm{ess}}(G_T)_x = \{(y, h, y) : y \in \overline{[x]} \text{ and } h \in K\}$. In particular, the set G'_x is equal to $(G_S)_x/{\sim_K}$, and the orbit $[x]_{G'}$ is equal to the orbit $[x]_{G_S}$. So we can identify $\ell^2(G'_x)$ with $\ell^2((G_S)_x/{\sim_K})$ and $\ell^2([x]_{G'})$ with $\ell^2([x]_{G_S})$. Consequently, we can regard the regular representation $\pi^{G'}_x$ of $C^*(G')$ for $x \in (G')^{(0)}$ as a representation on $\ell^2((G_S)_x/{\sim_K})$, and the representation $\varepsilon_x : C^*(G') \to \ell^2([x]_{G'})$ such that $\varepsilon_x(f)(\delta_y) = \sum_{\alpha\in G'_y} f(\alpha)\delta_{r(\alpha)}$ (see [BCFS14, Proposition 5.2]) as a representation on $\ell^2([x]_{G_S})$.

Proposition 2.6 of [SW16] shows that there is a homomorphism $\kappa : C^*(G_S) \to C^*(G')$ such that $\kappa(f)([\gamma]) = \sum_{\eta\in[\gamma]} f(\eta)$ for all $f \in C_c(G_S)$. Let $\gamma_z \in \mathrm{Aut}(C^*(G_S))$ be the automorphism such that $\gamma_z(f)(u, n, v) = z^n f(u, n, v)$ for $f \in C_c(G_S)$ as in Section 2.5. Direct calculation shows that, with the identifications of Hilbert spaces in the preceding paragraph, $\widetilde{\pi}^K_{(x,z)} = \pi^{G'}_x \circ \kappa \circ \gamma_z$ and $\widetilde{\pi}_{(x,z)} = \varepsilon_x \circ \kappa \circ \gamma_z$. Since $\overline{[x]} = (G')^{(0)}$, both $\pi^{G'}_x$ and ε_x are injective on $C_0((G')^{(0)})$. Hence [Sim20, Theorem 10.2.7] shows that they are both injective. Consequently, $\ker(\widetilde{\pi}^K) = \ker(\kappa \circ \gamma_z) = \ker(\widetilde{\pi}_{(x,z)})$ as required. □

The following technical lemma will be helpful in identifying elements in the kernels of the representations $\pi^H_{(x,z)}$.

Lemma 3.4. *Let H be a subgroup of $\mathbb{Z}^k$, fix $(x, z) \in X \times \mathbb{T}$, and let $\pi^H_{(x,z)}$ be the representation of Proposition 3.1. For $f \in C^*(G_T)$, we have $\pi^H_{(x,z)}(f) = 0$ if and only if for every pair of homogeneous bisections $B_1, B_2 \subseteq G_T$ and every pair of functions $h_i \in C_c(B_i)$, we have $\langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(h_1 f h_2) e_{[(x,0,x)]_H}\rangle = 0$. In particular, an ideal I is contained in $\ker(\pi^H_{(x,z)})$ if and only if*

$$\langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(f) e_{[(x,0,x)]_H}\rangle = 0,$$

for all $f \in I$.

Proof. If $\pi^H_{(x,z)}(f) = 0$, then each $\pi^H_{(x,z)}(h_1 f h_2) = 0$, so the "only if" implication is immediate. We just need to prove the "if" implication.

Fix $f \in C_c(G_T)$ and suppose that for every pair of homogeneous bisections $B_1, B_2 \subseteq G_T$ and every pair of functions $h_i \in C_c(B_i)$, we have $\langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(h_1 f h_2) e_{[(x,0,x)]_H}\rangle = 0$. It suffices to show that for all $\xi_1 = (y_1, n_1, x)$ and $\xi_2 = (y_2, n_2, x)$ in $(G_T)_x$, we have

$$\langle e_{[\xi_1]_H}, \pi^H_{(x,z)}(f)e_{[\xi_2]_H}\rangle = 0.$$

By definition of the topology on G_T there are open bisections $B_1 \subseteq c_T^{-1}(-n_1)$ and $B_2 \subseteq c_T^{-1}(n_2)$ such that $\xi_1^{-1} \in B_1$ and $\xi_2 \in B_2$, and by Urysohn's lemma we can find $h_i \in C_c(B_i)$ such that $h_1(\xi_1^{-1}) = 1 = h_2(\xi_2)$.

Let $\eta \in (G_T)_x$. Since h_1 is supported on a bisection containing ξ_1, the formula (3.1) implies that

$$\langle e_{[\eta]_H}, \pi^H_{(x,z)}(h_1)e_{[\xi_1]_H}\rangle = \sum_{(u,p,y_1)\in[\eta\xi_1^{-1}]_H} z^p h_1(u, p, y_1) = z^{-n_1} h_1(\xi_1^{-1}) = z^{-n_1}.$$

We deduce that $\pi^H_{(x,z)}(h_1)e_{[\xi_1]_H} = z^{-n_1}e_{[\eta]_H}$ where $[\eta]_H = [(x, 0, x)]_H$ and therefore that $\pi^H_{(x,z)}(h_1)^* e_{[(x,0,x)]_H} = z^{n_1}e_{[\xi_1]_H}$. A similar calculation shows that $\pi^H_{(x,z)}(h_2)e_{[(x,0,x)]_H} = z^{n_2}e_{[\xi_2]_H}$. Hence

$$0 = \langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(h_1fh_2)e_{[(x,0,x)]_H}\rangle = z^{n_2-n_1}\langle e_{[\xi_1]_H}, \pi^H_{(x,z)}(f)e_{[\xi]_H}\rangle,$$

and this shows that $\pi^H_{(x,z)}(f) = 0$.

For the final statement, the "only if" implication is trivial. For the "if" direction, fix $f \in I$. Then for any pair of homogeneous bisections $B_1, B_2 \subseteq G_T$ and any pair of functions $h_i \in C_c(B_i)$, we have $h_1fh_2 \in I$, and so $\langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(h_1fh_2)e_{[(x,0,x)]_H}\rangle = 0$ by hypothesis. Hence $\pi^H_{(x,z)}(f) = 0$ by the first statement. □

We shall need to know that averaging over subgroups of H preserves the kernel of $\pi^H_{(x,z)}$.

Lemma 3.5. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Fix nested subgroups $H \leq K$ of $\mathbb{Z}^k$ and a point $(x, z) \in X \times \mathbb{T}^k$. Let E_K be the conditional expectation on $C^*(G_T)$ given by*

$$E_K(f)(x, n, y) = \int_{K^\perp} w^n f(x, n, y)\, dw,$$

for all $f \in C_c(G_T)$ and $(x, n, y) \in G_T$. Then $E_K\big(\ker(\pi^H_{(x,z)})\big) \subseteq \ker(\pi^H_{(x,z)})$.

Proof. Let $\alpha\colon K^\perp \to \operatorname{Aut}(C^*(G_T))$ be the action given by $\alpha_w(f)(x, n, y) = w^n f(x, n, y)$, for all $f \in C_c(G_T)$ and $(x, n, y) \in G_T$. Then $E_K(f) = \int_{K^\perp} \alpha_w(f)\, dw$. Fix basis vectors $e_{[(u,m,x)]_H}, e_{([v,n,x)]_H}$ of $\ell^2((G_T)_x/{\sim_H})$. For any $f \in \ker(\pi^H_{(x,z)})$, we calculate, using [RW98, Lemma C.2] at the first equality, and that $H \leq K^\perp$ at the last equality:

$$\begin{aligned}
\langle e_{[(u,m,x)]_H}, \pi^H_{(x,z)}(E_K(f))e_{[(v,n,x)]_H}\rangle &= \int_{K^\perp} \sum_{h\in H} z^{h+m-n}w^{h+m-n} f(u, h+m-n, v)\, dw \\
&= \int_{K^\perp} w^{m-n} \sum_{h\in H} z^{h+m-n} f(u, h+m-n, v)\, dw \\
&= \int_{K^\perp} w^{m-n}\langle e_{[(u,m,x)]_H}, \pi^H_{(x,z)}(f)e_{[(v,n,x)]_H}\rangle\, dw \\
&= 0.
\end{aligned}$$

Therefore, $\pi^H_{(x,z)}(E_K(f)) = 0$ and the claim follows. □

4. The map π is continuous

We show in this section that the map from $X \times \mathbb{Z}^k$ to the primitive-ideal space of the C^*-algebra of a Deaconu–Renault groupoid determined by the irreducible representations $\pi^{\mathbb{Z}^k}_{(x,z)}$ from the preceding section is continuous. We will use the following notation throughout the remainder of the paper.

Notation 4.1. Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For each $x \in X$ and $z \in \mathbb{T}^k$, we write

$$\pi_{(x,z)} := \pi^{\mathbb{Z}^k}_{(x,z)}\colon C^*(G_T) \to B(\ell^2((G_T)_x/{\sim_{\mathbb{Z}^k}})) \tag{4.1}$$

for the representation obtained from Proposition 3.1 applied with $H = \mathbb{Z}^k$. By [SW16, Theorem 3.2] and Remark 3.2, the map

$$\pi\colon X \times \mathbb{T}^k \to \operatorname{Prim}(C^*(G_T)) \tag{4.2}$$

defined by $\pi(x,z) := \ker(\pi_{(x,z)})$ is a surjection.

With the notation above, we have $\ker(\pi_{(x,z)}) = \ker(\pi_{(x',z')})$ if and only if $\overline{[x]} = \overline{[x']}$ and z and z' determine the same character of $c(\mathcal{I}^{\mathrm{ess}}_x) = c(\mathcal{I}^{\mathrm{ess}}_{x'})$.

Suppose that $\mathcal{B} = \{B_i : i \in I\}$ is a family of bisections of G_T. We write $\mathcal{B}^{\mathrm{ess}}$ for the intersection

$$\mathcal{B}^{\mathrm{ess}} := \big(\textstyle\bigcup \mathcal{B}\big) \cap \mathcal{I}^{\mathrm{ess}} = \big(\textstyle\bigcup_{i\in I} B_i\big) \cap \mathcal{I}^{\mathrm{ess}}.$$

This is an algebraic bundle of subsets of $\mathbb{Z}^k$ (its fibres are not necessarily groups, and it need not be particularly well behaved topologically; for example, it is unlikely to be locally compact). For $x \in X$, we write $\mathcal{B}^{\mathrm{ess}}_x$ for the fibre $\mathcal{B}^{\mathrm{ess}} \cap G_x$ of this bundle over x. We then write

$$(\mathcal{B}^{\mathrm{ess}})^\perp = \{(x,z) \in (s(\mathcal{B}^{\mathrm{ess}}) \times \mathbb{T}^k) : z^{c(\gamma)} = 1 \text{ for all } \gamma \in \mathcal{B}^{\mathrm{ess}}_x\}.$$

Algebraically, this is a bundle over $s(\mathcal{B}^{\mathrm{ess}})$ of subgroups of $\mathbb{T}^k$, though it need not be topologically well-behaved. We think of it as the bundle of annihilators of the fibres of $\mathcal{B}^{\mathrm{ess}}$.

Given a subset $W \subseteq s(\mathcal{B}^{\mathrm{ess}}) \times \mathbb{T}^k$, we define the *$\mathcal{B}$-saturation* of W to be the set

$$W \cdot (\mathcal{B}^{\mathrm{ess}})^\perp = \{(x, wz) : (x,w) \in W \text{ and } (x,z) \in (\mathcal{B}^{\mathrm{ess}})^\perp\}.$$

Equivalently,

$$W \cdot (\mathcal{B}^{\mathrm{ess}})^\perp = \bigcup_{(x,w)\in W} \{(x,z) : z^{c(\gamma)} = w^{c(\gamma)} \text{ for all } \gamma \in \mathcal{B}^{\mathrm{ess}}_x\}.$$

Theorem 4.2. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Let $\pi\colon X \times \mathbb{T}^k \to \operatorname{Prim}(C^*(G_T))$ be as in Notation 4.1. Let $A \subseteq \operatorname{Prim}(C^*(G_T))$ be an open subset. Suppose that $(x,z) \in \pi^{-1}(A)$ and that $(B_\alpha)_{\alpha\in\mathcal{J}_x}$ is a family of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for each $\alpha \in \mathcal{J}_x$. Then there exist an open neighbourhood $U \subseteq B_x \cap X$ of x and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that the $\mathcal{B}$-saturation $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp$ of $U \times V$ is contained in $\pi^{-1}(A)$.*

Remark 4.3. The condition in the conclusion of the theorem that $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \pi^{-1}(A)$ can be restated as follows: whenever $x_1 \in U$, $z_1 \in V$, and $z_2 \in \mathbb{T}^k$ satisfy $(z_1)^h = (z_2)^h$ for every $h \in c(\mathcal{J}_x) \cap c(\mathcal{B}^{\mathrm{ess}}_{x_1})$, we have $(x_1, z_2) \in \pi^{-1}(A)$.

Proof of Theorem 4.2. We prove the contrapositive; that is, we consider a point $(x,z) \in X \times \mathbb{T}^k$ and a sequence $(x_i, z_i, z'_i)_{i\in\mathbb{N}} \in X \times V \times \mathbb{T}^k$ satisfying

(i) $x_i \to x$ and $z_i \to z$ as $i \to \infty$,
(ii) for every $i \in \mathbb{N}$, we have $(z_i)^h = (z_i')^h$ for all $h \in c(\bigcup \mathcal{B} \cap \mathcal{I}_{x_i}^{\mathrm{ess}})$, and
(iii) for every $i \in \mathbb{N}$, we have $(x_i, z_i') \notin \pi^{-1}(A)$,

and we prove that $(x, z) \notin \pi^{-1}(A)$.

Since A is open in the hull-kernel topology, there is an ideal $I \subseteq C^*(G_T)$ such that $A = \{P \in \mathrm{Prim}(C^*(G_T)) : I \not\subseteq P\}$. Condition (iii) implies that each $(x_i, z_i') \notin \pi^{-1}(A)$, and hence that each $\ker(\pi_{(x_i,z_i')}) \notin A$; so

$$I \subseteq \ker(\pi_{(x_i,z_i')}) \quad \text{for all } i. \tag{4.3}$$

Let H be the subgroup of $\mathbb{Z}^k$ generated by

$$\textstyle\bigcup_{i=1}^{\infty} \left(\bigcap_{j=i}^{\infty} c(\mathcal{B}_{x_i}^{\mathrm{ess}})\right) = \{h \in c(\mathcal{J}_x) : (x_i, h, x_i) \in B_{(x,h,x)} \cap \mathcal{I}^{\mathrm{ess}} \text{ for large } i\}. \tag{4.4}$$

Since subgroups of $\mathbb{Z}^k$ are finitely generated, by discarding finitely many terms in the sequence $(x_i, z_i, z_i')_i$ and relabelling, we may assume that

$$H \subseteq c(\mathcal{B}_{x_i}^{\mathrm{ess}}) \subseteq c(\mathcal{I}_{x_i}^{\mathrm{ess}}) \text{ for all } i. \tag{4.5}$$

Let $\pi_{(x,z)}^H : C^*(G_T) \to \mathcal{B}(\ell^2([x]) \otimes \ell^2(\mathbb{Z}^k/H))$ be the representation of Proposition 3.1. We will show that

$$I \subseteq \ker(\pi_{(x,z)}^H) \subseteq \ker(\pi_{(x,z)}), \tag{4.6}$$

giving $(x, z) \notin \pi^{-1}(A)$ as required. Since $\ker(\pi_{(x,z)}) = \ker(\pi_{(x,z)}^{\mathbb{Z}^k})$ (see Remark 3.2), the second inclusion in (4.6) follows from Proposition 3.1, so we just have to establish that $I \subseteq \ker(\pi_{(x,z)}^H)$.

By the final statement of Lemma Lemma 3.4 it suffices to prove that

$$\langle e_{[(x,0,x)]_H}, \pi_{(x,z)}^H(f) e_{[(x,0,x)]_H}\rangle = 0, \tag{4.7}$$

for all $f \in I$.

Fix $f \in I$ and $\varepsilon > 0$, and choose $g \in C_c(G_T)$ such that $\|f - g\| < \varepsilon/3$. Since g has compact support, there is a finite subset $F \subseteq \mathbb{Z}^k$ such that

$$\mathrm{supp}(g) \subseteq \bigcup_{h \in F} c^{-1}(h).$$

We have

$$\langle e_{[(x,0,x)]_H}, \pi_{(x,z)}^H(g) e_{[(x,0,x)]_H}\rangle = \sum_{h \in H \cap F} z^h g(x, h, x). \tag{4.8}$$

By (i), there exists $i \in \mathbb{N}$ such that

$$\left| \sum_{h \in H \cap F} z^h g(x, h, x) - \sum_{h \in H \cap F} (z_j)^h g(x_j, h, x_j) \right| < \varepsilon/3 \tag{4.9}$$

for all $j \geq i$.

We now make a few observations. The set

$$\{h \in \mathbb{Z}^k : (z_i)^h = (z_i')^h \text{ and } (x_i, h, x_i) \in \mathcal{I}^{\mathrm{ess}} \text{ for large } i\}$$

is a subgroup of $\mathbb{Z}^k$ that contains H by (ii). After possibly discarding finitely many terms of the sequence (x_i, z_i, z_i') and re-indexing, we may therefore assume that $(z_i)^h = (z_i')^h$ and $(x_i, h, x_i) \in \mathcal{I}^{\mathrm{ess}}$ for all $i \in \mathbb{N}$ and all $h \in H \cap F$.

Since F is finite, by passing to a subsequence of the $(x_i, z_i, z_i')_i$ and re-indexing, we may further assume that for every $h' \in F$ either $(x_i, h', x_i) \in \mathcal{I}^{\mathrm{ess}}$ for all $i \in \mathbb{N}$ or $(x_i, h', x_i) \notin \mathcal{I}^{\mathrm{ess}}$ for all $i \in \mathbb{N}$.

Suppose that $h' \in F$ satisfies $(x_i, h', x_i) \in \mathcal{I}^{\mathrm{ess}}$ for all $i \in \mathbb{N}$ and that there is a subsequence $(x_{i_j})_{j \in \mathbb{N}}$ of $(x_i)_{i \in \mathbb{N}}$ such that $(x_{i_j}, h', x_{i_j}) \in \mathrm{supp}(g)$ for all j. Since $\mathrm{supp}(g)$

is compact, every subsequence of $(x_{i_j}, h', x_{i_j})_j$ has a convergent subsequence, and since $x_i \to x$, the limit is (x, h', x). Hence $(x_{i_j}, h', x_{i_j}) \to (x, h', x)$. In particular, $h' \in \mathcal{J}_x$ and $(x_{i_j}, h', x_{i_j}) \in \bigcup \mathcal{B}$ for large j, so $h' \in H$.

By the preceding paragraph, if $h' \in F$ satisfies $(x_i, h', x_i) \in \mathcal{I}^{\text{ess}}$ for all i but $h' \notin H$, then $g(x_i, h, x_i) = 0$ for large i. So for each such h', by discarding finitely many terms of the sequence (x_i, z_i, z_i') and relabelling again, we may assume that $g(x_i, h, x_i) = 0$ for all $i \in \mathbb{N}$, whenever $h' \in F \setminus H$ satisfies $(x_i, h', x_i) \in \mathcal{I}^{\text{ess}}$ for all i.

For each $i \in \mathbb{N}$, let E_i be the conditional expectation on $C^*(G_T)$ satisfying

$$E_i(f')((w,p,y)) = \int_{c(\mathcal{I}^{\text{ess}}_{x_i})^{\perp}} z^p f((w,p,y))\, dz$$

for $f' \in C^*(G_T)$ and $(w, p, y) \in G_T$.

Since $(z_i)^h = (z_i')^h$ and $(x_i, h, x_i) \in \mathcal{I}^{\text{ess}}$ for $h \in H \cap F$, and $g(x_i, h, x_i) = 0$ for $h \in \{h' \in F : (x_i, h', x_i) \in \mathcal{I}^{\text{ess}} \text{ for all } i \in \mathbb{N}\} \setminus H$,

$$\begin{aligned}
\sum_{h \in H \cap F} (z_i)^h g(x_i, h, x_i) &= \sum_{h \in H \cap F} (z_i')^h g(x_i, h, x_i) \\
&= \sum_{h \in F \cap c(\mathcal{I}^{\text{ess}}_{x_i})} (z_i')^h g(x_i, h, x_i) \\
&= \langle \delta_{x_i}, \pi_{(x_i, z_i')}(E_i(g))\delta_{x_i} \rangle. \qquad (4.10)
\end{aligned}$$

Since $H \subseteq \mathcal{I}^{\text{ess}}_{x_i}$ for all i (see (4.5)), Lemma 3.5 shows that $E_i(\ker(\pi^H_{(x_i, z_i')})) \subseteq \ker(\pi^H_{(x_i, z_i')})$ for all i. So it follows from (4.3) that that $E_i(f) \in \ker(\pi^H_{(x_i, z_i')})$ for all i. Since $\|f - g\| < \varepsilon/3$ and each E_i is a contraction, we deduce that

$$\left| \langle e_{x_i}, \pi^H_{(x_i, z_i')}(E_i(g)) e_{x_i} \rangle \right| < \varepsilon/3.$$

Combining this with (4.8), (4.9), and (4.10), we conclude that

$$\left| \langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(g) e_{[(x,0,x)]_H} \rangle \right| < 2\varepsilon/3,$$

and since $\|f - g\| < \varepsilon/3$ it follows that

$$\left| \langle e_{[(x,0,x)]_H}, \pi^H_{(x,z)}(f) e_{[(x,0,x)]_H} \rangle \right| < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary this proves the claim (4.7). □

This result allows us to infer that the map π is continuous. Elementary examples (such as the dumbbell graph—see Section 10.1) show that it is not typically open.

Corollary 4.4. *The map $\pi \colon X \times \mathbb{T}^k \to \operatorname{Prim}(C^*(G_T))$ of Notation 4.1 is continuous.*

Proof. Pick an open set $A \subseteq \operatorname{Prim}(C^*(G_T))$ and take $(x, z) \in \pi^{-1}(A)$. Since G_T is étale, there exists a collection $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_x}$ of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for all $\alpha \in \mathcal{J}_x$. Theorem 4.2 implies that there are open sets $U \subseteq X$ and $V \subseteq \mathbb{T}^k$ containing x and z, respectively, such that $(U \times V) \cdot (\mathcal{B}^{\text{ess}})^{\perp} \subseteq \pi^{-1}(A)$. Since $(\mathcal{B}^{\text{ess}})^{\perp}$ is a bundle over $s(\mathcal{B}^{\text{ess}})$ of subgroups of $\mathbb{T}^k$ it contains $s(\mathcal{B}^{\text{ess}}) \times \{1\}$. Since B_0 is a neighbourhood of $(x, 0, x)$, it contains an open neighbourhood $(x, 0, x) \in U' \subseteq G_T^{(0)} \subseteq \mathcal{I}^{\text{ess}}$, so $U' \times \{1\} \subseteq s(\mathcal{B}^{\text{ess}}) \times \{1\}$. In particular $(U \cap U') \times V = (U \times V) \cdot (U' \times \{1\})$ is an open subset of $X \times \mathbb{T}^k$ and we have $(x, z) \in (U \cap U') \times V \subseteq \pi^{-1}(A)$. Hence $\pi^{-1}(A)$ is open in $X \times \mathbb{T}^k$, and therefore π is continuous. □

5. The sandwiching lemma for Deaconu–Renault groupoids

In [BCS22, Lemma 3.3], we proved that for any étale groupoid G and any ideal $I \subseteq C^*(G)$, the set

$$U = \{x \in G^{(0)} : j(f)(x) \neq 0 \text{ for some } f \in I \cap C_0(G^{(0)})\}$$

is the unique smallest open invariant set such that $I \subseteq I_U = C^*(G|_U)$, and

$$V = \{x \in G^{(0)} : j(a)(x) \neq 0 \text{ for some } a \in I\}$$

is the unique largest open invariant set such that $C^*(G|_V) = I_V \subseteq I$. We refer to U and V as the sandwich sets related to I.

In this section, we identify the sandwich sets for an ideal I of the C^*-algebra of a Deaconu–Renault groupoid G_T, and relate them to the open set $\{(x, z) \in X \times \mathbb{T}^k : I \not\subseteq \ker(\pi_{(x,z)})\}$ corresponding to a representation $\pi_{(x,z)}$ as in Notation 4.1. This serves to relate the sandwiching lemma to Katsura's results for singly-generated dynamical systems in Section 10.

We first need to know that if the groupoid G_T admits a unit x with dense orbit, then the direct sum of the representations $\pi_{(x,z)}$ as z ranges over $\mathbb{T}^k$ is faithful.

Lemma 5.1. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For $x \in X$ and $z \in \mathbb{T}^k$ let $\pi_{(x,z)}$ be as in Notation 4.1. Suppose that $x \in X$ satisfies $\overline{[x]} = X$. Then $\bigoplus_{z\in\mathbb{T}^k} \pi_{(x,z)}$ is a faithful representation of $C^*(G_T)$. In particular, writing λ_x for the regular representation of $C^*(G_T)$ on $\ell^2((G_T)_x)$, we have $\ker\big(\bigoplus_{z\in\mathbb{T}^k} \pi_{(x,z)}\big) = \ker(\lambda_x)$.*

Proof. We identify the groupoid G_T with the groupoid of the topological higher-rank graph Λ defined by $\Lambda^n = X \times \{n\}$ for all n, range and sources maps given by $s(x, n) = (T^n(x), 0)$ and $r(x, n) = (x, 0)$ and factorisation rules $(x, m)(T^m(x), n) = (x, m + n) = (x, n)(T^n(x), m)$. The isomorphism $C^*(G_T) \cong C^*(\Lambda)$ induced by this identification carries $C_0(G_T^{(0)})$ to $C_0(\Lambda^0)$. Hence, by [CLSV11, Corollary 5.21], it suffices to show that $\bigoplus_{z\in\mathbb{T}^k} \pi_{(x,z)}$ is faithful on $C_0(X)$ and that there is an action β of $\mathbb{T}^k$ on $\bigoplus_{z\in\mathbb{T}^k} \ell^2([x])$ such that $\beta_w \circ (\bigoplus_{z\in\mathbb{T}^k} \pi_{(x,z)}) = (\bigoplus_{z\in\mathbb{T}^k} \pi_{(x,z)}) \circ \gamma_w$ for all $w \in \mathbb{T}^k$.

For the first statement, observe that since $\overline{[x]} = X$, if $f \in C_0(X)$ is nonzero, then there exists $y \in [x]$ such that $f(y) \neq 0$. Thus, for any z we have $\langle e_y, \pi_{(x,z)}(f)e_y\rangle = f(y) \neq 0$. So $\bigoplus_z \pi_{(x,z)}(f) \neq 0$.

For the second, to keep notation straight, identify $\bigoplus_{z\in\mathbb{T}^k} \ell^2([x])$ with $\ell^2([x]\times\mathbb{T}^k)$ so that the copy of $\ell^2([x])$ corresponding to $z \in \mathbb{T}^k$ is identified with $\ell^2([x] \times \{z\}) \subseteq \ell^2([x] \times \mathbb{T}^k)$.

For each $z \in \mathbb{T}^k$ write $\mathcal{H}_z$ for the summand of $\bigoplus_{z\in\mathbb{T}^k} \ell^2([x])$ corresponding to z, and denote the canonical orthonormal basis for $\mathcal{H}_z$ by $\{e^z_y : y \in [x]\}$. For each $w \in \mathbb{T}^k$, let $U_w : \bigoplus_z \mathcal{H}_z \to \bigoplus_z \mathcal{H}_z$ be the unitary given by $U_w e^z_y = e^{wz}_y$. By definition, $\pi_{(x,z)} = \pi_{(x,1)} \circ \gamma_z$, regarded as representations on $\ell^2([x])$. So regarding $\pi_{(x,z)}$ as a representation on $\mathcal{H}_z$ and $\pi_{(x,1)}$ as a representation on $\mathcal{H}_1$, we have $\pi_{(x,z)} = \mathrm{Ad}_{U_z} \circ \pi_{(x,1)} \circ \gamma_z$. Hence

$$\begin{aligned}\pi_{(x,z)} \circ \gamma_w &= \mathrm{Ad}_{U_z} \circ \pi_{(x,1)} \circ \gamma_z \circ \gamma_w\\ &= \mathrm{Ad}_{U_z} \circ \mathrm{Ad}_{U^*_{wz}} \circ \pi_{(x,wz)}\\ &= \mathrm{Ad}_{U^*_w} \circ \pi_{(x,z)}.\end{aligned}$$

So $\beta_w = \mathrm{Ad}_{U^*_w}$ gives the desired action. □

Now we characterise the sandwich sets for an ideal in $C^*(G_T)$.

Proposition 5.2. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For $x \in X$ and $z \in \mathbb{T}^k$, let*

$\pi_{(x,z)}$ be as in Notation 4.1. Let I be an ideal of $C^*(G_T)$, and let U and V be the sandwich sets related to I. The set

$$W = \{(x,z) \in X \times \mathbb{T}^k : I \not\subseteq \ker(\pi_{(x,z)})\} \tag{5.1}$$

is open and

$$X \setminus V = \{x \in X : (\{x\} \times \mathbb{T}^k) \cap W = \varnothing\} \quad \text{and} \quad U = \{x \in X : \{x\} \times \mathbb{T}^k \subseteq W\}.$$

Proof. Since I is an ideal in $C^*(G_T)$, the set $\{P \in \operatorname{Prim} C^*(G_T) : I \subseteq P\}$ is open, and W is the preimage of this open set under $\pi\colon X \times \mathbb{T}^k \to \operatorname{Prim} C^*(G_T)$ which is continuous by Corollary 4.4.

Observe that if $f \in C_c(G_T|_V)$, $y \in X \setminus V$, and $z \in \mathbb{T}^k$, then we have $\pi_{(y,z)}(f) = 0$, so by continuity, we have $I_V \subseteq \ker(\pi_{(y,z)})$ for all z. In particular, $I \subseteq I_V \subseteq \ker(\pi_{(y,z)})$, which implies that $(y,z) \notin W$ for all $y \in X \setminus V$ and $z \in \mathbb{T}^k$. That is, $X \setminus V \subseteq \{x \in X : (\{x\} \times \mathbb{T}^k) \cap W = \varnothing\}$. For the reverse containment, let $x \in X$ and suppose that $(\{x\} \times \mathbb{T}^k) \cap W = \varnothing$. Then $I \subseteq \bigcap_{z \in \mathbb{T}^k} \ker(\pi_{(x,z)})$, so $I \subseteq \ker(\lambda_x)$ where λ_x is the regular representation of $C^*(G_T)$ by Lemma 5.1. If $j\colon C^*(G_T) \to C_0(G_T)$ is Renault's map, then $j(a)|_{(G_T)_x} = 0$ for all $a \in I$. In particular, $j(I)(x) = \{0\}$, so that $x \notin V$.

For the second statement, first observe that $C_0(U) \subseteq \ker(\pi_{(x,z)})$ if and only if $x \notin U$. Since $C_0(U) \subseteq I_U \subseteq I$, if $x \in U$ then $I \not\subseteq \ker(\pi_{(x,z)})$ for all $z \in \mathbb{T}^k$. Hence if $x \in U$ then $\{x\} \times \mathbb{T}^k \subseteq W$. Thus $U \subseteq \{x \in X : x \times \mathbb{T}^k \subseteq W\}$.

For the reverse containment, let $O = \{x \in X : \{x\} \times \mathbb{T}^k \subseteq W\}$. Then $X \setminus O$ is the image of $(X \times \mathbb{T}^k) \setminus W$ under the projection map which is closed because $\mathbb{T}^k$ is compact, so O is open in X. It is invariant by [SW16, Theorem 3.2]. Since U is the largest open invariant set such that $I_U \subseteq I$, so to see that $O \subseteq U$ it suffices to show that $I_O \subseteq I$. Recall from [SW16, Theorem 3.2] that $\operatorname{Prim}(C^*(G_T)) = \{\ker(\pi_{(x,z)}) : x \in X, z \in \mathbb{T}^k\}$ as a set. Since every ideal of a separable C^*-algebra is the intersection of the primitive ideals that contain it, we have

$$I = \bigcap_{\{(x,z) : I \subseteq \ker(\pi_{(x,z)})\}} \ker(\pi_{(x,z)}) = \bigcap_{(x,z) \in (X \times \mathbb{T}^k) \setminus W} \ker(\pi_{(x,z)}).$$

Let $K = X \setminus O$ and observe that $(X \times \mathbb{T}^k) \setminus W \subseteq (X \times \mathbb{T}^k) \setminus (O \times \mathbb{T}^k) = K \times \mathbb{T}^k$. So

$$I \supseteq \bigcap_{(x,z) \in K \times \mathbb{T}^k} \ker(\pi_{(x,z)}) = \bigcap_{x \in K} \Big(\bigcap_{z \in \mathbb{T}^k} \ker(\pi_{(x,z)}) \Big) = \bigcap_{x \in K} \ker \Big(\bigoplus_{z \in \mathbb{T}^k} \ker(\pi_{(x,z)}) \Big).$$

For each x, write λ_x for the regular representation of $C^*(G_T)$ on $\ell^2((G_T)_x)$. Then Lemma 5.1 gives

$$I \supseteq \bigcap_{x \in K} \ker(\lambda_x) = \ker \Big(\bigoplus_{x \in K} \lambda_x \Big).$$

This is precisely the regular representation of $C^*(G_T|_K)$, and since Deaconu–Renault groupoids are amenable, it is faithful on $C^*(G_T|_K)$. So by, for example, [Sim20, Proposition 4.3.2], we have $\ker \Big(\bigoplus_{x \in K} \lambda_x \Big) = I_{X \setminus K} = I_O$. Therefore, $I_O \subseteq I$ as claimed. □

By [BCS22, Theorem 3.5], in order to understand the ideals of the C^*-algebras of an action $T\colon \mathbb{N}^k \curvearrowright X$ it suffices to describe the purely non-dynamical ideals with full support; this is the ideals I satisfying $I \cap C_0(X) = \{0\}$ and $\operatorname{supp}(I) = G_T$. The following corollary describes an obstruction to the existence of such an ideal (cf. [BCS22, Section 4]).

Corollary 5.3. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Suppose that I is an ideal of $C^*(G_T)$ such that $I \cap C_0(X) = \{0\}$ and $\operatorname{supp}(I) = G_T$. For every $x \in X$ and every family $(B_\alpha)_{\alpha \in \mathcal{J}_x}$ of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for each $\alpha \in \mathcal{J}_x$, there is a* *neighbourhood U of X such that $\mathcal{B}_y^{\mathrm{ess}} \neq \{y\}$ for all $y \in U$. In particular, $\mathcal{I}_x^{\mathrm{ess}}(G_T) \neq \{x\}$ for all $x \in X$.*

Proof. Let W be as in (5.1) and let $W_x = \{z \in \mathbb{T}^k : (x, z) \in W\}$ for every $x \in X$. Since $I \cap C_0(X) = \{0\}$ and $\mathrm{supp}(I) = G_T$, the sandwich sets for I are $U = \varnothing$ and $V = X$, so each W_x is a nonempty and proper subset of $\mathbb{T}^k$ by Proposition 5.2.

Fix $x \in X$, and a family $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_x}$ of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for each $\alpha \in \mathcal{J}_x$. Since W_x is nonempty, there exists $z \in \mathbb{T}^k$ such that $(x, z) \in W$. Theorem 4.2 shows that there exist an open neighbourhood $U \subseteq B_x \cap X$ of x and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq W$. In particular, for each $y \in U$ we have $(\mathcal{B}_y^{\mathrm{ess}})^\perp \cdot z \subseteq W_y \subsetneq \mathbb{T}^k$. This forces $(\mathcal{B}_y^{\mathrm{ess}})^\perp \subsetneq \mathbb{T}^k$ and hence $\mathcal{B}_y^{\mathrm{ess}} \neq \{y\}$.

For the final statement, just observe that the preceding paragraph implies in particular that $\mathcal{B}_x^{\mathrm{ess}} \neq \{x\}$, and since $\mathcal{B}_x^{\mathrm{ess}} \subseteq \mathcal{I}_x^{\mathrm{ess}}(G_T)$, the result follows. □

We now obtain an obstruction ideal, in the sense of Ara–Lolk [AL18, Definition 7.11] for a Deaconu–Renault groupoid.

Corollary 5.4. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Let $W' = s(\mathcal{I}^{\mathrm{ess}}(G_T) \setminus X)^\circ$, and let W be the set of points $x \in X$ such that for every family $(B_\alpha)_{\alpha \in \mathcal{J}_x}$ of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for each $\alpha \in \mathcal{J}_x$, there is a neighbourhood U of X such that $\mathcal{B}_y^{\mathrm{ess}} \neq \{y\}$ for all $y \in U$. Then $W \subseteq W'$, and if I is an ideal of $C^*(G_T)$ such that $I \cap C_0(X) = \{0\}$, then $I \subseteq I_W$.*

Proof. The inclusion $W \subseteq W'$ is clear so we just need to prove the second containment. Let U and V be the sandwich sets for I, and note that $U = \varnothing$ because $I \cap C_0(X) = \{0\}$. Moreover, $I \subseteq I_V \cong C^*(G_T|_V)$, so we can regard I as an ideal of $C^*(G_T|_V)$ with $I \cap C_0(V) = \{0\}$ and $\mathrm{supp}(I) = G_T|_V$. Now Corollary 5.3 implies that $V \subseteq W$. □

Remark 5.5. The definition of W in Corollary 5.4 is more technical than that of W', but W can be strictly smaller, so it provides a better estimate. For example, in the instance of the dumbbell graph of Example 2.7, the essential isotropy at every unit is nontrivial, and so the set W' of Corollary 5.4 is all of E^∞. However, as we will see in Section 10.1 below, it is straightforward to construct a family $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_{e^\infty}}$ of bisections such that for $n \neq 0$, we have $B_{(e^\infty, n, e^\infty)} \cap \mathcal{I}(G_E) = \{(e^\infty, n, e^\infty)\}$. So the set W of Corollary 5.4 is the (open) orbit of g^∞, which is the support of the minimal obstruction ideal described in [BCS22, Definition 4.3].

6. Harmonious families of bisections

Our main result requires the concept of a harmonious family of bisections based at a unit. Recall from Notation 2.4 that for an etale groupoid G, we write $\mathcal{J}_x = \overline{\mathcal{I}^{\mathrm{ess}}(G)}_x$. We emphasise that harmonious families of bisections are meaningful for any étale groupoid but we shall only study the case of groupoids G_T of actions $T\colon \mathbb{N}^2 \curvearrowright X$ by local homeomorphisms.

Definition 6.1. A *harmonious family of bisections based at a unit* $x \in X$ is a collection $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_x}$ of open bisections of G_T satisfying the following conditions:

(i) $B_x \subseteq X$;
(ii) $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for all $\alpha \in \mathcal{J}_x$;
(iii) $(B_\alpha \cap \mathcal{I}^{\mathrm{ess}})^{-1} = B_{\alpha^{-1}} \cap \mathcal{I}^{\mathrm{ess}}$ for all α;
(iv) $B_\alpha(B_\beta \cap \mathcal{I}^{\mathrm{ess}}) \subseteq B_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{J}_x$; and
(v) for each $\alpha \in \mathcal{J}_x$ there exists a compact set $K_\alpha \subseteq G^{(0)}$ such that $s(B_\alpha) = K_\alpha \cap B_x$.

In particular, $\bigcup\mathcal{B} \subseteq s^{-1}(B_x)$. We say a unit $x \in X$ *admits a harmonious family of bisections* if there exist a harmonious family of bisections based at x and we frequently just say that $(B_\alpha)_{\alpha\in\mathcal{J}_x}$ *is a harmonious family of bisections*; it is implicit that it is based at x.

Remark 6.2. If $\mathcal{B} = (B_\alpha)_{\alpha\in\mathcal{J}_x}$ is a harmonious family of bisections in G_T, then $\mathcal{B}_y^{\mathrm{ess}}$ is a subgroup of $(G_T)_y$ for every $y \in B_x \subseteq X$, and determines a subgroup

$$H_{\mathcal{B}}(y) := c(\mathcal{B}_y^{\mathrm{ess}}) \leq c(\mathcal{J}_x) \leq \mathbb{Z}^k.$$

We have

$$H_{\mathcal{B}}(y) = \{n \in \mathbb{Z}^k : (x, n, x) \in \mathcal{J}_x \text{ and } (y, n, y) \in B_{(x,n,x)} \cap \mathcal{I}^{\mathrm{ess}}\}.$$

Remark 6.3. Condition (v) in Definition 6.1 is a technical condition that captures two special cases:

(1) if there are bisections B_α satisfying (i)–(iv) that are all compact open sets, then Condition (v) is satisfied since we can take $K_\alpha = s(B_\alpha)$ for all α; and
(2) if there are bisections B_α satisfying (i)–(iv) such that $s(B_\alpha) = B_x$ for all $\alpha \in \mathcal{J}_x$ (such as when T is an action by homeomorphisms), then for any precompact open U such that $x \in U \subseteq \overline{U} \subseteq B_x$, the sets $(B_\alpha U)_\alpha$ also satisfy (i)–(iv), and they satisfy (v) since we can take $K_\alpha = \overline{U}$ for all α.

The condition is used in the proof of Proposition 7.2 to show that the functions $1_{B_\alpha}\phi$ are continuous.

Example 6.4. (1) Any isolated unit admits harmonious families of bisections.
(2) If G_T is a Deaconu–Renault groupoid that is strongly effective, then $\mathcal{J}_x = \{x\}$ for every unit x, and G_T admits a harmonious family of bisections.

For the following, recall that a subset C of the groupoid G_T is homogeneous, if $c(C)$ is a singleton.

Lemma 6.5. *Let X be a second-countable locally compact Hausdorff space and suppose $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Suppose that $\mathcal{B} = (B_\alpha)_{\alpha\in\mathcal{J}_x}$ is a harmonious family of bisections based at $x \in X$, and that C is an open and homogeneous bisection such that $x \in s(C)$. Let $\gamma \in C$ be the unique element such that $s(\gamma) = x$. Then $(CB_\alpha C^{-1})_{\alpha\in\mathcal{J}_x}$ is a harmonious family of bisections at $r(\gamma)$. In particular, if $x \in X$ admits a harmonious family of bisections then every element in the orbit of x admits a harmonious family of bisections.*

Proof. For any $\alpha \in \mathcal{J}_x$, we have

$$\gamma\alpha\gamma^{-1} \in CB_h(x)C^{-1}.$$

Since $\mathcal{I}^{\mathrm{ess}}(G_T)$ is a normal subgroupoid by Lemma 2.2 the map $\alpha \mapsto \gamma\alpha\gamma^{-1}$ is a group isomorphism between $\mathcal{I}^{\mathrm{ess}}(G_T)_x$ and $\mathcal{I}^{\mathrm{ess}}(G_T)_z$, so $(CB_\alpha C^{-1})_{\alpha\in\mathcal{J}_x}$ is a family of bisections indexed by $\mathcal{J}_z$.

For each $\beta \in \mathcal{J}_z$, let $C_\beta := CB_{\gamma^{-1}\beta\gamma}C^{-1}$ and write $\mathcal{C} = (C_\beta)_{\beta\in\mathcal{J}_z} = (CB_\alpha C^{-1})_{\alpha\in\mathcal{J}_x}$.

The set CB_xC^{-1} is contained in X and contains $z := r(\gamma)$, which is Definition 6.1(i).

Fix $\beta \in \mathcal{J}_z$ and let $\alpha = \gamma^{-1}\beta\gamma$. Then $\alpha \in \mathcal{J}_x$, and we have $\beta = \gamma\alpha\gamma^{-1} \in C_\beta$. For any $\zeta \in C_\beta = CB_\alpha C^{-1}$, there exist $\eta \in B_\alpha$ and $\rho, \tau \in C$ such that $\zeta = \rho\eta\tau^{-1}$. Since C is homogeneous, we have $c(\rho) = c(\tau)$, and so $c(\zeta) = c(\rho) + c(\eta) - c(\tau) = c(\eta) \in c(B_\alpha) = \{c(\alpha)\}$. Applying this to $\zeta = \beta$ we see that $c(\beta) = c(\alpha)$ as well, and so $C_\beta \subseteq c^{-1}(c(\beta))$, giving Definition 6.1(ii).

Next fix $\alpha \in \mathcal{J}_x$ and $\eta \in CB_\alpha C^{-1} \cap \mathcal{I}^{\mathrm{ess}}$. Then $\eta^{-1} \in \mathcal{I}^{\mathrm{ess}}$ by Lemma 2.2. We need to verify that $\eta^{-1} \in C_{(\gamma\alpha\gamma^{-1})^{-1}}$. Write $\eta = \xi\theta\xi^{-1}$. Then by Lemma 2.2 again, $\theta \in \mathcal{I}^{\mathrm{ess}}$.

We have $\theta^{-1} \in B_{\alpha^{-1}}$ by definition of a harmonious family of bisections, so we see that $\eta^{-1} = \xi\theta^{-1}\xi^{-1} \in CB_{\alpha^{-1}}C^{-1}$, which is Definition 6.1(iii).

Since $C^{-1}C = s(C)$, for $\beta, \eta \in \mathcal{J}_z$, we have $C_\beta C_\eta = CB_{\gamma^{-1}\beta\gamma}C^{-1}CC_{\gamma^{-1}\eta\gamma}C^{-1} \subseteq CB_{\gamma^{-1}\beta\gamma\gamma^{-1}\eta\gamma}C^{-1} = B_{\gamma^{-1}\beta\eta\gamma} = C_{\beta\eta}$, giving Definition 6.1(iv).

Choose compact subsets $K_\alpha \in X$ satisfying $s(B_\alpha) = K_\alpha \cap B_x$ for every $\alpha \in \mathcal{J}_x$. Let $K'_{\gamma\alpha\gamma^{-1}} = CK_\alpha C^{-1}$ for each $\alpha \in \mathcal{J}_x$. Then

$$CK_\alpha(x)C^{-1} = r \circ (s|_C)^{-1}(K_\alpha(x)).$$

Since $r \circ (s|_C)^{-1}$ is the canonical partial homeomorphism associated to the bisection C, we see that $K'_\alpha(z)$ is a compact subset of X. Let $\beta := \gamma\alpha\gamma^{-1}$. We claim that $s(C_\beta) = K'_\beta \cap C_z$.

Computation shows that $s(B_\beta) = r \circ (s|_C)^{-1}(s(B_\alpha))$ and $CB_xC^{-1} = r \circ (s|_C)^{-1}(B_x)$, so

$$K'_\beta C_z = r \circ (s|_C)^{-1}(K_\alpha) \cap r \circ (s|_C)^{-1}(B_x) = r \circ (s|_C)^{-1}(s(B_\alpha)) = s(C_\beta).$$

This gives Definition 6.1(v).

For the final statement, suppose that x admits a harmonious family of bisections $\mathcal{B}$, fix $z \in [x]$ and $\gamma \in (G_T)_x^z$. Since G_T is étale, there is an open bisection C containing γ. Now the preceding paragraphs show that $(CB_\alpha C^{-1})_{\alpha \in \mathcal{J}_x}$ is a harmonious family of bisections based at z. □

The main obstacle in applying our strongest results is establishing the existence of harmonious families of bisections at all points for a given action of $\mathbb{N}^k$ by local homeomorphisms. Next we outline some techniques for constructing harmonious families of bisections that apply to a large class of examples among them actions by commuting homeomorphisms, a single local homeomorphism, and 2-graphs in Section 10. We leave it as an open problem to determine if every Deaconu–Renault groupoid admits harmonious families of bisections.

The first existence result applies e.g. to actions of commuting homeomorphisms, cf. Section 10.3.

Lemma 6.6. *Let X be a second-countable locally compact Hausdorff spaces and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Let $x \in X$. Suppose there are open bisections $(B_\alpha)_{\alpha \in \mathcal{J}_x}$ satisfying $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ and $B_\alpha B_\beta = B_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{J}_x$. Then x admits a harmonious family of bisections.*

Proof. We have $B_\alpha B_{\alpha^{-1}} = B_x = B_{\alpha^{-1}} B_\alpha$ for each α, and this implies that $B_x \subseteq r(B_\alpha)$. In particular, $B_x \subseteq X$ and this is (i) in Definition 6.1. By assumption, we have $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ and this is (ii). Since $B_x B_\alpha = B_{x\alpha} = B_\alpha$, we have $r(B_\alpha) \subseteq B_x$, and combined with the first sentence this gives equality. Hence the B_α are a subset of the group $\mathcal{B}(X)$ of open bisections with range and source equal to B_x, and they constitute the range of the homomorphism $\alpha \mapsto B_\alpha$. So $\{B_\alpha : \alpha \in \mathcal{J}_x\}$ is a subgroup of $\mathcal{B}(X)$. Since the inverse operation in $\mathcal{B}(X)$ is implemented by pointwise inverses in G, we obtain $B_{\alpha^{-1}} = B_\alpha^{-1}$, and we already have $B_\alpha B_\beta = B_{\alpha\beta}$ for all α, β, so (iii) and (iv) are trivial. The final condition (v) is automatic by Remark 6.3(2) □

Corollary 6.7. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Let $x \in X$. Suppose that $M \subseteq \mathbb{Z}^k$ satisfies $|M| = \operatorname{rank}(c(\mathcal{J}_x))$ and $c(\mathcal{J}_x) = \operatorname{span}_{\mathbb{Z}}(M)$. Suppose that there is an open neighbourhood $B_x \subseteq X$ of x and a collection of open bisections $(B_\alpha)_{\alpha \in M}$ such that*

(1) *each $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$,*
(2) *$B_\alpha B_\beta = B_\beta B_\alpha$ for all $\alpha, \beta \in M \cup -M$, and*
(3) *$r(B_\alpha) = s(B_\alpha) = B_x$ for all $\alpha \in M$.*

Then x admits a harmonious family of bisections.

Proof. The hypotheses guarantee that there is a well-defined map $\gamma \mapsto B_\gamma$ from the group generated by M to the collection of open bisections of G_T such that for every function $n : M \to \mathbb{Z}$, we have

$$B_{\prod_{\alpha\in M} \alpha^{n(\alpha)}} = \prod_{\alpha\in M_+(x)} B_\alpha^{n(\alpha)}.$$

An induction using (2) and (3) shows that these B_γ satisfy the hypotheses of Lemma 6.6, and the result follows. ☐

Our next existence result will be applied to the case of 2-graphs in Section 10.5.

Lemma 6.8. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Let $x \in X$. Suppose that $M \subseteq \mathcal{J}_x$ generates $\mathcal{J}_x$ as a group and that $|M| = \operatorname{rank}(\mathcal{J}_x)$. Suppose that $\{B_\alpha : \alpha \in M\}$ is a family of compact open bisections satisfying*

(1) *$\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for all $\alpha \in M$,*
(2) *$B_\alpha B_\beta = B_\beta B_\alpha$ for all $\alpha, \beta \in M$, and*
(3) *$B_\alpha B_\beta^{-1} \subseteq B_\beta^{-1} B_\alpha$ for all $\alpha, \beta \in M$.*

For each $\gamma \in \mathcal{J}_x$ and $\alpha \in M$, let $m_\alpha(\gamma)$ be the integers such that $\gamma = \prod_{\alpha\in M} \alpha^{m_\alpha(\gamma)}$. Taking the convention that the empty product is equal to X, the sets

$$B_\gamma := \Big(\prod_{m_\alpha(\gamma)<0} B_\alpha^{m_\alpha(\gamma)}\Big)\Big(\prod_{m_\alpha(\gamma)>0} B_\alpha^{m_\alpha(\gamma)}\Big),$$

indexed by $\gamma \in \mathcal{J}_x$ constitute a harmonious family of bisections at x.

Proof. The bisections $(B_\alpha)_\alpha$ pairwise commute, so the formula for B_γ is well-defined. We must verify the five conditions of Definition 6.1. We have $B_x \subseteq X$, which is (i), by definition. Since each $\alpha \in B_\alpha$, we have $\gamma \in B_\gamma$ for all γ, and since each $B_\alpha \subseteq c^{-1}(\alpha)$ and c is a homomorphism on $\mathcal{J}_x$ we have $B_\gamma \subseteq c^{-1}(\gamma)$ for all γ; this gives (ii).

To see (iii), note that $m_{\alpha(\gamma^{-1})} = -m_\alpha(\gamma)$, and so

$$\begin{aligned} B_{\gamma^{-1}} &= \prod \Big(\prod_{m_\alpha(\gamma^{-1})<0} B_\alpha^{m_\alpha(\gamma^{-1})}\Big)\Big(\prod_{m_\alpha(\gamma^{-1})>0} B_\alpha^{m_\alpha(\gamma^{-1})}\Big) \\ &= \prod \Big(\prod_{m_\alpha(\gamma)>0} B_\alpha^{m_\alpha(\gamma)}\Big)^{-1}\Big(\prod_{m_\alpha(\gamma)<0} B_\alpha^{m_\alpha(\gamma^{-1})}\Big)^{-1} = B_\gamma^{-1} \end{aligned}$$

for all $\gamma \in \mathcal{J}_x$. Since $\mathcal{I}^{\mathrm{ess}}$ is self-inverse, it follows that $(B_\gamma \cap \mathcal{I}^{\mathrm{ess}})^{-1} = B_\alpha^{-1} \cap \mathcal{I}^{\mathrm{ess}}$ for all $\gamma \in \mathcal{J}_x$, which gives (iii).

Finally, for (iv), First note that a simple induction using that $B_\alpha B_\beta^{-1} \subseteq B_\beta^{-1} B_\alpha$ for all $\alpha, \beta \in M$ shows that for any pair of functions $p, q : M \to \mathbb{N}$ we have

$$\Big(\prod_{\alpha\in M} B_\alpha^{p(\alpha)}\Big)\Big(\prod_{\alpha\in M} B_\alpha^{-q(\alpha)}\Big) \subseteq \Big(\prod_{\alpha\in M} B_\alpha^{-q(\alpha)}\Big)\Big(\prod_{\alpha\in M} B_\alpha^{p(\alpha)}\Big).$$

We deduce that for $\gamma, \delta \in \mathcal{J}_x$, we have

$$\begin{aligned} B_\gamma B_\delta &= \Big(\prod_{m_\alpha(\gamma)<0} B_\alpha^{m_\alpha(\gamma)}\Big)\Big(\prod_{m_\alpha(\gamma)>0} B_\alpha^{m_\alpha(\gamma)}\Big)\Big(\prod_{m_\alpha(\delta)<0} B_\alpha^{m_\alpha(\delta)}\Big)\Big(\prod_{m_\alpha(\delta)>0} B_\alpha^{m_\alpha(\delta)}\Big) \\ &\subseteq \Big(\prod_{m_\alpha(\gamma)<0} B_\alpha^{m_\alpha(\gamma)}\Big)\Big(\prod_{m_\alpha(\delta)<0} B_\alpha^{m_\alpha(\delta)}\Big)\Big(\prod_{m_\alpha(\gamma)>0} B_\alpha^{m_\alpha(\gamma)}\Big)\Big(\prod_{m_\alpha(\delta)>0} B_\alpha^{m_\alpha(\delta)}\Big) \\ &= B_{\gamma\delta}. \end{aligned}$$

Hence $B_\gamma(B_\delta \cap \mathcal{I}^{\mathrm{ess}}) \subseteq B_\gamma B_\delta \subseteq B_{\gamma\delta}$, giving (iv).

Since the B_α are compact open, so are the B_γ, and so condition (v) is automatic. ☐

Example 6.9. If G_T is a minimal Deaconu–Renault groupoid, then $\mathcal{J}(G_T) = \mathcal{I}^\circ(G_T)$ by [KPS16, Proposition 2.1]. As in Section 2.5, $H(x) = c(\mathcal{I}^\circ(G_T)_x)$ is constant with respect to x, and writing H for this group, $\mathcal{J}_x = \mathcal{I}^{\mathrm{ess}}_x = \mathcal{I}^\circ_x = \{x\} \times H \times \{x\}$ for all x. So for $n \in H$, the set $B(n) \coloneqq \{(x, n, x) : x \in X\}$ is a bisection. These $B(n)$ pairwise commute, and each commute with any subset of $G_T^{(0)}$. So for each $x \in X$ and $n \in H$, choosing a precompact open neighbourhood W_x of x and putting $B_{(x,n,x)} \coloneqq W_x B(n)$ for each n determines a harmonious family of bisections $\mathcal{B}^x = (B_\alpha)_{\alpha \in \mathcal{J}_x}$.

To make use of results like Lemma 6.8 or Corollary 6.7, we need to be able to identify free generators of a given subgroup of $\mathbb{Z}^k$. When working with Deaconu–Renault groupoids, it is often helpful to work with minimal collections of generators in $\mathbb{N}^k$. So we prove that every rank-k subgroup of $\mathbb{Z}^k$ admits k free generators in $\mathbb{N}^k$ that are minimal with respect to the usual algebraic order. This is surely well known, but we could not find a reference. We found the key idea behind the proof on Math StackExchange [Kot12].

We make use of the usual lattice order $\le$ on $\mathbb{Z}^k$; so $m \le n$ if $n - m \in \mathbb{N}^k$. For $i < k$ we also identify $\mathbb{Z}^i$ with the subgroup of $\mathbb{Z}^k$ consisting of elements whose final $k-i$ coordinates are zero. So $\mathbb{N}^i \setminus \mathbb{N}^{i-1} = \mathbb{N}^{i-1} \times (\mathbb{N} \setminus \{0\}) \times \{0_{k-i}\}$.

Lemma 6.10. *Let $H \subseteq \mathbb{Z}^k$ be a subgroup of rank k. Then there exist $m^1, \dots, m^k \in \mathbb{N}^k \cap H$ such that*

(1) *each m^i is a minimal element of $\mathbb{N}^k \cap H \setminus \{0\}$,*
(2) *for each $i \le k$, we have $m^i \in \mathbb{N}^i \setminus \mathbb{N}^{i-1}$,*
(3) *$H = \bigoplus_i \mathbb{Z} m^i$.*

Proof. Since the rank of H is k, any k elements that generate H are free abelian generators, so it suffices to establish that there exist $m^1, \dots, m^k \in \mathbb{N}^k \cap H$ satisfying (1) and (2) such that $H = \operatorname{span}_{\mathbb{Z}}\{m^1, \dots, m^k\}$.

We argue by induction on k. For $k = 1$, the result is trivial. So suppose inductively that every rank-$(k-1)$ subgroup H' of $\mathbb{Z}^{k-1}$ has such generators. Let $\pi \colon \mathbb{Z}^k \to \mathbb{Z}$ be the homomorphism $\pi(n) = n_k$ onto the kth coordinate, so $\ker(\pi) = \mathbb{Z}^{k-1}$. Since $\pi(H) \le \mathbb{Z}$, we have $\pi(H) = a\mathbb{Z}$ for some $a \in \mathbb{N}$. Fix $\widetilde{m}^k \in H$ satisfying $\pi(\widetilde{m}^k) = a$.

Let $H' = H \cap \mathbb{Z}^{k-1}$. For any $h \in H$, we have $a \mid \pi(h)$, and so $h' = h - \frac{\pi(h)}{a}\widetilde{m}^k \in H'$, and so $h = h' + \frac{\pi(h)}{a}\widetilde{m}^k \in \mathbb{Z}\widetilde{m}^k + H'$. So H is generated by $\{\widetilde{m}^k\} \cup H'$. This implies in particular that $\operatorname{rank}(H) \le 1 + \operatorname{rank}(H')$, so $\operatorname{rank}(H') \ge k-1$. Hence $\operatorname{rank}(H') = k-1$ because $H' \subseteq \mathbb{Z}^{k-1}$. By the inductive hypothesis, there are generators $m^1, \dots, m^{k-1}$ of $H \cap \mathbb{N}^{k-1}$ satisfying (1) and (2). In particular, $H = \operatorname{span}_{\mathbb{Z}}\{m^1, \dots, m^{k-1}, \widetilde{m}^k\}$ where each m^i is in $\mathbb{N}^i \setminus \mathbb{N}^{i-1}$.

Since each $m^i_i \ne 0$, there exist $a_1, \dots, a_{k-1} \ge 0$ such that $a_i m^i_i + \widetilde{m}^k_i \ge 0$. Since each $m^i_j \ge 0$, we deduce that $\big(\sum_i a_i m^i\big) + \widetilde{m}^k \in \mathbb{N}^k$. Hence $C \coloneqq \big(\widetilde{m}^k_i + \operatorname{span}_{\mathbb{Z}}\{m^1, \dots, m^{k-1}\}\big) \cap \mathbb{N}^k$ is nonempty, and therefore has a minimal element. We fix $a_1, \dots, a_k$ such that $m^k \coloneqq \widetilde{m}^k + \sum_{i<k} a_i m^i_i$ is a minimal element of C. This implies that $m^i \not\le m^k$ for $i < k$. Then $m^k \in H \cap \mathbb{N}^k$, and $m^k_k = \widetilde{m}^k_k > 0$. We have $\widetilde{m}^k = m^k + \sum_{i<k}(-a_i)m^i \in \operatorname{span}_{\mathbb{Z}}\{m^1, \dots, m^k\}$. Hence $\operatorname{span}_{\mathbb{Z}}\{m^1, \dots, m^{k-1}, m^k\} = \operatorname{span}_{\mathbb{Z}}\{m^1, \dots, m^{k-1}, \widetilde{m}^k\} = H$.

It remains to show that $m^1, \dots, m^k$ are minimal in $H \cap \mathbb{N}^k \setminus \{0\}$. Fix $i < k$. Since $m^1, \dots, m^{k-1} \subseteq \mathbb{N}^{k-1}$ and $m^k_k > 0$, we have $\{p \in H \cap \mathbb{N}^k : p < m^i\} = \{p \in H' \cap \mathbb{N}^k : p < m^i\} = \{0\}$ by the inductive hypothesis. So m^i is minimal in $H \cap \mathbb{N}^k \setminus \{0\}$. To see that m^k is minimal, suppose that $p \in H \cap \mathbb{N}^k$ satisfies $0 < p \le m^k$. If $p \in H'$ then there exists $j < k$ such that $m^j \le p \le m^k$ which is impossible as observed in the preceding paragraph. So $p \in H \setminus H'$. Then $0 < \pi(p) \in a\mathbb{Z}$ and $\pi(p) \le a$, forcing $\pi(p) = a$. Write $p = \sum_i b_i m^i$. Since $\pi(m^i) = 0$ for $i < k$, we have $b_k = 1$. Hence

$p \in m^k + \operatorname{span}_{\mathbb{Z}}\{m^1, \ldots, m^{k-1}\} \cap \mathbb{N}^k = \widetilde{m}^k + \operatorname{span}_{\mathbb{Z}}\{m^1, \ldots, m^{k-1}\} \cap \mathbb{N}^k = C$. Since m^k is minimal in C, we obtain $p = m^k$. □

7. The primitive-ideal space

With the concept of harmonious families of bisections available, we can now state our second main theorem, which describes a family of subsets of $X \times \mathbb{T}^k$ that are preimages of open subsets of $\operatorname{Prim}(C^*(G_T))$.

Theorem 7.1. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For $x \in X$ and $z \in \mathbb{T}^k$, let $\pi_{(x,z)}$ be as in Notation 4.1. Let $(x_0, z_0) \in X \times \mathbb{T}^k$ and suppose that $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_{x_0}}$ is a harmonious family of bisections, and that $V \subseteq \mathbb{T}^k$ is an open neighbourhood of z_0. Then the set*

$$A(\mathcal{B}, V) := \{\ker(\pi_{(x,z)}) : x \in B_{x_0}, z \in V H_{\mathcal{B}}(x)^\perp\} \subseteq \operatorname{Prim}(C^*(G_T)) \tag{7.1}$$

is an open neighbourhood of $\ker(\pi_{(x_0,z_0)})$ in $\operatorname{Prim}(C^*(G_T))$.

The following technical proposition is the engine-room in the proof of Theorem 7.1. We think of this result as a kind of noncommutative Urysohn lemma, and we state it separately so that we can use it later to describe generators for the ideal of $C^*(G_T)$ corresponding to a given open subset of $X \times \mathbb{T}^k$ in Proposition 8.2. We thank Johannes Christensen and Sergiy Neshveyev for pointing out an error in the original proof of this result, and for helpful subsequent conversations.

Proposition 7.2. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For $x \in X$ and $z \in \mathbb{T}^k$, let $\pi_{(x,z)}$ be as in Notation 4.1. Let $(x_0, z_0) \in X \times \mathbb{T}^k$ and suppose $\mathcal{B} = (B_\alpha)_{\alpha \in \mathcal{J}_{x_0}}$ is a harmonious family of bisections, and that $V \subseteq \mathbb{T}^k$ is an open neighbourhood of z_0. For any $(x, z) \in B_{x_0} \times V$ there exist a function $\phi \in C_c(B_{x_0}, [0,1])$ such that $\phi(x) = 1$, a function $\psi \in C^\infty(\mathbb{T}^k)$ such that $\psi(z) = 1$ and $\psi|_{\mathbb{T}^k \setminus V} = 0$, and an element $h_0 \in \mathbb{Z}^k$ such that the h_0-perturbation ψ_{h_0} satisfies*

$$\sum_{h \in H_{\mathcal{B}}(x)} z^h \widehat{\psi}_{h_0}(h) \neq 0.$$

For any such ϕ, ψ and h_0 the series

$$\sum_{(x_0,h,x_0) \in \mathcal{J}_{x_0}} \widehat{\psi}_{h_0}(h)(1_{B_{(x_0,h,x_0)}} \phi)$$

converges to an element f of $C^*(G_T)$ such that $\pi_{(x,zw)}(f) \neq 0$ for every $w \in H_{\mathcal{B}}(x)^\perp$ and such that $f \in Q$ for every primitive ideal $Q \notin \pi((B_{x_0} \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp)$.

Proof. By Urysohn's lemma there exists $\phi \in C_c(X, [0,1])$ such that $\phi(x) = 1$ and $\phi|_{X \setminus B_{x_0}} = 0$. By, for example, [Fol99, Chapter 8], there exists $\psi \in C^\infty(\mathbb{T}^k)$ such that $\psi(z) = 1$ and $\psi|_{\mathbb{T}^k \setminus V} = 0$. By Lemma 2.9, there exists $h_0 \in \mathbb{Z}^k$ such that

$$\sum_{h \in H_{\mathcal{B}}(x)} z^h \widehat{\psi}_{h_0}(h) \neq 0.$$

For each $\alpha \in \mathcal{J}_{x_0}$, by definition of a harmonious family of bisections, there is a compact subset K_α of X such that $s(B_\alpha) = K_\alpha \cap B_{x_0}$. We claim that this implies that $1_{B_\alpha}\phi$ is continuous. Since B_α is open it suffices to show that if $(\beta_n)_n$ is a sequence in B_α and $\beta_n \to \beta$ then $(1_{B_\alpha}\phi)(\beta_n) \to (1_{B_\alpha}\phi)(\beta)$ as $n \to \infty$. Since each $\beta_n \in B_\alpha$, we have

$$(1_{B_\alpha}\phi)(\beta_n) = 1_{B_\alpha}(\beta_n)\phi(s(\beta_n)) = \phi(s(\beta_n)),$$

so we must show that

$$\phi(s(\beta_n)) \to \begin{cases} \phi(s(\beta)) & \text{if } \beta \in B_\alpha \\ 0 & \text{if } \beta \notin B_\alpha. \end{cases}$$

Since $\phi \circ s$ is continuous, it therefore suffices to show that if $\beta \notin B_\alpha$ then $\phi(s(\beta)) = 0$. So suppose that $\beta \notin B_\alpha$. Since K_α is compact and the $s(\beta_n)$ belong to K_α, we have $s(\beta) \in K_\alpha$. We claim that $s(\beta) \notin s(B_\alpha)$. To see this, suppose for contradiction that $s(\beta) = s(\beta')$ for some $\beta' \in B_\alpha$. Since s restricts to a homeomorphism $B_\alpha \to s(B_\alpha)$ and $s(\beta_n) \to s(\beta')$ we have $\beta_n \to \beta'$. Since $\beta_n \to \beta$ by hypothesis, and since G_T is Hausdorff, this forces $\beta = \beta'$, contradicting $\beta \notin B_\alpha$. Thus $s(\beta) \notin B_\alpha = K_\alpha \cap B_{x_0}$. We saw that $s(\beta) \in K_\alpha$, so we deduce that $s(\beta) \notin B_{x_0}$. Since ϕ vanishes on $X \setminus B_{x_0}$ by construction, we then have $\phi(s(\beta)) = 0$ as required, so $1_{B_\alpha}\phi$ is continuous.

For each $\alpha \in \mathcal{J}_{x_0}$, it follows from [Sim20, Corollary 9.3.4] that $\|1_{B_\alpha}\phi\|_{C^*(G_T)} = \|\phi\|_\infty = 1$. Since ψ is smooth, its Fourier coefficients are absolutely summable [Fol99, Chapter 8]. Hence the series $\sum_{(x_0,h,x_0)\in\mathcal{J}_{x_0}} \widehat{\psi}_{h_0}(h)(1_{B_{(x_0,h,x_0)}}\phi)$ converges to an element f of $C^*(G_T)$. Since Renault's map $j\colon C^*(G_T) \to C_0(G_T)$ of [Ren80, Proposition II.4.2] is continuous, we have

$$j(f)(y',h',x') = \begin{cases} \phi(x')\widehat{\psi}_{h_0}(h) & \text{if } (y',h',x') \in \bigcup\mathcal{B}, \\ 0 & \text{otherwise,} \end{cases} \tag{7.2}$$

for all $(y',h',x') \in G_T$. Fix $w \in H_{\mathcal{B}}(x)^\perp$. We must show that $\pi_{(x,zw)}(f) \neq 0$. Let $K := c_T(\mathcal{I}_x^{\mathrm{ess}})$. By Lemma 3.3, it suffices to show that $\pi^K_{(x,zw)}(f) \neq 0$. For this, note that, by definition, $[(x,0,x)]_K = \{(x,h,x) : h \in K\} = \mathcal{I}_X^{\mathrm{ess}}$, and by (7.2), if $j(f)(x,h,x) \neq 0$ then $(x,h,x) \in \bigcup\mathcal{B}$. So if $(x,h,x) \in [(x,0,x)]_K$ and $j(f)(x,h,x) \neq 0$, then $(x,h,x) \in \mathcal{I}_x^{\mathrm{ess}} \cap \bigcup\mathcal{B} = \mathcal{B}_x^{\mathrm{ess}}$; in particular $h \in H_{\mathcal{B}}(x)$. Hence

$$\langle e_{[x,0,x]_K}, \pi^K_{(x,zw)}(f)e_{[x,0,x]_K}\rangle = \sum_{h\in H_{\mathcal{B}}(x)} z^h w^h \widehat{\psi}_{h_0}(h) = \sum_{h\in H_{\mathcal{B}}(x)} z^h \widehat{\psi}_{h_0}(h) \neq 0.$$

Hence $\pi^K_{(x,zw)}(f) \neq 0$ as required.

It remains to show that $f \in Q$ for every primitive ideal $Q \notin \pi((B_{x_0} \times V)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp)$. To see this, fix such a Q. By [SW16, Theorem 3.2], there exists $(x_1,z_1) \in X \times \mathbb{T}^k$ such that $Q = \ker(\pi_{(x_1,z_1)})$, and it suffices to show that

$$\langle e_{y_2}, \pi_{(x_1,z_1)}(f)e_{y_1}\rangle = 0 \tag{7.3}$$

for all $y_1, y_2 \in [x_1]$.

Fix $y_1, y_2 \in [x_1]$. Let $H \coloneqq c(\mathcal{I}_{y_1}) = \{h \in \mathbb{Z}^k : (y_1,h,y_1) \in G_T\}$. Then $H_{\mathcal{B}}(y_1) \leq H$. Fix a complete set $R \subseteq H$ of representatives of the cosets of $H_{\mathcal{B}}(y_1)$ in H. Then $H = \bigsqcup_{r\in R}(r + H_{\mathcal{B}}(y_1))$. Fix $h_1, h_2 \in \mathbb{Z}^k$ such that (y_i,h_i,x_1) are in G_T. Then $(y_2, h_2 - h_1, y_1) \in G_T$.

We have

$$\langle e_{y_2}, \pi_{(x_1,z_1)}(f)e_{y_1}\rangle = \sum_{(y_2,h,y_1)\in G_T} z_1^h j(f)((y_2,h,y_1)) = \sum_{h\in H} z_1^{h+h_2-h_1} j(f)((y_2,h+h_2-h_1,y_1)).$$

By definition of f, for each h the number $j(f)(y_2, h + h_2 - h_1, y_1)$ is equal to either $\phi(y_1)\widehat{\psi}_{h_0}(h)$ or to 0. By [Fol99, Chapter 8] as before, the series $\sum_{h\in\mathbb{Z}^k} \widehat{\psi}_{h_0}(h)$ is absolutely convergent, and it follows that the series $\sum_{h\in H} z_1^{h+h_2-h_1} j(f)((y_2, h+h_2-h_1, y_1))$ is also absolutely convergent. So we can rearrange its terms to obtain

$$\langle e_{y_2}, \pi_{(x_1,z_1)}(f)e_{y_1}\rangle = \sum_{r\in R}\sum_{h\in H_{\mathcal{B}}(y_1)} z_1^{h+r+(h_2-h_1)} j(f)((y_2, h+r+(h_2-h_1), y_1)).$$

To see that this is zero, fix $r \in R$. It suffices to show that

$$\sum_{h\in H_{\mathcal{B}}(y_1)} z_1^{h+r+(h_2-h_1)} j(f)((y_2, h+r+(h_2-h_1), y_1)) = 0.$$

The formula (7.2) shows that $j(f)((y_2, h+r+(h_2-h_1), y_1)) = 0$ for all h if $y_1 \notin B_{x_0}$, so we may assume that $y_1 \in B_{x_0}$. Letting $h_2' := r + h_2$, we have

$$\begin{aligned}\sum_{h\in H_{\mathcal{B}}(y_1)} & z_1^{h+r+(h_2-h_1)} j(f)((y_2, h+r+(h_2-h_1), y_1)) \\ &= \sum_{h\in H_{\mathcal{B}}(y_1)} z_1^{h+(h_2'-h_1)} j(f)\big(y_2, h+(h_2'-h_1), y_1\big).\end{aligned} \tag{7.4}$$

If $(y_2, h_2'-h_1, y_1) \notin \bigcup\mathcal{B}$, then Condition (iv) of Definition 6.1 implies that

$$(y_2, h+(h_2'-h_1), y_1) = (y_2, (h_2'-h_1), y_1)(y_1, h, y_1) \notin \bigcup\mathcal{B}$$

for all $h \in H_{\mathcal{B}}(y_1)$, so once again (7.2) shows that (7.4) vanishes. So we may assume that $(y_2, h_2'-h_1, y_1) \in \bigcup\mathcal{B}$.

Condition (iv) of Definition 6.1 then implies that $(y_2, h+(h_2'-h_1), y_1) \in \bigcup\mathcal{B}$ for all $h \in H_{\mathcal{B}}(y_1)$. Thus, using (7.2) again, for each $h \in H_{\mathcal{B}}(y_1)$, we have $j(f)\big(y_2, h+(h_2'-h_1), y_1\big) = \phi(y_1)\widehat{\psi}_{h_0}(h+(h_2'-h_1))$, and so (7.4) becomes

$$\begin{aligned}\sum_{h\in H_{\mathcal{B}}(y_1)} & z_1^{h+r+(h_2-h_1)} j(f)((y_2, h+r+(h_2-h_1), y_1)) \\ &= \phi(y_1)\sum_{h\in H_{\mathcal{B}}(y_1)} z_1^{h+(h_2'-h_1)}\widehat{\psi}_{h_0}(h+(h_2'-h_1)).\end{aligned}$$

To see that this is zero, let $\psi_{h_0+(h_1-h_2')}$ be the (h_1-h_2')-perturbation of ψ_{h_0}, and let $\chi_1 \in \widehat{H}_{\mathcal{B}(y_1)}$ be the character defined by $\chi_1(h) = z_1^h$, for all $h \in H_{\mathcal{B}}(y_1)$. Recall that $\Phi_{H_{\mathcal{B}}(y_1),\mathbb{Z}^k}$ is the averaging map of (2.2). We have

$$\phi(y_1)\sum_{h\in H_{\mathcal{B}}(y_1)} z_1^{h+r+(h_2-h_1)}\widehat{\psi}_{h_0}(h+r+(h_2-h_1)) \tag{7.5}$$
$$= z_1^{h_2'-h_1}\phi(y_1)\sum_{h\in H_{\mathcal{B}}(y_1)} z_1^h\widehat{\psi}_{h_0+(h_1-h_2')}(h)$$
$$= z_1^{h_2'-h_1}\phi(y_1)\Phi_{H_{\mathcal{B}}(y_1),\mathbb{Z}^k}(\psi_{h_0+(h_1-h_2')})(\chi_1). \tag{7.6}$$

In order to show that (7.6) is zero, it now suffices to show that $\chi_1 \notin \widehat{q}_{H_{\mathcal{B}}(y_1)}(V)$, because $\operatorname{supp}(\psi_{h_0+(h_1-h_2')}) = \operatorname{supp}(\psi) \subseteq V$ as discussed immediately after (2.1), and hence $\operatorname{supp}(\Phi_{H_{\mathcal{B}}(y_1)}(\psi_{h_0+(h_1-h_2')})) \subseteq \widehat{q}_{H_{\mathcal{B}}(y_1)}(V)$ as discussed just prior to Lemma 2.9. So it suffices to show that $z_1 w \notin V$ for all $w \in H_{\mathcal{B}}(y_1)^\perp$.

Suppose for contradiction that $w \in H_{\mathcal{B}}(y_1)^\perp$ satisfies $z_1 w \in V$. Then $z_1 \in VH_{\mathcal{B}}(y_1)^\perp$, and hence $(y_1, z_1) \in (B_{x_0}\times V)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp$. Since $y_1 \in [x_1]$, Theorem 3.2 of [SW16] implies that $\ker(\pi_{(y_1,z_1)}) = \ker(\pi_{(x_1,z_1)}) = Q$, contradicting $Q \notin \pi((B_{x_0}\times V)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp)$.

Thus (7.6) is zero, whence (7.4) is zero, giving (7.3). Hence $f \in Q$. □

Proof of Theorem 7.1. By definition of the topology on $\mathrm{Prim}(C^*(G_T))$, the closed sets are the sets $\{Q : Q \subseteq I\}$ indexed by ideals I of $C^*(G_T)$. We must show that the complement of $A(B, V)$ is closed, so we fix $x \in B_{x_0}$, $z \in V$ and $w \in H_{\mathcal{B}}(x)^\perp$ so that $P = \ker(\pi_{(x,zw)})$ is a typical point in $A(B, V)$. We must show that $\bigcap_{Q\in\mathrm{Prim}(C^*(G_T))\setminus A(B,V)} Q \not\subseteq P$. That is, we must find $f_P \in \bigcap_{Q\in\mathrm{Prim}(C^*(G_T))\setminus A(B,V)} Q$ such that $f_P \notin P$. But this is a direct application of Proposition 7.2. □

This now leads us to a complete description of the primitive ideal space when the action $T\colon \mathbb{N}^k \curvearrowright X$ admits sufficiently many harmonious families of bisections.

Definition 7.3. Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. We say that T *admits harmonious families of bisections* if every $x \in X$ admits a harmonious family of bisections in G_T.

Corollary 7.4. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits harmonious families of bisections. Let $\pi\colon X\times\mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ be the map of Notation 4.1. Let A be a subset of $\mathrm{Prim}(C^*(G_T))$ such that for every $(x_0, z_0) \in \pi^{-1}(A)$ there exist a harmonious family of bisections $\mathcal{B}$ at x_0 and an open neighbourhood $V_0 \subseteq \mathbb{T}^k$ of z_0 such that $(B_{x_0}\times V_0)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \pi^{-1}(A)$. Then A is open in $\mathrm{Prim}(C^*(G_T))$.*

Proof. Fix $(x_0, z_0) \in \pi^{-1}(A)$. By hypothesis there exist a harmonious family of bisections $\mathcal{B}$ based at x_0 and an open neighbourhood $V_0 \subseteq \mathbb{T}^k$ of z_0 such that $(B_{x_0} \times V_0) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \pi^{-1}(A)$. Let $V := V_0 \cdot (H_{\mathcal{B}}(x_0))^\perp$, the $H(x_0)$-saturation of V_0 and let $A(\mathcal{B}, V) \subseteq \mathrm{Prim}(C^*(G_T))$ be the corresponding basic open neighbourhood of $\ker(\pi_{(x_0,z_0)})$ as in (7.1). It suffices to show that $A(\mathcal{B}, V) \subseteq A$. Let $(x, z) \in \pi^{-1}A(\mathcal{B}, V)$. Then there exist $z' \in V_0$ and $w \in H(x)^\perp$ such that $z = z'w$. Since $(B_{x_0} \times V_0) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \pi^{-1}(A)$, we have $(x, z) \in \pi^{-1}(A)$. Hence $A(\mathcal{B}, V) \subseteq A$ as required. □

Combining Theorem 4.2 and Corollary 7.4, we obtain the following complete description of the hull-kernel topology.

Corollary 7.5. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits harmonious families bisections. Let $\pi\colon X \times \mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ be the map of Notation 4.1. A subset $A \subseteq \mathrm{Prim}(C^*(G_T))$ is open if and only if for every $(x, z) \in \pi^{-1}(A)$ there exist a harmonious family of bisections $\mathcal{B}$ based at x and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $(x, z) \in B_0 \times V$ and $(B_0 \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \pi^{-1}(A)$.*

Corollary 7.6. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits harmonious families of bisections. Let $\pi\colon X\times\mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ be the map of Notation 4.1. For each $x \in X$, fix a harmonious family of bisections $\mathcal{B}(x) = (B(x)_\alpha)_{\alpha\in\mathcal{J}_x}$ at x. Then the collection*

$$\big\{\pi\big((U \times V) \cdot (\mathcal{B}(x)^{\mathrm{ess}})^\perp\big) : x \in X, x \in U \subseteq_{open} B(x)_x, V \subseteq_{open} \mathbb{T}\big\}$$

is a base for the topology on $\mathrm{Prim}(C^*(G_T))$.

Finally, we mention a few immediate examples to illustrate how the base for the topology recovers well known results.

Example 7.7. (1) If G_T is strongly effective, then as in Example 6.4(2) $\mathcal{B} := \{G_T^{(0)}\}$ defines a harmonious family of bisections at each x. We then have $(U \times V) \cdot (\mathcal{B}(x)^{\mathrm{ess}})^\perp = U\times\mathbb{T}^k$, and Corollary 7.6 reduces to the statement that the primitive ideal space is homeomorphic to the quasi-orbit space of G_T.

(2) If G_T is minimal, then as in Example 6.9 we have $H(x) = H(y) = H$ for all $x, y \in X$, and G_T admits harmonious families of bisections. For the harmonious family of bisections $W_x B(n)$ of Example 6.9, we have $(U \times V) \cdot (\mathcal{B}(x)^{\mathrm{ess}})^{\perp} = U \times (VH^{\perp})$ whenever $U \subseteq W_x$. The quasi-orbit space is a point, so $\ker(\pi_{(x,z)}) = \ker(\pi_{(y,w)})$ if and only if $zH^{\perp} = wH^{\perp}$, and so Corollary 7.6 establishes that $\mathrm{Prim}(C^*(G_T))$ is homeomorphic to $H^{\perp}$. This recovers [SW16, Theorem 4.7] for T minimal.
(3) In particular, if G_T is minimal and effective, then the base for the topology is just a singleton so $C^*(G_T)$ is simple.

8. The lattice of ideals of $C^*(G_T)$

In this section we apply the results and ideas from the preceding section to describe the lattice of ideals of $C^*(G_T)$. We provide a partial description for arbitrary T, but in order to obtain a complete description, we must assume that T admits harmonious families of bisections.

Let X be a second-countable locally compact Hausdorff space. Suppose that $T \colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Recall from Section 2.4 that for each ideal I in $C^*(G_T)$,

$$A_I := \{P \in \mathrm{Prim}(C^*(G_T)) : I \not\subseteq P\}$$

is open in the hull-kernel topology on $\mathrm{Prim}(C^*(G_T))$. So Corollary 4.4 implies that with respect to the map $\pi \colon X \times \mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ of Notation 4.1, the map

$$\theta \colon I \mapsto \pi^{-1}(A_I) \tag{8.1}$$

is an injective lattice homomorphism from the ideals of $C^*(G_T)$ into the open subsets of $X \times \mathbb{T}^k$ ordered by inclusion.

The range of this lattice homomorphism is difficult to describe in general, but Theorem 4.2 describes a (fairly technical) invariance condition that every A_I must satisfy. The range of θ consists of open sets W that are π-saturated in the sense that $W = \pi^{-1}(\pi(W))$. However, not every open π-saturated subset of X need be in the range of θ; if it were, then π would be a quotient map, and [SW16, Remark 3.3 and Example 3.4] show that this is not the case in general.

To describe the range of θ, and to describe generators of the ideal $\theta^{-1}(W)$ for a given set W, we must assume that T admits harmonious families of bisections. The description of the range follows directly from Corollary 7.5.

Lemma 8.1. *Let X be a second-countable locally compact Hausdorff space and suppose that $T \colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits harmonious families of bisections. Let θ be the lattice homomorphism of (8.1). A subset $W \subseteq X \times \mathbb{T}^k$ is in the image of θ if and only if*

(1) *W is open and π-saturated, and*
(2) *whenever $(x, z) \in W$, then there exist a harmonious families of bisections $\mathcal{B}$ at x and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $(B_x \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^{\perp} \subseteq W$.*

We may now describe generators for the ideal $\theta^{-1}(A)$ corresponding to a given set satisfying the conditions of Lemma 8.1. The generating elements are exactly the elements of $C^*(G_T)$ we obtain from the noncommutative Urysohn lemma, Proposition 7.2.

Proposition 8.2. *Let X be a second-countable locally compact Hausdorff space and suppose that $T \colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits harmonious families of bisections. Let θ be the lattice homomorphism of (8.1). Suppose that $W \subseteq X \times \mathbb{T}^k$*

satisfies the conditions of Lemma 8.1. Then

$$\theta^{-1}(W) = \bigcap_{(x,z)\in(X\times\mathbb{T}^k)\setminus W} \ker(\pi_{(x,z)}).$$

For each $(x_0, z_0) \in W$, fix a harmonious families of bisections $\mathcal{B}^{(x_0,z_0)}$ at x_0 and an open neighbourhood $V^{(x_0,z_0)}$ of z_0 as in Condition 2 of Lemma 8.1, and let $f^{(x_0,z_0)}$ be any element obtained from Proposition 7.2 applied to (x_0, z_0), $\mathcal{B}^{(x_0,z_0)}$, and $V^{(x_0,z_0)}$. Then $\theta^{-1}(W)$ is generated as an ideal by $\{f^{(x_0,z_0)} : (x_0, z_0) \in W\}$.

Proof. Fix a set W in the range of θ. By [RW98, Proposition A.17(a)] the ideal $\theta^{-1}(W)$ is equal to the intersection of the primitive ideals that contain it. Hence Lemma 8.1 gives the first statement.

Let $I = \theta^{-1}(W)$. By definition of θ, we have $W = \{(x, z) \in X \times \mathbb{T}^k : I \not\subseteq \ker(\pi_{(x,z)})\}$. Let J be the ideal generated by the $f^{(x_0,z_0)}$. Since every ideal is the intersection of the primitive ideals containing it and since π is surjective, it suffices to show that for $(x, z) \in X \times \mathbb{T}^k$, we have $J \subseteq \ker(\pi_{(x,z)})$ if and only if $(x, z) \notin W$.

So fix $(x, z) \in X \times \mathbb{T}^k$. First suppose that $(x, z) \notin W$. Then for each $(x_0, z_0) \in W$,

$$A(\mathcal{B}^{(x_0,z_0)}, V^{(x_0,z_0)}) = (B^{(x_0,z_0)}_{x_0} \times V^{(x_0,z_0)}) \cdot ((\mathcal{B}^{(x_0,z_0)})^{\mathrm{ess}})^{\perp} \subseteq W,$$

so $(x, z) \notin A(\mathcal{B}^{(x_0,z_0)}, V^{(x_0,z_0)})$. Hence, by Proposition 7.2, $f^{(x_0,z_0)} \in \ker(\pi_{(x,z)})$. Since $(x_0, z_0) \in W$ was arbitrary, it follows that all the generators of J belong to the kernel of $\pi_{(x,z)}$. So $J \subseteq \ker(\pi_{(x,z)})$ as required. Now suppose that $(x, z) \in W$. Then $\pi_{(x,z)}(f^{(x,z)}) \neq 0$ by Proposition 7.2. Since $f^{(x,z)}$ is a generator of the ideal J, it belongs to J and so $J \not\subseteq \ker(\pi_{(x,z)})$ as required. □

9. Convergence of primitive ideals

Now we apply our results on the primitive ideal space of $C^*(G_T)$ to describe convergence of primitive ideals. We consider only systems $T\colon \mathbb{N}^k \curvearrowright X$ on second-countable spaces, so the C^*-algebras $C^*(G_T)$ are separable, and the primitive ideal space $\mathrm{Prim}(C^*(G_T))$ is second-countable (cf. e.g. [RW98, p. 231]), so it suffices to consider convergent *sequences*.

The map $\pi\colon X \times \mathbb{T}^k \to \mathrm{Prim}(C^*(G_T))$ from Notation 4.1 is continuous, so if $(x_i, z_i)_i \to (x, z)$ in $X \times \mathbb{T}^k$, then $\pi(x_i, z_i) \to \pi(x, z)$ in $\mathrm{Prim}(C^*(G_T))$. This is however far from a complete descrition: many divergent sequences in $X \times \mathbb{T}^k$ descend to convergent sequences in $\mathrm{Prim}(C^*(G_T))$.

We first describe a weaker sufficient condition for convergence of a sequence of primitive ideals. In order to do this, we need to extend the notation of Remark 6.2. Suppose that X is a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Fix $x \in X$, and let $\mathcal{B} := (B_\alpha)_{\alpha\in\mathcal{J}_x}$ be a collection of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for all $\alpha \in \mathcal{J}_x$. For each $y \in X$, we consider the subgroup

$$H_{\mathcal{B}}(y) := \mathrm{span}_{\mathbb{Z}}\, c\Big(\bigcup \mathcal{B}^{\mathrm{ess}}_y\Big)$$

in $\mathbb{Z}^k$ generated by the values of c on the intersection of $\bigcup \mathcal{B}$ with $\mathcal{I}^{\mathrm{ess}}_y$. That is,

$$H_{\mathcal{B}}(y) = \mathrm{span}_{\mathbb{Z}}\{c(\alpha) : \alpha \in \mathcal{J}_x \text{ and } B_\alpha \cap \mathcal{I}^{\mathrm{ess}}_y \neq \varnothing\}.$$

If $\mathcal{B}$ is a harmonious family of bisections and $y \in B_x$, then $\mathcal{B}^{\mathrm{ess}}_y$ is a group by Conditions (iv) and (iii) of Definition 6.1, so this new definition of $H_{\mathcal{B}}(y)$ agrees with the one given in Remark 6.2.

Now given a subgroup $H \subseteq \mathbb{Z}^k$, we let $\widehat{q}_H\colon \mathbb{T}^k \to \widehat{H}$ be the canonical quotient map.

Definition 9.1. Suppose that $(H_n)_n$ is a sequence of subgroups of $\mathbb{Z}^k$. A sequence $(z_n)_n$ in $\mathbb{T}^k$ *converges to* $z \in \mathbb{T}^k$ *along* $(H_n)_n$ if for every open neighbourhood $V \subseteq \mathbb{T}^k$ of z in $\mathbb{T}^k$ there exists $N \in \mathbb{N}$ such that $\widehat{q}_{H_n}(z_n) \in \widehat{q}_{H_n}(V)$ for all $n \geq N$.

The following sufficient condition for convergence of primitive ideals applies to any Deaconu–Renault system.

Proposition 9.2. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For $x \in X$ and $z \in \mathbb{T}^k$, let $\pi_{(x,z)}$ be as in Notation 4.1. Let $(x,z) \in X \times \mathbb{T}^k$ and let $\mathcal{B} = (B_\alpha)_{\alpha\in\mathcal{J}_x}$ be a collection of open bisections such that $\alpha \in B_\alpha \subseteq c^{-1}(c(\alpha))$ for all α. If $(x_i, z_i)_i$ is a sequence in $\in X \times \mathbb{T}^k$ satisfying $x_i \to x$ and that $z_i \to z$ along $H_{\mathcal{B}}(x_i)$, then* $\ker(\pi_{(x_i,z_i)}) \to \ker(\pi_{(x,z)})$ *in* $\operatorname{Prim}(C^*(G_T))$.

Proof. Fix an open set $A \subseteq \operatorname{Prim}(C^*(G_T))$ that contains $\ker(\pi_{(x,z)})$. We must show that $\ker(\pi_{(x_i,z_i)}) \in A$ for large i. By Theorem 4.2 there exist an open neighbourhood $U \subseteq X$ of x and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $(U \times V) \cdot (\mathcal{B}^{\text{ess}})^\perp \subseteq A$. Since $x_i \to x$ we see that $x_i \in U$ for large i, so it suffices to show that $z_i \in V \cdot (\mathcal{B}^{\text{ess}}_{x_i})^\perp$ for large i.

For each i, let $H_i := H_{\mathcal{B}}(x_i)$. Since $z_i \to z$ along $(H_i)_i$, we have $\widehat{q}_{H_i}(z_i) \in \widehat{q}_{H_i}(V)$ for large i. So there exist $z_i' \in V$ such that for large i, we have $(z_i)^h = (z_i')^h$ for all $h \in H_i$. That is, $w_i := z_i\overline{z_i'} \in (\mathcal{B}^{\text{ess}}_{x_i})^\perp$ for large i, and it follows that $z_i = w_i z_i' \in V \cdot (\mathcal{B}^{\text{ess}}_{x_i})^\perp$ for large i as required. $\square$

We can also identify a necessary condition that is valid for all Deaconu–Renault systems. Recall that the *quasi-orbit space* $\mathcal{Q}(G)$ of an étale groupoid G is the set $\{\overline{[x]} : x \in G^{(0)}\}$ of orbit closures in G, in the quotient topology induced by the surjection $\mathcal{Q}\colon x \mapsto \overline{[x]}$ from $G^{(0)}$ to $\mathcal{Q}(G)$.

Lemma 9.3. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. For $x \in X$ and $z \in \mathbb{T}^k$, let $\pi_{(x,z)}$ be as in Notation 4.1. Then the quasi-orbit map $q\colon \operatorname{Prim}(C^*(G_T)) \to \mathcal{Q}(G_T)$ given by $q(\ker(\pi_{(x,z)})) = \overline{[x]}$ for all $(x,z) \in X \times \mathbb{T}^k$ is continuous. In particular, if $\ker(\pi_{(x_n,z_n)}) \to \ker(\pi_{(x,z)})$ in $\operatorname{Prim}(C^*(G_T))$, then $\overline{[x_n]} \to \overline{[x]}$ in $\mathcal{Q}(G_T)$.*

Proof. Take a closed subset K of $\mathcal{Q}(G_T)$ and consider the preimage

$$W = q^{-1}(K) = \{\ker(\pi_{(x,z)}) : \overline{[x]} \in K, z \in \mathbb{T}^k\}.$$

We aim to show that W is closed in $\operatorname{Prim}(C^*(G_T))$. Take $\ker(\pi_{(y,w)})$ in the closure of W. By definition of the hull-kernel topology, this means that

$$\bigcap_{\overline{x}\in K, z\in\mathbb{T}^k} \ker(\pi_{(x,z)}) \subseteq \ker(\pi_{(y,w)}).$$

It suffices to show that $\overline{[y]} \in K$, so assume for contradiction that this is not the case. Then y is not in the preimage $Y = \mathcal{Q}^{-1}(K)$ which is closed and invariant. By Urysohn's lemma, there is a function $f \in C_c(X, [0,1])$ satisfying $f(y) = 1$ and $f|_Y = 0$. But then $f \in \ker(\pi_{(x,z)})$ whenever $\overline{[x]} \in K$ and $f \notin \ker(\pi_{(y,w)})$, and this contradicts the inclusion above. $\square$

Bönicke and Li [BL20, Remark 3.16] observe that $\mathcal{Q}$ is a continuous and open surjection and that $\mathcal{Q}(G)$ is T0 and Baire. When X is second-countable, the quasi-orbit space $\mathcal{Q}(G_T)$ is also second-countable. Therefore, the quasi-orbit map is *sequence-covering* in the sense of Siwiec, see [Siw71, Proposition 2.4] (Siwiec assumes that all spaces are Hausdorff but the proof is valid even if the codomain is not Hausdorff). Indeed, since the quasi-orbit map is not just an almost-open map but a *bona fide* open map, the proof of [Siw71, Proposition 2.4] establishes the following strong sequence-covering condition:

Lemma 9.4. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms. Suppose that $x \in X$ and that $(x_n)_n$ is a sequence in X such that $\overline{[x_n]} \to \overline{[x]}$ in $\mathcal{Q}(G_T)$. Then there is a sequence $(x'_n)_n$ in X such that $x'_n \to x$ and $\overline{[x'_n]} = \overline{[x_n]}$ for all n.*

Proof. Since $\mathcal{Q}$ is open, $\mathcal{Q}(U)$ is a neighbourhood of $\overline{[x]}$ in $\mathcal{Q}(G)$ for each neighbourhood U of x. Now putting $y_n := \overline{[x_n]} \in \mathcal{Q}(G)$ for each n, the proof of [Siw71, Proposition 2.4] starting from the fifth sentence "Let $F_n = f^{-1}(y_n)\ldots$," establishes the result. □

We will now use Lemma 9.4 together with harmonious families of bisections to obtain a complete description of convergence of primitive ideals.

Theorem 9.5. *Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N}^k \curvearrowright X$ is an action by local homeomorphisms that admits bisection families. For $x \in X$ and $z \in \mathbb{T}^k$, let $\pi_{(x,z)}$ be as in Notation 4.1. Take $(x,z) \in X \times \mathbb{T}^k$, let $\mathcal{B} = \big(B_\alpha\big)_{\alpha\in\mathcal{J}_x}$ be a harmonious family of bisections at x, and let $P = \ker(\pi_{(x,z)})$. Let $(P_n)_n$ be a sequence in $\operatorname{Prim}(C^*(G_T))$. Then $P_n \to \ker(\pi_{(x,z)})$ in $\operatorname{Prim}(C^*(G_T))$ if and only if there is a sequence $(x'_n, z_n)_n$ in $X \times \mathbb{T}^k$ satisfying*
(i) $\ker(\pi_{(x'_n,z_n)}) = P_n$ *for all* $n \in \mathbb{N}$,
(ii) $x'_n \to x$ *in* X,
(iii) $z_n \to z$ *along* $(H_{\mathcal{B}}(x'_n))_n$.

Proof. First suppose that $P_n \to P$. Choose a sequence $(x_n, z_n)_n$ in $X \times \mathbb{T}^k$ such that $P_n = \ker(\pi_{(x_n,z_n)})$ for all n. Then $\overline{[x_n]} \to \overline{[x]}$ in $\mathcal{Q}(G_T)$ by Lemma 9.3. By Lemma 9.4, there is a convergent sequence $(x'_n)_n$ in X such that $\overline{[x'_n]} = \overline{[x_n]}$ for all n and $x'_n \to x$. We must show that $z_n \to z$ along $(H_{\mathcal{B}}(x'_n))_n$. For this, fix an open set $V \subseteq \mathbb{T}^k$ containing z. Theorem 7.1 shows that the set $A = A(\mathcal{B}, V)$ of (7.1) is an open neighbourhood of P, so $P_n \in A$ for large n. Hence for large n there exists $z''_n \in V$ such that $(z_n)^h = (z''_n)^h$ for all $h \in H_{\mathcal{B}}(x'_n)$. So $z_n \to z$ along $\big(H_{\mathcal{B}}(x'_n)\big)_n$.

Conversely, suppose that there is a sequence (x'_n, z_n) satisfying the three conditions in the theorem. Assume for contradiction that $P_n \not\to P$ and choose an open neighbourhood A of $\pi(x,z)$ and a subsequence $(P_{n_i})_i$ of $(P_n)_n$ such that $P_{n_i} \notin A$ for all i. By Theorem 4.2, there exist an open $U \subseteq X$ containing x and an open $V \subseteq \mathbb{T}^k$ containing z such that $(U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \pi^{-1}(A)$. In particular, $P_{n_i} \notin \pi((U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp)$ for all i. Since (x'_n, z_n) satisfies the conditions of the theorem, we have $x'_{n_i} \in U$ for large i. Since $z_n \to z$ along $(H_{\mathcal{B}}(x_n))_n$, we have $z_{n_i} \to z$ along $(H_{\mathcal{B}}(x_{n_i}))_i$, which forces $z_{n_i} \in V \cdot H_{\mathcal{B}}(x'_{n_i})^\perp = V \cdot H_{\mathcal{B}}(x_{n_i})^\perp$ for large i. Hence $(x_{n_i}, z_{n_i}) \in (U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp$ for large i, contradicting that $\ker(\pi_{(x_{n_i},z_{n_i})}) = P_{n_i} \notin A$. □

10. Examples

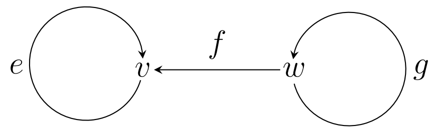
In this section, we provide a number of examples. Firstly, we apply our theorems to recover the primitive-ideal space of the C^*-algebra of the "dumbbell graph" of Example 2.7, and then also those of C^*-algebras of arbitrary row-finite graphs with no sources, and of crossed products of locally compact Hausdorff spaces by actions of $\mathbb{Z}^k$. These examples are intended to be illustrative. The results are not new.

Next we outline how our results and techniques relate to those of Katsura in [Kat21]. Much of the work on the present paper had been completed when Katsura's work was posted on the arXiv, so naturally we were interested to determine how the two approaches relate. It turns out that our techniques can be used in Katsura's setting, but not by a straightforward application of our main theorem. Combining the two approaches seems like an avenue for future exploration.

Finally, we demonstrate that our results and hypotheses are checkable for all 2-graph groupoids. The question of how to describe the primitive-ideal spaces of C^*-algebras of 2-graphs was the initial motivation for our line of investigation, so we detail how our main results answer this question.

10.1. **The Dumbell graph.** The ideal structure of the C^*-algebra of the dumbell graph is relatively simple, but illuminating. It is also well-known—it is an example of the original results of an Huef and Raeburn on primitive-ideal spaces of Cuntz–Krieger algebras [aHR97], and is analysed explicitly in [SW16, Example 3.4].

First recall that the dumbell graph is the graph E depicted below



As discussed in Example 2.7, the essential isotropy in G_E is

$$\{(e^\infty, n, e^\infty) : n \in \mathbb{Z}\} \cup \{(e^m f g^\infty, n, e^m f g^\infty) : m \geq 0, n \in \mathbb{Z}\} \cup \{(g^\infty, n, g^\infty) : n \in \mathbb{Z}\}.$$

The unit space is a clopen subset homeomorphic to $\mathbb{N} \cup \{\infty\}$, with (e^∞, n, e^∞) identified with the point at infinity, and the remainder of $\mathcal{I}^{\mathrm{ess}}$ is discrete.

In particular, there are just two orbits, namely $[e^\infty] = \{e^\infty\}$, and $[g^\infty] = E^\infty \setminus \{e^\infty\}$. Their orbit closures are $\{e^\infty\}$ and E^∞ respectively. For $x \in E^\infty$ and $z \in \mathbb{T}$, let $\pi_{(x,z)}$ be as in Notation 4.1. Then the primitive ideals of $C^*(E)$ are $\{\ker(\pi_{(e^\infty,z)}), \ker(\pi_{(g^\infty,z)}) : z \in \mathbb{T}\}$. For each of $x = e^\infty$ and $x = g^\infty$, the group $H(x) = c(\mathcal{I}^{\mathrm{ess}}_x)$ is $\mathbb{Z}$, so two elements $z, z' \in \mathbb{T}$ induce the same character of $H(x)$ if and only if they are equal. Hence $\pi : (x, z) \to \ker(\pi_{(x,z)})$ is a bijection $\{e^\infty, g^\infty\} \times \mathbb{T} \to \mathrm{Prim}(C^*(E))$.

To describe the topology, first observe that since g^∞ is an isolated point in E^∞, we obtain a harmonious family of bisections $\mathcal{B}^g$ at g^∞ by putting $B_{(g^\infty,n,g^\infty)} = \{(g^\infty, n, g^\infty)\}$. By Theorem 7.1, the sets $\{A(\mathcal{B}^g, V) : V \subseteq \mathbb{T} \text{ is open}\}$ are a basis for the topology on the clopen subset $\{\ker(\pi_{(g^\infty,z)} : z \in \mathbb{T}\}$, and we deduce that this subset is homeomorphic to $\mathbb{T}$ via $\ker(\pi_{(g^\infty,z)} \mapsto z$.

Now consider e^∞. To lighten notation, we write $\alpha(n) \coloneqq (e^\infty, n, e^\infty)$ for each $n \in \mathbb{Z}$. We define bisections $C_{\alpha(n)}$ as follows:

$$C_{\alpha(n)} \coloneqq \begin{cases} Z(e^n, v) & \text{if } n > 0 \\ E^\infty & \text{if } n = 0 \\ Z(v, e^{-n}) & \text{if } n < 0 \end{cases}$$

We claim that $\mathcal{C} \coloneqq (C_{\alpha(n)})_{n \in \mathbb{Z}}$ is a harmonious family of bisections for e^∞. Clearly each $C_{\alpha(n)}$ is an open bisection. Conditions (i) and (ii) are immediate from the definition of the $C_{\alpha(n)}$. Since each $C_{\alpha(n)}$ is a compact open bisection, it satisfies (v) with $K_{\alpha(n)} = s(C_{\alpha(n)})$. For conditions (iii) and (iv), it suffices to show that for $n \neq 0$, we have $C_{\alpha(n)} \cap \mathcal{I}^{\mathrm{ess}} = \{\alpha(n)\}$. To see this, observe that if $x \in r(C_{\alpha(n)})$, then $(x, n, \sigma^n(x))$ is the unique element of $C_{\alpha(n)}$, so it suffices to fix $x \in r(C_{\alpha(n)}) \setminus \{e^\infty\}$ and show that $\sigma^n(x) \neq x$. For this, note that by definition of $C_{\alpha(n)}$ we have $x = e^n x'$ and $\sigma^n(x) = x'$ for some x'. Since $x \neq e^\infty$, we deduce that there exists $k \geq 0$ such that $x = e^{n+k} f g^\infty$, and hence $x' = e^k f g^\infty \neq x$.

Now given $z \in \mathbb{T}^k$, Theorems 4.2 and 7.1 imply that the sets $(U \times V) \cdot (\mathcal{C}^{\mathrm{ess}})^\perp$ ranging over open neighbourhoods U of e^∞ and V of z are the pre-images of a neighbourhood basis for $\ker(\pi_{(e^\infty,z)})$. Since, for any unit $y \neq e^\infty$ in U, we have $\mathcal{C}^{\mathrm{ess}}_y \cap (G_E)_y = \{y\}$, we see that $(\mathcal{C}^{\mathrm{ess}})_y^\perp = \{y\} \times \mathbb{T}$ for $y \neq e^\infty$. Since $\mathcal{C}^{\mathrm{ess}}_{e^\infty} \cap (G_E)_{e^\infty} = \{\alpha(n) : n \in \mathbb{Z}\}$, we have $(\mathcal{B}^{\mathrm{ess}})_y^\perp = \{e^\infty\} \times \{1\}$. So for any neighbourhood U of e^∞ and any neighbourhood V of z, we have

$$(U \times V) \cdot (\mathcal{C}^{\mathrm{ess}})^\perp = \big(\{e^\infty\} \times V\big) \cup \big((U \setminus \{e^\infty\}) \times \mathbb{T}\big).$$

Since $\ker(\pi_{(x,z)}) = \ker(\pi_{(g^\infty,z)})$ for $x \in E^\infty \setminus \{e^\infty\}$, it follows that a basic open neighbourhood of $\ker(\pi_{(e^\infty,z)})$ has the form

$$\{\ker(\pi_{(e^\infty,w)}) : w \in V\} \cup \{\ker \pi_{(g^\infty,w)} : w \in \mathbb{T}\}.$$

To summarise, if we put

$$P_v := \{\ker(\pi_{(g^\infty,z)}) : z \in \mathbb{T}\} \quad \text{and} \quad P_w := \{\ker(\pi_{(e^\infty,z)}) : z \in \mathbb{T}\},$$

then $\mathrm{Prim}(C^*(E)) = P_v \sqcup P_w$; the subset P_w is open; the map $\ker(\pi_{(e^\infty,z)}) \mapsto z$ is a homeomorphism of P_v in the relative topology onto $\mathbb{T}$; the map $\ker(\pi_{(g^\infty,z)}) \mapsto z$ is a homeomorphism of P_w in the relative topology onto $\mathbb{T}$; and for any z, the closure of the point $\ker(\pi_{(g^\infty,z)})$ is $P_v \cup \ker(\pi_{(g^\infty,z)})$.

Remark 10.1. As mentioned in [SW16, Example 3.4], this example demonstrates that the map $\pi \colon (x,z) \mapsto \ker(\pi_{(x,z)})$ is not an open map, since the image of $W := E^\infty \times \{z \in \mathbb{T} : \mathrm{Re}(z) > 0\}$ is not open. Indeed, π is not even a quotient map: $\pi(W) = \{\ker(\pi_{(e^\infty,z)}), \ker(\pi_{(g^\infty,z)}) : \mathrm{Re}(z) > 0\}$, and since $\pi^{-1}(\ker(\pi_{(g^\infty,z)})) = \{\ker(\pi_{(\alpha g^\infty,z)}) : \alpha \in E^* w\}$ it follows that $\pi^{-1}(\pi(W)) = W$; so $\pi(W)$ is a subset of $\mathrm{Prim}(C^*(E))$ that is not open but whose preimage in $E^\infty \times \mathbb{T}$ is open.

10.2. **Graph groupoids.** The analysis of the dumbbell graph above extends to arbitrary row-finite graphs with no sources. Our results here are not new—they are special cases of the theorems of [HS04], and also appear in [CS16]. We include them to indicate how our results relate to those papers.

The orbit closures are indexed by the maximal tails T as in [HS04], but for our picture, we prefer to describe them in terms of orbit closures. The correspondence is as follows. Given an infinite path x with orbit-closure

$$\overline{[x]} = \{y : \text{ for each } m \geq 0 \text{ there exists } n \geq 0 \text{ such that } r(y_m)E^* r(x_n) \neq \varnothing\},$$

the corresponding maximal tail T_x is the set of vertices v such that $vE^* r(x_n) \neq \varnothing$ for some n. Conversely, given a maximal tail T, enumerate the vertices of T as $(v_1, v_2, \dots)$. Let $w_1 := v_1$. Condition (3) for maximal tails ensures that there exists w_2^0 such that $w_1 E^* w_2^0$ and $v_2 E^* w_2^0$ are nonempty. Condition (2) for maximal tails ensures that there exists $e \in w_2^0 E^1$ with $s(e) \in T$. So $w_2 \in T$ has the property that $w_1 E^* w_2 \setminus E^0$ and $v_2 E^* w_2$ are nonempty. Repeating this procedure, we obtain a sequence w_n such that $w_1 = v_1$ and $w_n E^* w_{n+1} \setminus E^0$ and $v_{n+1} E^* w_{n+1}$ are both nonempty for all n. For each n, fix $\mu_n \in w_n E^* w_{n+1}$. Then $x = \mu_1 \mu_2 \dots$ is an infinite path. We clam that $T = T_x$. Since each $v_n E^* w_n \neq \varnothing$, we see that $v_n \in T_x$ for all n, and so $T \subseteq T_x$. For the reverse containment, fix $v \in T_x$. Then $vE^* r(x_i) \neq \varnothing$ for some i, say $\alpha \in vE^* r(x_i)$. Since each $|\mu_j| \geq 1$ there exists n such that $l := |\mu_1 \dots \mu_n| \geq i$. So $\mu_1 \dots \mu_n = x_1 \dots x_i x_{i+1} \dots x_l$ and in particular $\alpha x_i x_{i+1} \dots x_l \in vE^* r(x_{l+1})$. So $v \in T$ by condition (1) for maximal tails. So $T_x \subseteq T$ as required.

If x is an infinite path such that every cycle in ET_x has an entrance in T_x, then $\mathcal{I}_x^{\mathrm{ess}} = \{x\}$, so constructing a bisecton family at x is trivial (just take $B_x = Z(r(x))$). If x is an infinite path and there is a cycle $\mu \in ET_x$ with no entrance in ET_x, then $x = \alpha\mu^\infty$ for some α, and we may as well take $\alpha = r(\mu)$ and $x = \mu^\infty$, and assume that μ is the cycle of minimal length such that $\mu^\infty = x$ (that is, that μ is not a multiple of a shorter cycle). In this instance, $H(x) = |\mu|\mathbb{Z}$, and $\mathcal{J}_x = \{(\mu^\infty, n|\mu|, \mu^\infty) : n \in \mathbb{Z}\}$. We write $\alpha(n) := (\mu^\infty, n|\mu|, \mu^\infty)$ for all n. Define

$$B_{\alpha(n)} := \begin{cases} Z(\mu^n, r(\mu)) & \text{if } n > 0 \\ E^\infty & \text{if } n = 0 \\ Z(r(\mu), \mu^{-n}) & \text{if } n < 0. \end{cases} \tag{10.1}$$

We claim that $B_{\alpha(n)} \cap \mathcal{I}^{\text{ess}} = \{\alpha(n)\}$ for all n. To see this, suppose that $x \in r(B_{\alpha(n)}) \setminus \{\mu^\infty\}$. Since μ has no entrance in ET_x, there exists a minimum $i \in \mathbb{N}$ such that $s(x_i) \notin T$. We have $x = \mu^n x'$ for some x', and since the vertices on μ belong to t, we have $i > n|\mu|$. Hence the minimum $j \in \mathbb{N}$ such that $s(x'_j) \notin T$ is $j = i - n|\mu| \neq i$, and so $x' \neq x$. Since $(x, n|\mu|, x')$ is the unique element of $B_{\alpha(n)}$ with range x, we deduce that $B_{\alpha(n)} \cap \mathcal{I}(G_E)_x = \varnothing$. Since $x \in r(B_{\alpha(n)}) \setminus \{\mu^\infty\}$ was arbitrary, we deduce that $B_{\alpha(n)} \cap \mathcal{I}(G_E) = B_{\alpha(n)} \cap \mathcal{I}(G_E)_{\mu^\infty} = \{\alpha(n)\}$ as claimed. It now follows just as in the dumbbell graph that $\mathcal{B} = (B_{\alpha(n)})_{n \in \mathbb{Z}}$ is a harmonious family of bisections at μ^∞.

Let $S \subseteq E^\infty$ be a set containing one representative for each orbit closure. We define a partial order $\leq$ on S by $y \leq x$ if and only if $y \in \overline{[x]}$. Write $S = S_a \sqcup S_p$ where S_a consists of precisely the elements of S such that every cycle in the associated maximal tail has an entrance. For each $x \in S_p$ we can choose a cycle μ of minimal length with no entrance in the tail corresponding to x, and may assume that $x = \mu^\infty$. For $x \in E^\infty$ and $z \in \mathbb{T}$, let $\pi_{(x,z)}$ be as in Notation 4.1. Then

$$\operatorname{Prim}(C^*(E)) = \{\ker(\pi_{(x,z)}) : x \in S\},$$

and that $\ker(\pi_{(x,z)}) = \ker(\pi_{(x,w)})$ for all z, w if $x \in S_a$, whereas $\ker(\pi_{(\mu^\infty,z)}) = \ker(\pi_{(\mu^\infty,w)})$ if and only if $z^{|\mu|} = w^{|\mu|}$ when $\mu^\infty \in S_p$.

To describe point closures, we argue precisely as in the dumbbell-graph example to see that for any infinite path $y \in S$ and any z, the closure of $\ker(\pi_{(y,z)})$ is $\ker(\pi_{(y,z)}) \cup \{\ker(\pi_{(y',w)}) : y' < y \text{ and } w \in \mathbb{T}\}$.

We can recover [CS16, Theorem 4.1], which describes the closure operation in the hull-kernel topology as follows. Fix a set Y of pairs (x, z) consisting of an infinite path x and an element $z \in \mathbb{T}$. To describe the corresponding set of pairs (T, z) consisting of a maximal tail and an element of $\mathbb{T}$ in [CS16, Theorem 4.1], for a maximal tail T, we define $\operatorname{Per}(T)$ to be equal to n if T contains a cycle with no entrance and the minimal length of such a cycle is n, and to be 0 otherwise. Then the set of pairs appearing in [CS16, Theorem 4.1] is $\{(T_x, z^{\operatorname{Per}(T_x)}) : (x, z) \in Y\}$. We must show that $\ker(\pi_{(y,w)})$ belongs to the closure of $\{\ker(\pi_{(x,z)}) : (x, z) \in Y\}$ if and only if $T_y \subseteq \bigcup_{(x,z) \in Y} T_x$ and if $\operatorname{Per}(T_y) \neq 0$ and the cycle μ with no entrance in T_y has no entrance in $\bigcup_{(x,z) \in Y} T_x$ then $z^{\operatorname{Per}(T_y)} \in \overline{\{z^{\operatorname{Per}(T_x)} : (x, z) \in Y \text{ and } T_x = T_y\}}$.

First observe that if $T_y \not\subseteq \bigcup_{(x,z) \in Y} T_x$ then there is a neighbourhood U of y that does not intersect $\bigcup_{(x,z) \in Y} [x]$, so for any harmonious family of bisections $\mathcal{B}$ at y, the set $\pi\big((U \times \mathbb{T}^k) \cdot (\mathcal{B}^{\text{ess}})^\perp\big)$ is a neighbourhood of $\ker(\pi_{(y,w)})$ that is disjoint from $\{\ker(\pi_{(x,z)}) : (x, z) \in Y\}$. So we suppose that $T_y \subseteq \bigcup_{(x,z) \in Y} T_x$ and prove that $\ker(\pi_{(y,w)}) \in \overline{\{\ker(\pi_{(x,z)}) : (x, z) \in Y\}}$ if and only if, if $\operatorname{Per}(T_y) \neq 0$ and the cycle μ with no entrance in T_y has no entrance in $\bigcup_{(x,z) \in Y} T_x$ then $z^{\operatorname{Per}(T_y)} \in \overline{\{z^{\operatorname{Per}(T_x)} : (x, z) \in Y \text{ and } T_x = T_y\}}$.

First suppose that every cycle in T_y has an entrance. We saw in Example 2.8 that in a graph groupoid, $\mathcal{I}^{\text{ess}} \setminus E^\infty$ is discrete, so $\mathcal{J}_y = \mathcal{I}^{\text{ess}}_y = \{y\}$. Hence a harmonious family of bisections at y is just an open neighbourhood B_y of y in E^∞. For any open $V \subseteq \mathbb{T}$, the corresponding neighbourhood $A(\mathcal{B}, V)$ of $\ker(\pi_{(y,w)})$ is $\{\ker(\pi_{(x,z)}) : x \in U, z \in \mathbb{T}\}$, so we see that $\ker(\pi_{(y,w)}) \in \overline{\{\ker(\pi_{(x,z)}) : (x, z) \in Y\}}$ precisely if $U \cap \bigcup_{(x,z) \in Y} [x]$ is nonempty for every neighbourhood U of y; that is, if and only if y is in the closure of $\bigcup_{(x,z)\in Y}[x]$, which is precisely if $T_y \subseteq \bigcup_{(x,z)\in Y} T_x$.

Now suppose that there is a cycle with no entrance in T_y, and let μ be such a cycle of minimal length. Then $y = \alpha\mu^\infty$ for some α and since $\ker(\pi_{(\alpha\mu^\infty,z)}) = \ker(\pi_{(\mu^\infty,z)})$ for all z, we may as well assume that $y = \mu^\infty$. We must consider two cases.

Case 1: suppose that μ has an entrance in $\bigcup_{(x,z)\in Y} T_x$. Then there is a sequence (x_n, z_n) such that each $[x_n] = [x'_n]$ for some $(x'_n, z_n) \in Y$, and there is no cycle with an entrance in each T_{x_n}. For the harmonious family of bisections $\mathcal{B}$ at y described in (10.1), we then have $H_{\mathcal{B}}(x_n) = \{0\}$ for all n. So for any open U containing y and contained in B_y and any open $V \subseteq \mathbb{T}$, we have $x_n \times \mathbb{T}$ eventually contained in $(U \times V) \cdot (\mathcal{B}(y)^{\mathrm{ess}})^\perp$. Thus $\ker(\pi_{(y,w)}) \in \overline{\{\ker(\pi_{(x,z)}) : (x,z) \in Y\}}$ by Corollary 7.6.

Case 2: suppose that μ has no entrance in $\bigcup_{(x,z)\in Y} T_x$. Let $\mathcal{B}$ be the harmonious family of bisections at y described in (10.1). Since μ has no entrance in any T_x, the sets $W_y := \{x : [x] = [y]\}$ and $W' := \{x : r(\mu) \notin T_x\}$ satisfy $\{x : (x,z) \in Y\} = W_y \sqcup W'$. The set W_y is nonempty because $T_y \not\subseteq \bigcup_{(x,z)\in Y} T_x$. Since $y = \mu^\infty$, we have $\pi\big((U \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp\big) \cap \{\ker(\pi_{(x,z)}) : x \in W'\} = \varnothing$, and so $\ker(\pi_{(y,w)}) \in \overline{\{\ker(\pi_{(x,z)}) : (x,z) \in Y\}}$ if and only if

$$\begin{aligned}\ker(\pi_{(y,w)}) &\in \overline{\{\ker(\pi_{(x,z)}) : (x,z) \in Y \text{ and } x \in W_y\}}\\ &= \overline{\{\ker(\pi_{(y,z)}) : (x,z) \in Y \text{ and } x \in W_y\}}.\end{aligned}$$

The set $\{\ker(\pi_{(y,z)}) : z \in \mathbb{T}\}$ is homeomorphic to $\mathbb{T}$ via the map induced by $z \mapsto z^{|\mu|}$, so

$$\begin{aligned}\ker(\pi_{(y,w)}) &\in \overline{\{\ker(\pi_{(y,z)}) : (x,z) \in Y \text{ and } x \in W_y\}}\\ \iff w^{|\mu|} &\in \overline{\{z^{|\mu|} : (x,z) \in Y \text{ and } T_x = T_y\}}\end{aligned}$$

as required.

10.3. **Crossed products by free abelian groups.** The primitive ideals of crossed products of spaces by free abelian group are well understood, see e.g. [Wi07, Theorem 8.39]. Here we describe how to recover their structure from our results.

Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{Z}^k \curvearrowright X$ is an action by homeomorphisms of X. The groupoid $G_T := \{(x,n,y) \in X \times \mathbb{Z}^k \times X : T^n(y) = x\}$ is identical to the Deaconu–Renault groupoid of the restriction of T to $\mathbb{N}^k$, and the map $(x,n,y) \mapsto (n,y)$ is an isomorphism of G_T onto $\mathbb{Z}^k \times X$.

Note that if $y \in \overline{[x]}$, then $T^{n_i}x \to y$ for some sequence $(n_i)_i$, so $T^{-n_i}y \to x$. This shows that $\overline{[x]} = \overline{[y]}$ if and only if $y \in \overline{[x]}$. Moreover, if $T^n x = x$, then for every $m \in \mathbb{Z}^k$, we have $T^n(T^m x) = T^m T^n x = T^m x$, and so $T^n y = y$ for all $y \in [x]$. By continuity, it follows that $T^n y = y$ for all $y \in \overline{[x]}$. Therefore, $\mathcal{I}^{\mathrm{ess}}(G_T) = \mathcal{I}(G_T)$ which is closed so $\mathcal{J}_x = \mathcal{I}^{\mathrm{ess}}_x = \mathcal{I}(G_T)_x$ for every unit $x \in X$. In particular, all isotropy is essential isotropy.

For every periodic point $x \in X$ and every $n \in \mathbb{Z}^k$ with $T^n x = x$, the sets

$$B^x_{(x,n,x)} := \{(T^n y, n, y) : y \in X\}$$

constitute a harmonious family of bisections at x, cf. Lemma 6.6. Hence T admits harmonious families of bisections.

We write $\mathrm{Stab}(x) = \{n \in \mathbb{Z}^k : T^n x = x\}$ for the stabiliser group. For $x \in X$ and $z \in \mathbb{T}^k$, let $\pi_{(x,z)}$ be as in Notation 4.1. The primitive ideals of $C^*(G_T) \cong C_0(X) \rtimes \mathbb{Z}^k$ are the kernels $\{\ker(\pi_{(x,z)}) : x \in X, z \in \mathbb{Z}\}$, and we have $\ker(\pi_{(x,w)}) = \ker(\pi_{(y,z)})$ precisely if $y \in \overline{[x]}$ and $w^n = z^n$ for all $n \in \mathrm{Stab}(x)$. A neighbourhood base at $\ker(\pi_{(x,z)})$ is described by Theorems 4.2 and 7.1 as follows. Given open neighbourhoods $x \in U \subseteq X$ and $z \in V \subseteq \mathbb{T}^k$, the set

$$A^x(U,V) = \{\ker(\pi_{(y,w)}) : y \in U \text{ and } w^n = z^n \text{ for all } n \in \operatorname{Stab}(y) \cap \operatorname{Stab}(x)\}$$

is an open neighbourhood of $\ker(\pi_{(x,z)})$, and the collection

$$\{A^x(U,V) : x \in U \subseteq_{\text{open}} X \text{ and } z \in V \subseteq_{\text{open}} \mathbb{T}^k\}$$

is a neighbourhood base at x.

By Theorem 9.5, a sequence $\big(\ker(\pi_{(x_i,z_i)})\big)_i$ of primitive ideals of $C_0(X) \rtimes \mathbb{Z}^k$ converges to $\ker(\pi_{(x,z)})$ if and only if for every subsequence $\big(\ker(\pi_{(x_{i_j},z_{i_j})})\big)_j$ there is a sequence $(y_j, w_j)_j$ in $X \times \mathbb{T}^k$ such that

(1) $y_j \in \overline{[x_{i_j}]}$ and $(w_j)^n = z_{i_j}^n$ for all $n \in \operatorname{Stab}(x_{i_j})$ for all j;
(2) $y_j \to x$; and
(3) z_i converges to z along $\big(\operatorname{Stab}(x_i) \cap \operatorname{Stab}(x)\big)_i$ in the sense of Definition 9.1.

10.4. **Singly generated dynamical systems.** This paper was fairly advanced when Katsura's work [Kat21] was posted to the arXiv. In it, Katsura completely describes the ideals of the C^*-algebra of a singly generated dynamical system, which are merely (partially defined) local homeomorphisms on a locally compact Hausdorff space, and this had been one of our main objectives. Katsura applied the technology of C^*-correspondences and his adaptation of Cuntz–Pimsner algebras [Kat04] to define the C^*-algebras of topological graphs. Here we outline how we can recover his results in the setting of globally defined local homeomorphisms of second-countable locally compact Hausdorff spaces.

Let X be a second-countable locally compact Hausdorff space and suppose that $T\colon \mathbb{N} \curvearrowright X$ is an action by local homeomorphisms. We will use ideas and terminology from our previous work [BCS22]. Our result [BCS22, Theorem 3.5] shows that the problem of describing the ideal structure of $C^*(G_T)$ can be reduced to considering nested open invariant subsets $U \subseteq V \subseteq X$ and describing all ideals J of $C^*((G_T)|_{V\setminus U})$ satisfying $J \cap C_0(V \setminus U) = \{0\}$ and $\operatorname{supp}(J) = V \setminus U$. We called such ideals purely non-dynamical with full support.

Since U and V are invariant, so is $V\setminus U$, and $(G_T)|_{V\setminus U}$ is precisely the Deaconu–Renault groupoid of the restricted action $T\colon \mathbb{N} \curvearrowright (V\setminus U)$. So we may as well replace X with $V\setminus U$; our task is then to describe all of the ideals J of $C^*(G_T)$ such that $J \cap C_0(X) = \{0\}$ and $\operatorname{supp}(J) = G_T$. Note that the open invariant sets in our sense are precisely the T-invariant sets in the sense of [Kat21, Definition 3.1].

Assume that G_T admits a purely non-dynamical ideal J with full support. By [BCS22, Lemma 5.2] (cf. [AL18, Example 7.6]) the groupoid G_T is jointly effective where it is effective. For each $p \in \mathbb{N}$, let

$$\mathcal{P}_p := \bigcup\{V \subseteq X : V \text{ is open and } T^p(y) = y \text{ for all } y \in V\},$$

and let

$$P = \bigcup_{p>0} P_p.$$

Then [BCS22, Lemma 5.1] shows that complement in X of the points at which G_T is effective in the sense of [BCS22, Definition 4.1] is precisely $P' := \{x \in X : [x] \cap P \neq \varnothing\}$. Hence [BCS22, Theorem 4.12] (cf. [AL18, Theorem 7.12]) implies that J is contained in the ideal $I_{P'}$. Since $\operatorname{supp}(J) = G_T$ we deduce that $P' = X$, and so P is a full open subset of X. Hence by [MRW87, Example 2.7], the inclusion $C^*(G_T|_P) \hookrightarrow C^*(G_T)$ defines a Morita equivalence, and in particular induces a bijection between ideals of $C^*(G_T|_P)$ and $C^*(G_T)$.

For each P_n, the map T^{n-1} is an inverse for $T|_{P_n}$, so T is injective on each P_n and hence on P. Since T is a local homeomorphism, it follows that it restricts to a homeomorphism of P. So $G_T|_P$ is the transformation groupoid associated to the homeomorphism $T|_P$ on P, so it admits harmonious families of bisections, and its ideals are described by Section 10.3 above.

Remark 10.2. To relate our discussion to Katsura's work, observe that given an action $T\colon \mathbb{N} \curvearrowright X$, the representations $\pi_{x,z}$ of $C^*(G_T)$ from [SW16], whose kernels coincide with those of the representations $\pi_{(x,z)}$ of Notation 4.1 by Remark 3.2, are precisely the representations described by Katsura in [Kat21, Definition 2.6]. For an ideal I of $C^*(G_T)$, the set $W = \{(x,z) : I \not\subseteq \ker(\pi_{(x,z)})\}$ is the complement of the set $Y = Y_I$ of [Kat21, Definition 5.1]. So Proposition 5.2 shows that the sandwich sets U and V of [BCS22, Lemma 3.3] are the sets $\{x : Y_x = \varnothing\}$ and $\{x : Y_x = \mathbb{T}\}$, respectively, that feature in Katsura's definition [Kat21, Definition 5.7].

10.5. **Rank-two graphs.** A significant goal for this project was to obtain a systematic approach to calculating the primitive-ideal spaces of the C^*-algebras of k-graphs, and in particular to explicitly calculate the primitive ideal spaces of C^*-algebras of 2-graphs. In this section, we show that 2-graph groupoids always admit harmonious families of bisections, and therefore that Proposition 8.2 and Theorem 9.5 can be used to compute the ideal spaces of arbitrary 2-graph algebras. This is the first general computation of the ideal lattices of 2-graph C^*-algebras.

Let Λ be a row-finite rank-two graph with no sources as in [KP00]. The infinite path space $X = \Lambda^\infty$ is second-countable and locally compact and Hausdorff, and the two translation operations $T = (T^{\varepsilon_1}, T^{\varepsilon_2})$ are local homeomorphisms [KP00, Remarks 2.5], so $T\colon \mathbb{N}^2 \curvearrowright X$ is an action of commuting local homeomorphisms. We will show that any point in the path space admits a harmonious family of bisections.

Fix $x \in X$. If $\mathcal{J}_x = \{x\}$ then $\mathcal{B} = \{B_x\}$ where $B_x = X$ is trivially a harmonious family of bisections at x. Suppose that $\mathcal{J}_x$ is a rank-one subgroup of $\mathbb{Z}^2$ with generator $h \in \mathbb{Z}^2$. Let $B_h \subseteq X \times \{h\} \times X$ be any compact open bisection containing (x, h, x). Then the association $nh \mapsto (B_h)^n$ defines a harmonious family of bisections at x.

Now suppose that $\mathcal{J}_x$ is a rank-two subgroup of $\mathbb{Z}^2$. By Lemma 6.10, there exists $h_1, h_2 \in c(\mathcal{J}_x) \cap \mathbb{N}^2$ that generate $c(\mathcal{J}_x)$. Fix $n \in \mathbb{N}^2$ such that

$$T^{h_1+n}x = T^n x = T^{h_2+n}x. \tag{10.2}$$

We first construct an explicit harmonious family of bisections in the situation where $n = 0$.

Consider the open sets $U_i \coloneqq Z(x_{[0,h_i]})$ in X and define open bisections

$$B_{h_i} \coloneqq Z(U_i, h_i, 0, X) \tag{10.3}$$

which contain (x, h_i, x), for $i = 1, 2$.

We record some technical properties.

Lemma 10.3. *The bisections defined in (10.3) commute, and they satisfy*

$$B_{h_2}B_{h_1}^{-1} \subseteq B_{h_1}^{-1}B_{h_2}, \tag{10.4}$$

$$B_{h_1}B_{h_2}^{-1} \subseteq B_{h_2}^{-1}B_{h_1}. \tag{10.5}$$

Proof. We first verify that the B_{h_i} commute. Take composable elements $(y_1, h_1, y_2) \in B_{h_1}$ and $(y_2, h_2, y_3) \in B_{h_2}$ so that

$$(y_1, h_1 + h_2, y_3) = (y_1, h_1, y_2)(y_2, h_2, y_3) \in B_{h_1}B_{h_2}.$$

This means that $T^{h_1}y_1 = y_2 \in U_1$ and $y_3 = T^{h_2}y_2 = T^{h_1+h_2}y_1$. In particular, $y_1 \in Z(x_{[0,h_1+h_2]})$ so by the factorisation property we have $T^{h_2}y_1 \in U_1$. It now follows that

$$(y_1, h_1 + h_2, y_3) = (y_1, h_2, T^{h_2}y_1)(T^{h_2}y_1, h_1, T^{h_1+h_2}y_1) \in B_{h_2}B_{h_1}$$

showing that $B_{h_1}B_{h_2} \subseteq B_{h_2}B_{h_1}$. A similar argument shows that $B_{h_2}B_{h_1} \subseteq B_{h_1}B_{h_2}$.

Next we verify Equation (10.4); a similar argument applies to Equation (10.5). Take composable elements $(y_1, h_1, h_2) \in B_{h_1}$ and $(y_2, -h_2, y_3) \in B_{h_2}$ so that

$$(y_1, h_1 - h_2, y_3) = (y_1, h_1, h_2)(y_2, -h_2, y_3) \in B_{h_1}B_{h_2}^{-1}.$$

In particular, $y_1 = x_{[0,h_1]}y_2$ and $y_3 = x_{[0,h_2]}y_2$. Consider $z := x_{[0,h_1+h_2]}y_2$ and observe that $(y_1, -h_2, z) \in B_{h_2}{}^{-1}$ and $(z, h_1, y_3) \in B_{h_1}$ so we have

$$(y_1, h_1 - h_2, y_3) = (y_1, -h_2, z)(z, h_1, y_3) \in B_{h_2}^{-1}B_{h_1},$$

giving $B_{h_1}B_{h_2}^{-1} \subseteq B_{h_2}^{-1}B_{h_1}$. □

Corollary 10.4. *Let B_{h_1} and B_{h_2} be as in (10.3). For $h \in \mathcal{J}_x$, write $h = m_1(h)h_1 + m_2(h)h_2$ with $m_1(h), m_2(h) \in \mathbb{Z}$. Define*

$$B_h := \begin{cases} B_{h_1}^{m_1(h)}B_{h_2}^{m_2(h)} & \text{if } m_1(h) \leq m_2 \\ B_{h_2}^{m_2(h)}B_{h_1}^{m_1(h)} & \text{if } m_2(h) < m_1, \end{cases} \tag{10.6}$$

with the convention that $B_{h_i}^0 = \Lambda^\infty$. Then $B_{(x,h,x)} := B_h$ defines a harmonious family of bisections at x.

Proof. This follows immediately from Lemma 10.3 and Lemma 6.8. □

Remark 10.5. The argument above breaks into two parts corresponding to the situation where $\mathcal{J}_x$ is singly generated and where $\mathcal{J}_x$ is generated as a group by its intersection with $c^{-1}(\mathbb{N}^k)$. The point of restricting to 2-graphs is that these two cases cover all possibilities thanks to Lemma 6.10. In general, the arguments presented above could be run verbatim to prove the following statement: Let Λ be a k-graph and suppose that for each $x \in \Lambda^\infty$, the group $\mathcal{J}_x$ either has rank 1, or is generated by its intersection with $c^{-1}(\mathbb{N}^k)$. Then G_Λ admits harmonious families of bisections. In particular, if $\operatorname{rank}(\mathcal{J}_x) \in \{0, 1, k\}$ for all x, then G_Λ admits harmonious families of bisections.

10.6. **A countable subshift.** The following example is an adaptation of [AL18, Example 7.9]. Ara and Lolk show that this example is not relatively strongly topologically free (cf. [AL18, Definition 7.4]) and hence not amenable to their techniques for studying ideal structure. This example does however admit harmonious families of bisections.

For $n \in \mathbb{N}$, define $x_n, y_n \colon \mathbb{N}^2 \to \{0, 1\}$ by

$$x_n(a, b) = \begin{cases} 1 & \text{if } b \leq n, \\ 0 & \text{if } b > n \end{cases} \quad \text{and} \quad y_n(a, b) = \begin{cases} 1 & \text{if } a \leq n, \\ 0 & \text{if } a > n \end{cases}$$

for all $(a, b) \in \mathbb{N}^2$. Let $\bar{0}, \bar{1} \colon \mathbb{N}^2 \to \{0, 1\}$ be constantly 0 and 1, respectively. Then $(x_n)_n$ and $(y_n)_n$ converge to $\bar{1}$ as $n \to \infty$, and $X = \{\bar{0}, \bar{1}, x_n, y_n : n \in \mathbb{N}\}$ is a compact and Hausdorff subspace of $\{0, 1\}^{\mathbb{N}^2}$. The coordinate translations on $\{0, 1\}^{\mathbb{N}^2}$ restrict to an action $T \colon \mathbb{N}^2 \curvearrowright X$ given by

$$T^{(k,l)}x(a, b) = x(a + k, b + l),$$

for all $(k, l) \in \mathbb{N}^2$, $(a, b) \in \mathbb{N}^2$, and $x \in X$. Note that $\bar{0}$ and $\bar{1}$ are fixed points of T, and that $T^{(1,0)}(x_{n+1}) = x_n$ and $T^{(1,0)}y_n = y_n$, and $T^{(0,1)}x_n = x_n$ and $T^{(0,1)}y_{n+1} = g_n$, for all $n \in \mathbb{N}$.

The only point in X that is not isolated is $\bar{1}$. The set

$$U_{(1,0)} = \{\bar{0}, \bar{1}, x_{n+1}, y_n : n \in \mathbb{N}\}$$

is an open neighborhood of $\bar{1}$ in X and $T^{(1,0)}$ restricts to a homeomorphism on $U_{(1,0)}$; similarly,

$$U_{(0,1)} = \{\bar{0}, \bar{1}, x_n, y_{n+1} : n \in \mathbb{N}\}$$

is an open neighborhood of $\bar{1}$ in X, and $T^{(0,1)}$ restricts to a homeomorphism on $U_{(0,1)}$. This shows that $T\colon \mathbb{N}^2 \curvearrowright X$ is an action by local homeomorphisms.

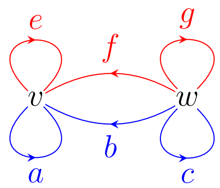
We will now verify that T admits harmonious families of bisections. Every isolated point admits a harmonious family of bisections so we only need to construct one for $\bar{1}$. The sets

$$B_{(1,0)} = Z(U_{(1,0)}, (1,0), (0,0), X) \quad \text{and} \quad B_{(0,1)} = Z(U_{(0,1)}, (0,1), (0,0), X)$$

are homogeneous compact open bisections that contain $(\bar{1}, (1,0), \bar{1})$ and $(\bar{1}, (0,1), \bar{1})$, respectively. In fact, the bisections satisfy the conditions of Lemma 6.8, so $\bar{1}$ admits a harmonious family of bisections.

Remark 10.6. While we have proved that the preceding example admits harmonious families of bisections by constructing them by hand, we could also establish this by proving that it is conjugate to the shift space of a 2-graph as follows, and then using the general results of Section 10.5 above.

Consider the 2-coloured graph



By [KP00, Section 6], there is a unique 2-graph Λ with this skeleton and satisfying the factorisation rules

$$ae = ea, \quad af = fc, \quad bg = eb, \quad \text{and} \quad cg = gc.$$

Writing $T_\Lambda\colon \mathbb{N}^2 \curvearrowright \Lambda^\infty$ for the shift action, it is routine to check that the formulas

$$\theta(x_v) = \bar{1}, \quad \theta(a^n b x_w) = y_{n+1}, \quad \theta(e^n f x_w) = x_{n+1}, \quad \text{and} \quad \theta(x_w) = \bar{0},$$

define a conjugacy θ from $(\Lambda^\infty, T_\Lambda)$ to (X, T).

11. Ideals of higher-rank graphs

In this section, we present a catalogue of all ideals in the C^*-algebra of a higher-rank graph Λ whose groupoid admits harmonious families of bisections, in terms of open subsets D of $\Lambda^0 \times \mathbb{T}^k$. This extends the catalogue of ideals that are gauge-equivariant. Although the condition that characterises which subsets $D \subseteq \Lambda^0 \times \mathbb{T}^k$ correspond to ideals is quite technical, in practice the description is usable, as we show by example at the end of the section.

Our strategy is as follows. We begin by describing the collection $\mathcal{A}_\Lambda$ of subsets of $\Lambda^\infty \times \mathbb{T}^k$ that correspond to ideals of $C^*(\Lambda)$ via Corollary 7.5. We then describe maps δ and α, the first of which carries subsets of $\Lambda^\infty \times \mathbb{T}^k$ to subsets of $\Lambda^0 \times \mathbb{T}^k$ and the second of which takes subsets of $\Lambda^0 \times \mathbb{T}^k$ to subsets of $\Lambda^\infty \times \mathbb{T}^k$. We show that $\alpha \circ \delta$ is the identity map on $\mathcal{A}_\Lambda$, and deduce that the ideals of $C^*(\Lambda)$ are indexed by the elements of $\delta(\mathcal{A}_\Lambda)$. We then identify exactly which subsets of $\Lambda^0 \times \mathbb{T}^k$ belong to $\delta(\mathcal{A}_\Lambda)$.

Recall from the beginning of Section 4 that if $\mathcal{B} = (B_\gamma)_{\gamma \in \mathcal{J}_x}$ is a harmonious family of bisections based at $x \in \Lambda^\infty$, then $\mathcal{B}^{\mathrm{ess}}$ denotes its intersection with the essential isotropy of G_Λ, and that $(\mathcal{B}^{\mathrm{ess}})^\perp$ is the group bundle $\{(y, z) \in B_x \times \mathbb{T}^k : z^{c(\gamma)} = 1 \text{ for all } \gamma \in \mathcal{B}^{\mathrm{ess}}_y\}$, which is a sub-bundle of the group bundle $B_x \times \mathbb{T}^k$.

Notation 11.1. Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. We write $\mathcal{A}_\Lambda$ for the collection of all subsets $A \subseteq \Lambda^\infty \times \mathbb{T}^k$ such that

(A1) if $(x, z) \in A$, and $(x', z') \in X \times \mathbb{T}^k$ satisfy $\overline{[x]} = \overline{[x']}$, and $(\overline{z}z')^{c(\gamma)} = 1$ for all $\gamma \in \mathcal{I}^{\mathrm{ess}}_x$, then $(x', z') \in A$; and

(A2) for each $(x, z) \in A$, then there exist a harmonious family of bisections $\mathcal{B} = (B_g)_{g \in \mathcal{J}_x}$ based at x and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $(B_x \times V) \cdot (\mathcal{B}^{\mathrm{ess}})^\perp \subseteq A$.

It follows from Corollary 7.5 applied to the k-graph groupoid G_Λ associated to the shift action $T \colon \mathbb{N}^k \curvearrowright \Lambda^\infty$ of [KP00] that the map π of Notation 4.1 is a bijection from $\mathcal{A}_\Lambda$ to the set of open subsets of $\mathrm{Prim}(C^*(G_\Lambda))$.

For the following, recall that if $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$, then $x(n)$ denotes the vertex $x(n) = r(T^n(x)) \in \Lambda^0$

Lemma 11.2. *Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. Suppose that $A \in \mathcal{A}_\Lambda$. For $x \in \Lambda^\infty$, $n \in \mathbb{N}^k$ and $z \in \mathbb{T}^k$ we have $(x, z) \in A$ if and only if $(T^n(x), z) \in A$. If $(x, z) \in A$ then there exist $n \in \mathbb{N}^k$ and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $Z(x(n)) \times V \subseteq A$.*

Proof. For $x \in \Lambda^\infty$, we have $[x] = [T^n(x)]$. Hence (A1) implies that $(x, z) \in A$ if and only if $(T^n(x), z) \in A$.

Suppose that $(x, z) \in A$. Since π is continuous by Corollary 4.4, the set $A \subseteq X \times \mathbb{T}^k$ is open. So there exist $n \in \mathbb{N}^k$ and an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $Z(x(0, n)) \times V \subseteq A$. Since $[x(0, n)y] = [y]$ for all $y \in Z(v)$, it follows from (A2) that $Z(x(n)) \times V \subseteq A$. □

Lemma 11.3. *Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. Given a subset $A \subseteq \Lambda^\infty \times \mathbb{T}^k$, define*

$$\delta(A) := \{(v, z) \in \Lambda^0 \times \mathbb{T}^k : Z(v) \times \{z\} \subseteq A\}.$$

Given a subset $D \subseteq \Lambda^0 \times \mathbb{T}^k$, define

$$\alpha(D) := \{(x, z) \in \Lambda^\infty \times \mathbb{T}^k : (x(n), z) \in D \textit{ for some } n \in \mathbb{N}^k\}.$$

Then $\alpha(\delta(A)) = A$ for all $A \in \mathcal{A}_\Lambda$.

Proof. Fix $(x, z) \in A$. By Lemma 11.2, there exist $n \in \mathbb{N}^k$ and an open neighbourhood V of z such that $Z(x(n)) \times V \subseteq A$. We then have $(x(n), z) \in x(n) \times V \subseteq \delta(A)$. By definition of α, we then have $(y, z) \in \alpha(\delta(A))$ whenever $y \in \Lambda^\infty$ satisfies $x(n)\Lambda y(m) \neq \varnothing$ for some $m \in \mathbb{N}^k$. Taking $m = n$ and $y = x$ we see that $(x, z) \in \alpha(\delta(A))$. That is $A \subseteq \alpha(\delta(A))$.

For the reverse containment, fix $(x, z) \in \alpha(\delta(A))$. Then there exists $n \in \mathbb{N}^k$ such that $(x(n), z) \in \delta(A)$. Hence $Z(x(n)) \times \{z\} \subseteq A$. In particular, $(\sigma^n(x), z) \in A$. Since $[\sigma^n(x)] = [x]$ and since A satisfies (A1), we deduce that $(x, z) \in A$. Hence $\alpha(\delta(A)) \subseteq A$. □

To describe the lattice of ideals of $C^*(\Lambda)$, it therefore suffices to describe the range of the map $\delta \colon \mathcal{A}_\Lambda \to \mathcal{P}(\Lambda^0 \times \mathbb{T}^k)$. We start with some elementary necessary conditions for membership of $\delta(\mathcal{A}_\Lambda)$.

Lemma 11.4. *Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. If $A \in \mathcal{A}_\Lambda$, then*

(1) *for $\lambda \in \Lambda$, if $(r(\lambda), z) \in \delta(A)$, then $(s(\lambda), z) \in \delta(A)$;*

(2) *for $n \in \mathbb{N}^k$, $v \in \Lambda^0$ and $z \in \mathbb{T}^k$, if $(s(\lambda), z) \in \delta(A)$ for all $\lambda \in v\Lambda^n$, then $(v, z) \in \delta(A)$; and*

(3) *for each $v \in \Lambda^0$, the set $\{z \in \mathbb{T}^k : (v, z) \in \delta(A)\}$ is open.*

Proof. (1) Suppose that $(r(\lambda), z) \in \delta(A)$. Then $Z(\lambda) \times \{z\} \subseteq Z(r(\lambda)) \times \{z\} \subseteq A$. Since $\overline{[\sigma^{d(\lambda)}(x)]} = \overline{[x]}$ for all $x \in \Lambda^\infty$, and since $Z(s(\lambda)) = \sigma^{d(\lambda)}(Z(\lambda))$, condition (A1) implies that $Z(s(\lambda)) \times \{z\} \subseteq A$, and hence $(s(\lambda), z) \in \delta(A)$.

(2) Fix $n \in \mathbb{N}^k$, $v \in \Lambda^0$ and $z \in \mathbb{T}^k$ and suppose that $(s(\lambda), z) \in \delta(A)$ for all $\lambda \in v\Lambda^n$. Then $Z(s(\lambda)) \times \{z\} \subseteq A$ for all $\lambda \in v\Lambda^n$. For each $\lambda \in v\Lambda^n$ and $x \in Z(s(\lambda))$, we have $\overline{[\lambda x]} = \overline{[x]}$. Hence condition (A1) implies that $\bigcup_{\lambda \in v\Lambda^n}(Z(\lambda) \times \{z\}) = \{(\lambda x, z) : \lambda \in \Lambda^n, x \in Z(s(\lambda))\} \subseteq A$. Since $Z(v) = \bigcup_{\lambda \in v\Lambda^n} Z(\lambda)$, it follows that $Z(v) \times \{z\} \subseteq A$ and hence $(v, z) \in \delta(A)$.

(3) Fix $v \in \Lambda^0$ and $z \in \mathbb{T}^k$ such that $(v, z) \in \delta(A)$. We must show that there exists an open neighbourhood $V \subseteq \mathbb{T}^k$ of z such that $\{v\} \times V \subseteq \delta(A)$. Since $(v, z) \in \delta(A)$, we have $Z(v) \times \{z\} \subseteq A$. For each $x \in Z(v)$, Lemma 11.2 gives $n_x \in \mathbb{N}^k$ and an open neighbourhood $V_x \subseteq \mathbb{T}^k$ of z such that $Z(x(0, n_x)) \times V_x \subseteq A$. The sets $Z(x(0, n_x))$ constitute an open cover of $Z(v)$, which is compact. So there is a finite $F \subseteq Z(v)$ such that $\bigcup_{x \in F} Z(x(0, n_x)) = Z(v)$. Let $V = \bigcap_{x \in F} V_x$. This is an open neighbourhood around z, and $Z(v) \times V = \bigcup_{x \in F}(Z(x(0, n_x)) \times V) \subseteq A$. Thus $\{v\} \times V \subseteq \delta(A)$. □

Lemma 11.5. *Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. Let D be subset of $\Lambda^0 \times \mathbb{T}^k$. Then $D \subseteq \delta(\alpha(D))$. If D satisfies conditions (1) and (2) of Lemma 11.4, then $D = \delta(\alpha(D))$.*

Proof. For the first statement, fix $(v, z) \in D$. For each $x \in Z(v)$, we have $(x(0), z) \in D$ and so by definition of $\alpha(D)$ we have $Z(v) \times \{z\} \subseteq \alpha(D)$. So by definition of δ, we have $(v, z) \in \delta(\alpha(D))$.

For the second statement, suppose that D satisfies conditions (1) and (2) of Lemma 11.4. We must show that $\delta(\alpha(D)) \subseteq D$. Fix $(v, z) \in \delta(\alpha(D))$. Then $Z(v) \times \{z\} \subseteq \alpha(D)$, and so for each $x \in Z(v)$, there exists $n_x \in \mathbb{N}^k$ such that $(x(n_x), z) \in D$. The sets $Z(x(0, n_x))_{x \in Z(v)}$ are an open cover of $Z(v)$ in Λ^∞, and since $Z(v)$ is compact, it follows that there is a finite subset $F \subseteq Z(v)$ such that $Z(v) = \bigcup_{x \in F} Z(x(0, n_x))$. Let $n = \bigvee_{x \in F} n_x$. For each $x \in F$ we have $Z(x(0, n_x)) = \bigcup_{\tau \in x(n_x)\Lambda^{n-n_x}} Z(x(0, n_x)\tau)$. Since each $x(0, n_x)\tau$ belongs to $v\Lambda^n$ and since the sets $Z(\eta), \eta \in \Lambda^n$ are mutually disjoint nonempty subsets of $Z(v)$, we deduce that $\{x(0, n_x)\tau : x \in F, \tau \in x(n_x)\Lambda^{n-n_x}\} = v\Lambda^n$. For $x \in F$ and $\tau \in x(n_x)\Lambda^{n-n_x}$, the fact that D satisfies Lemma 11.4(1) implies that $(s(\tau), z) \in D$, so $\{(s(\eta), z) : \eta \in v\Lambda^n\} \subseteq D$. Since D satisfies Lemma 11.4(2), we obtain $(v, z) \in D$. □

We now characterise the sets $D \subseteq \Lambda^0 \times \mathbb{T}^k$ that have the form $\delta(A)$ for some $A \in \mathcal{A}_\Lambda$. The condition is technical, but we will demonstrate by example that it can be checked in practice.

Throughout what follows, given a harmonious family of bisections $\mathcal{B} = (B_\gamma)_{\gamma \in \mathcal{J}_x}$ based at $x \in \Lambda^\infty$, and given $y \in B_x$, we write $(\mathcal{B}_y^{\mathrm{ess}})^\perp$ for the subgroup

$$(\mathcal{B}_y^{\mathrm{ess}})^\perp := \{z \in \mathbb{T}^k : z^{c(\gamma)} = 1 \text{ for all } \gamma \in \mathcal{B}_y^{\mathrm{ess}}\}.$$

Thus, the fibre $(\mathcal{B}^{\mathrm{ess}})_y^\perp$ of $(\mathcal{B}^{\mathrm{ess}})^\perp$ over the point y is $\{y\} \times (\mathcal{B}_y^{\mathrm{ess}})^\perp$.

Proposition 11.6. *Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. A set $D \subseteq \Lambda^0 \times \mathbb{T}^k$ has the form $\delta(A)$ for some $A \subseteq \Lambda^\infty \times \mathbb{T}^k$ satisfying (A1) and (A2) if and only if D satisfies (1) and (2) of Lemma 11.4 and for every $(v, z) \in D$ and every $x \in Z(v)$, there exist a bisection family $\mathcal{B} = (B_\gamma)_{\gamma \in \mathcal{J}_x}$ based at x and an open neighbourhood V of z in $\mathbb{T}^k$ such that for every $y \in B_x$ there exists $n \in \mathbb{N}^k$ such that $\{y(n)\} \times V(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq D$.*

In particular, α is a bijection from the collection of such sets D to $\mathcal{A}_\Lambda$.

Proof. If $D = \delta(A)$, then D satisfies (1)–(2) by Lemma 11.4. Fix $(v, z) \in D$. Since $D = \delta(A)$ we first observe that $Z(v) \times \{z\} \subseteq A$.

Fix $x \in Z(v)$. By condition (A2), there exist a harmonious family of bisections $\mathcal{B} = (B_\gamma)_{\gamma\in\mathcal{J}_x}$ based at x and an open neighbourhood $V_0 \subseteq \mathbb{T}^k$ of z such that $(B_x \times V_0)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp \subseteq A$. Since Λ^∞ is totally disconnected, we can fix a compact open neighbourhood C of x contained in B_x and apply Lemma 6.5 to obtain a harmonious family of bisections $\{CB_\gamma C : \gamma \in \mathcal{J}_x\}$ with CB_xC compact open. So we can assume without loss of generality that B_x is compact open. Let $K \subseteq V_0$ be a compact neighbourhood of z, and let V be the interior of K. We claim that $\mathcal{B}$ and V have the desired properties.

To see this, fix $y \in B_x$. We must find $n \in \mathbb{N}^k$ such that $\{y(n)\} \times V(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq D$. Since $y \in B_x$, we have

$$\{y\} \times K(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq \{y\} \times V_0(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq A.$$

By Lemma 11.2, for each $w \in K(\mathcal{B}_y^{\mathrm{ess}})^\perp$, there exists $n_w \in \mathbb{N}^k$ and a neighbourhood U_w of w such that $Z(y(n_w)) \times U_w \subseteq A$. Since $K(\mathcal{B}_y^{\mathrm{ess}})^\perp$ is compact, the open cover $\{U_w : w \in K(\mathcal{B}_y^{\mathrm{ess}})^\perp\}$ admits a finite subcover $\{U_w : w \in F\}$. Since each $Z(y(n_w)) \times U_w \subseteq A$, each $\{y(n_w)\} \times U_w \subseteq D$. Let $n = \bigvee_{w\in F} n_w$. Since D satisfies condition (1) of Lemma 11.4, we have $\{y(n)\} \times U_w \subseteq D$ for all $w \in F$. Hence,

$$\begin{aligned}\{y(n)\} \times V(\mathcal{B}_y^{\mathrm{ess}})^\perp &\subseteq \{y(n)\} \times K(\mathcal{B}_y^{\mathrm{ess}})^\perp\\ &\subseteq \{y(n)\} \times \big(\textstyle\bigcup_{w\in F} U_w\big) = \textstyle\bigcup_{w\in F}(\{y(n)\} \times U_w) \subseteq D.\end{aligned}$$

Now suppose that D satisfies (1)–(2) of Lemma 11.4 and that for every $(v, z) \in D$ and every $x \in Z(v)$, there exist a harmonious family of bisections $\mathcal{B} = (B_\gamma)_{\gamma\in\mathcal{J}_x}$ based at x and an open neighbourhood V of z in $\mathbb{T}^k$ such that for every $y \in B_x$ there exists $n \in \mathbb{N}^k$ such that $\{y(n)\} \times V(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq D$.

By Lemma 11.5, we have $D = \delta(\alpha(D))$, so we just need to show that $\alpha(D)$ belongs to $\mathcal{A}_\Lambda$. For condition (A1), suppose that $(x, z) \in \alpha(D)$ and that $\overline{[x']} = \overline{[x]}$ and that z and z' determine the same character of $c(\mathcal{I}_{x'}^{\mathrm{ess}})$. Since $(x, z) \in \alpha(D)$ there exists $n \in \mathbb{N}^k$ such that $(x(n), z) \in D$. We have $T^n(x) \in [x] \subseteq \overline{[x]} = \overline{[x']}$. Hence $\overline{[x']} \cap Z(x(n)) \neq \varnothing$. Since $Z(x(n))$ is open, it follows that $[x'] \cap Z(x(n)) \neq \varnothing$. Hence there exists $m \in \mathbb{N}^k$ such that $x(n)\Lambda x'(m) \neq \varnothing$. Since $(x(n), z) \in D$, Lemma 11.4(1) ensures that $(x'(m), z) \in D$. Hence $(x', z) \in \alpha(D)$. By hypothesis on D, there exist a harmonious family of bisections $\mathcal{B}$ based at x', a neighbourhood V of z and $p \in \mathbb{N}^k$ such that $\{x'(p)\} \times V(\mathcal{B}_{x'}^{\mathrm{ess}})^\perp \subseteq D$. Since D satisfies condition (1) of Lemma 11.4, we then have $\{x'(p \vee m)\} \times V(\mathcal{B}_{x'}^{\mathrm{ess}})^\perp \subseteq D$, so by replacing p with $p \vee m$ we can assume that $p \geq m$. By definition we have $\mathcal{B}_{x'}^{\mathrm{ess}} = \bigcup \mathcal{B} \cap \mathcal{I}_{x'}^{\mathrm{ess}} \subseteq \mathcal{I}_{x'}^{\mathrm{ess}}$, so $(\mathcal{B}_{x'}^{\mathrm{ess}})^\perp \supseteq (\mathcal{I}_{x'}^{\mathrm{ess}})^\perp$. Since z and z' determine the same characters of $c(\mathcal{I}_{x'}^{\mathrm{ess}})$, we therefore have $z' \in zc(\mathcal{I}_{x'}^{\mathrm{ess}})^\perp \subseteq V(\mathcal{B}_{x'}^{\mathrm{ess}})^\perp$, and hence $(x'(p), z') \in D$. Therefore, $(x', z') \in \alpha(D)$.

For (A2), fix $(x, z) \in \alpha(D)$. By hypothesis, there exist a harmonious family of bisections $\mathcal{B}$ based at x, and a neighbourhood V of z such that for every $y \in B_x$ there exists $n_y \in \mathbb{N}^k$ such that $\{y(n_y)\} \times V(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq D$. In particular, for $y \in B_x$, we have

$$\big((B_x \times V)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp\big)_y = \{y\} \times V(\mathcal{B}_y^{\mathrm{ess}})^\perp \subseteq \alpha(D).$$

That is, $(B_x \times V)\cdot(\mathcal{B}^{\mathrm{ess}})^\perp \subseteq \alpha(D)$.

The final statement follows from Lemmas 11.3 and 11.5. □

The following complete description of the ideal structure of the higher-rank graph C^*-algebra whose groupoid G_Λ admits harmonious families of bisections now follows directly from Corollary 7.5 and Proposition 11.6. We identify $C^*(\Lambda)$ with the groupoid C^*-algebra $C^*(G_\Lambda)$.

Corollary 11.7. *Let Λ be a row-finite higher-rank graph with no sources whose groupoid G_Λ admits harmonious families of bisections. Let $\mathcal{D}_\Lambda$ denote the collection of subsets D of $\Lambda^0 \times \mathbb{T}^k$ such that*

(D1) *for $\lambda \in \Lambda$, if $(r(\lambda), z) \in \delta(A)$, then $(s(\lambda), z) \in \delta(A)$;*
(D2) *for $n \in \mathbb{N}^k$, $v \in \Lambda^0$ and $z \in \mathbb{T}^k$, if $s(v\Lambda^n) \times \{z\} \subseteq \delta(A)$ then $(v, z) \in \delta(A)$; and*
(D3) *for every $(v, z) \in D$ and every $x \in Z(v)$, there exist a bisection family $\mathcal{B} = (B_\gamma)_{\gamma \in \mathcal{J}_x}$ and an open neighbourhood V of z in $\mathbb{T}^k$ such that for every $y \in B_x$ there exists $n \in \mathbb{N}^k$ such that $\{y(n)\} \times V(\mathcal{B}^{\mathrm{ess}}_y)^\perp \subseteq D$.*

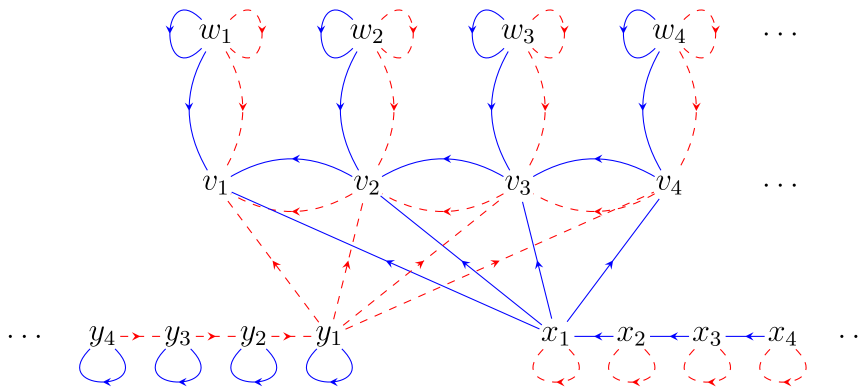
Let $\alpha\colon \mathcal{D}_\Lambda \to \mathcal{A}_\Lambda$ be the restriction of the map of Lemma 11.3, and for $(x, z) \in \Lambda^\infty \times \mathbb{T}^k$, let $\pi_{(x,z)}$ be the representation $C^*(G_\Lambda)$ of Remark 3.2. Then

$$D \mapsto \bigcap_{(x,z)\in(\Lambda^\infty\times\mathbb{T}^k)\setminus\alpha(D)} \ker(\pi_{(x,z)})$$

is a bijection between $\mathcal{D}_\Lambda$ and the collection of ideals of $C^*(\Lambda)$.

Remark 11.8. The map from $\mathcal{D}_\Lambda$ to the set of ideals of $C^*(\Lambda)$ described above generalises the well-known map [RSY03, Theorem 5.2] from saturated hereditary subsets of Λ^0 to gauge-invariant ideals of $C^*(\Lambda)$. Specifically, if $D \in \mathcal{D}_\Lambda$ and I is the corresponding ideal of $C^*(\Lambda)$, then I is gauge-invariant if and only if $D = H \times \mathbb{T}^k$ for some subset $H \subseteq \Lambda^0$, and then (D1) and (D2) say precisely that H is saturated and hereditary; specifically, it is the saturated hereditary set $\{v \in \Lambda^0 : p_v \in I\}$. Observe that if the complement of every saturated hereditary subgraph of Λ satisfies the aperiodicity condition (B) of [RSY03], then $\mathcal{I}^{\mathrm{ess}}(G_\Lambda)$ is just the unit space, so (D3) says that $\mathcal{D}_\Lambda$ consists of sets of the form $H \times \mathbb{T}^k$ for $H \subseteq \Lambda^0$, and (D1) and (D2) say that these sets H are saturated and hereditary; so we recover [RSY03, Theorem 5.3].

Example 11.9. We illustrate our results by applying them to the 2-graph Λ with the following skeleton.



This example appeared in [aHNS21] as an illustration of a 2-graph whose C^*-algebra is stably finite. We have chosen it as an illustration because it has a reasonably complex essential-isotropy structure and also a reasonably complex ideal structure that cannot be described using existing results (for example, it is not a pullback of a 1-graph, nor a product of 1-graphs).

We describe $\mathcal{D}_\Lambda$. It is convenient here to regard an element of $\mathcal{D}_\Lambda$ here as a function $D\colon \Lambda^0 \to \mathsf{Open}(\mathbb{T}^2)$ from the vertices of Λ to the open subsets of $\mathbb{T}^2$; a subset $D \subseteq \Lambda^0 \times \mathbb{T}^2$ is identified with the function $v \mapsto D(v) := \{z \in \mathbb{T}^2 : (v, z) \in D\}$.

For $D \in \mathcal{D}_\Lambda$, condition (D1) implies that:

- $D(v_i) \subseteq D(v_{i+1})$ for all i;
- $D(v_i) \subseteq D(w_i)$ for all i;
- $D(x_i) \subseteq D(x_{i+1})$ for all i;
- $D(y_i) \subseteq D(y_{i+1})$ for all i; and
- $\bigcup_j D(v_j) \subseteq D(x_1) \cap D(y_1)$.

The consequences of condition (D2) are as follows.

- Applied with $v = x_i$ and $n = (1, 0)$, condition (D2) forces $D(x_{i+1}) \subseteq D(x_i)$ for all i; combining this with the consequenes of (D1) above implies that D is constant on $\{x_i : i \in \mathbb{N}\}$.
- Applied with $v = y_i$ and $n = (0, 1)$, condition (D2) forces $D(y_{i+1}) \subseteq D(y_i)$ for all i; combining this with the consequenes of (D1) above implies that D is constant on $\{y_i : i \in \mathbb{N}\}$.
- Applied with $v = v_i$ and $n = (j, 0)$, condition (D2) implies that $D(v_{i+j}) \cap \big(\bigcap_{0 \le l < j} D(w_{i+l})\big) \cap D(x_1) \subseteq D(v_i)$ for all i; since each $D(w_{i+l})$ contains $D(v_{i+l})$, and since the $D(v_i)$ are increasing and contained in $D(x_1)$, this reduces to $D(v_{i+1}) \cap D(w_i) \subseteq D(v_i)$.

To understand the consequences of (D3), we must first understand the harmonious families of bisections in this 2-graph groupoid. To do this, first note that each $x_i\Lambda^\infty$, each $y_i\Lambda^\infty$ and each $w_i\Lambda^\infty$ is a singleton; we will write $\zeta(u)$ for the unique element of $u\Lambda^\infty$ for each $u \in \{x_i, y_i, z_i : i \in \mathbb{N}\}$. Since each $u\Lambda^\infty$ is clopen in Λ^∞, each $\zeta(u)$ is an isolated point. For a given vertex u and infinite path $\eta \in u\Lambda^\infty$, if $\mathcal{B}$ and $\mathcal{B}'$ are harmonious bisection families based at η such that $B_\alpha \subseteq B'_\alpha$ for all α, then the collection of functions D that satisfy (D3) with respect to $\mathcal{B}$ is larger than the collection that satisfy (D3) with respect to $\mathcal{B}'$. Combining this with Lemma 6.5 applied with $C = Z(u)$ we see that it suffices to consider harmonious families of bisections $\mathcal{B}^u$ based at each $\zeta(u)$ such that $B^u_{\zeta(u)} \subseteq u\Lambda^\infty = \{\zeta(u)\}$. For each u, there is only one such harmonious family of bisections, namely $\mathcal{B}^u = \{B^u_\alpha : \alpha \in \mathcal{J}_{\zeta(u)}\}$ given by $B^u_\alpha := \{\alpha\}$ for each $\alpha \in \mathcal{J}_{\zeta(u)} = \{(\zeta(u), p - q, \zeta(u)) : T^p(\zeta(u)) = T^q(\zeta(u))\}$. In particular,

$$\begin{aligned}
\mathcal{B}^{x_i} &= \{(\zeta(x_i), (0, l), \zeta(x_i)) : l \in \mathbb{Z}\},\\
\mathcal{B}^{y_i} &= \{(\zeta(x_i), (l, 0), \zeta(x_i)) : l \in \mathbb{Z}\},\\
\mathcal{B}^{w_i} &= \{(\zeta(x_i), (l_1, l_2), \zeta(x_i)) : (l_1, l_2) \in \mathbb{Z}^2\}.
\end{aligned}$$

So (D3) implies that

- each $D(x_i)$ is invariant under multiplication by $\mathbb{T} \times \{1\} \subseteq \mathbb{T}^2$, and
- each $D(y_i)$ is invariant under multiplication by $\{1\} \times \mathbb{T} \subseteq \mathbb{T}^2$.

Since each $((\mathcal{B}^{w_i})^{\mathrm{ess}})^\perp_{\zeta(w_i)} = \{1\}$, condition (D3) imposes no condition on the $D(w_i)$.

It remains to analyse the $D(v_i)$. For this, let $S := \{v_i : i \in \mathbb{N}\}$. Then ΛS is a sub-k-graph of Λ. There are infinitely many infinite paths in each $v_i\Lambda^\infty$, but all but one of these has the form $\mu\zeta(u)$ for some finite path u and some $u \in \{x_i, y_i, z_i : i \in \mathbb{N}\}$. The one remaining infinite path is the unique infinite path $\zeta(v_i)$ in $v_i\Lambda S$. Each $\overline{[\zeta(v_i)]} = \{\zeta(v_j) : j \in \mathbb{N}\} = (\Lambda S)^\infty$. We have $T^n(\zeta(v_i)) = T^m(\zeta(v_i))$ if and only if $m - n \in \mathbb{Z}(1, -1) \subseteq \mathbb{Z}^2$. So (D3) implies that there exists $l \in \mathbb{N}$ such that $D(v_j)$ is closed under multiplication by $\mathbb{Z}(1, -1)^\perp = \{(w, w) : w \in \mathbb{T}\} \subseteq \mathbb{T}^k$. Since we already established that $D(v_j) \subseteq D(x_j) = D(x_1)$ and since $D(x_1)$ is also closed under multiplication by $\mathbb{T} \times \{1\}$, we deduce that if any $D(v_j) \neq \varnothing$ then $D(x_i) = \mathbb{T}^2$ for all i. Similar considerations show that if any $D(v_j) \neq \varnothing$ then each $D(y_i) = \mathbb{T}^2$.

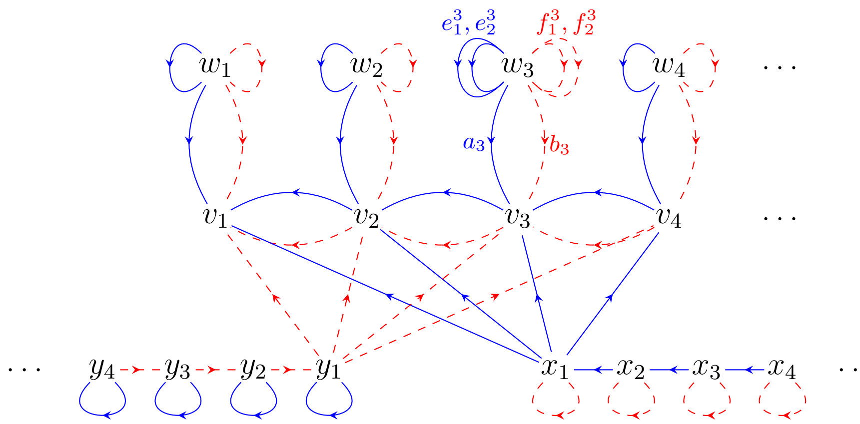
We are now in a position to describe all of $\mathcal{D}_\Lambda$. A function $D\colon \Lambda^0 \to \mathsf{Open}(\mathbb{T}^2)$ belongs to $\mathcal{D}_\Lambda$ if and only if either:

- $D(v_i) = \varnothing$ for all i; D is constant on $\{x_i : i \in \mathbb{N}\}$ and $D(x_1)$ is invariant under $\mathbb{T} \times \{1\}$; and D is constant on $\{y_i : i \in \mathbb{N}\}$ and $D(y_1)$ is invariant under $\{1\} \times \mathbb{T}$; or
- $D(x_i) = D(y_i) = \mathbb{T}^2$ for all i; each $D(v_i)$ is invariant under $\{(w, \overline{w}) : w \in \mathbb{T}\}$ and each $D(v_i) \subseteq D(v_{i+1})$; each $D(v_i) \subseteq D(w_i)$; and each $D(w_i) \cap D(v_{i+1}) \subseteq D(v_i)$.

Example 11.10. To illustrate how aperiodicity affects the ideal structure, suppose that we modify Example 11.9 as follows. Fix disjoint subsets $S_0, S_1 \subseteq \mathbb{N}$. For each $n \in S := S_0 \cup S_1$, add an additional red loop and an additional blue loop at w_n (in the picture below, $w_3 \in S$ while $w_1, w_2, w_4 \notin S$).

For $n \in S$ there are now multiple red-blue loops at w_n and multiple red-blue paths from w_n to v_n, so we must specify factorisation rules (see [HRSW13, Theorems 4.4 and 4.5]). For each $n \in S$, denote the blue loops at w_n by e_1^n, e_2^n, the red loops by f_1^n, f_2^n, the blue edge from w_n to v_n by a_n and the red edge from w_n to v_n by b_n. For all $n \in S$, impose the factorisation rules $a_n f_i^n = b_n e_i^n$ on red-blue paths from w_n to v_n. For blue-red loops at w_n, impose factorisation rules depending on whether $n \in S_0$ or $n \in S_1$:

$$e_i^n f_j^n = \begin{cases} f_j^n e_i^n & \text{if } n \in S_0. \\ f_i^n e_j^n & \text{if } n \in S_1 \end{cases}$$



Let Γ be the 2-graph with this skeleton and these factorisation rules. For $n \in S_0$, the subgraph $w_n \Gamma w_n$ is a cartesian product of two copies of the 1-graph B_2 that has one vertex and two loops. So the reduction of G_Γ to $Z(w_n)$ for $n \in S_0$ is the cartesian product of two copies of the standard groupoid $\mathcal{H}_2$ for the Cuntz algebra $\mathcal{O}_2$, which has trivial essential isotropy. Thus, for $n \in S_0$, and $x \in Z(w_n)$, we have $\mathcal{J}_x = \{0\}$.

For $n \in S_1$, the subgraph $w_n \Gamma w_n$ is the pullback of the same 1-graph B_2 by the functor $m \mapsto m_1 + m_2$ from $\mathbb{N}^2$ to $\mathbb{N}$. So by [KP00, Proposition 2.10], the reduction of G_Γ to $Z(w_n)$ for $n \in S_1$ is isomorphic to the cartesian product of $\mathcal{H}_2$ with the group $\mathbb{Z}$. For $n \in S_1$, and $x \in Z(w_n)$, we have $\mathcal{J}_x = \{(x, (m, -m), x) : m \in \mathbb{Z}\}$.

The set $\mathcal{D}_\Gamma$ differs from $\mathcal{D}_\Lambda$ in Example 11.9 only in that if $D \in \mathcal{D}_\Gamma$, then for $n \in S_0$, we have $D(w_n) \in \{\varnothing, \mathbb{T}^2\}$, and for $n \in S_1$, the set $D(w_n)$ is invariant for multiplication by $\mathbb{Z}(1, -1)^\perp = \{(w, w) : w \in \mathbb{T}\} \subseteq \mathbb{T}^2$. Aside from that, the constraints on D are as in Example 11.9. Writing $S_0^\circ = \{n \in S_0 : D(w_n) = \varnothing\}$ and $S_0^\bullet = \{n \in S_0 : D(w_n) = \mathbb{T}^2\}$, if $\bigcup D(v_i) \neq \varnothing$ so that $D(x_1) = D(y_1) = \mathbb{T}^2$, we have $D(v_n) = D(v_{n+1})$ for $n \in S_0^\bullet$.

References

[A-D97] C. Anantharaman-Delaroche, *Purely infinite C^*-algebras arising from dynamical systems*, Bull. Soc. Math. France **125** (1997), 199–225.

[AL18] P. Ara and M. Lolk, *Convex subshifts, separated Bratteli diagrams, and ideal structure of tame separated graph algebras*, Adv. Math. **328** (2018), 367–435.

[BHRS02] T. Bates, J.H. Hong, I. Raeburn and W. Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), 1159–1176.

[BL20] C. Bönicke and K. Li, *Ideal structure and pure infiniteness of ample groupoid C^*-algebras*, Ergodic Theory Dynam. Systems **40** (2020), 34–63.

[BS76] J.R. Boone and F. Siwiec, *Sequentially quotient mappings*, Czech. Math. J. **26** (1976), 174–182.

[BCS22] K.A. Brix, T.M. Carlsen and A. Sims, *Some results regarding the ideal structure of C^*-algebras of étale groupoids*, preprint 2022 (arXiv:2211.06126 [math.OA]).

[BCFS14] J.H. Brown, L.O. Clark, C. Farthing, and A. Sims, *Simplicity of algebras associated to étale groupoids*, Semigroup Forum **88** (2014), 433–452.

[CKSS14] T.M. Carlsen, S. Kang, J. Shotwell and A. Sims *The primitive ideals of the Cuntz–Krieger algebra of a row-finite higher-rank graph with no sources*, J. Funct. Anal **266** (2014), 2570–2589.

[CLSV11] T.M. Carlsen, N. Larsen, A. Sims and S.T. Vittadello, *Co-universal algebras associated to product systems, and gauge-invariant uniqueness theorems*, Proc. London Math. Soc. **103** (2011), 563–600.

[CS16] T.M. Carlsen, and A. Sims, *On Hong and Szymanski's description of the primitive-ideal space of a graph algebra*, in Operator algebras and Applications, Carlsen, Larsen, Neshveyev and Skau (eds), The Abel Symposium 2015, Springer 2016, 109–126.

[Cu81] J. Cuntz, *A class of C^*-algebras and topological Markov chains. II. Reducible chains and the Ext-functor for C^*-algebras*, Invent. Math. **63** (1981), 25–40.

[CK80] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.

[Dea95] V. Deaconu, *Groupoids associated with endomorphisms*, Trans. Amer. Math. Soc. **347** (1995), 1779–1786.

[DD63] J. Dixmier and A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[Ell76] G.A. Elliott, *On the classification of inductive limits of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

[Ell93] G.A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. reine angew. Math. **443** (1993), 179–219.

[Exe11] R. Exel, *Non-Hausdorff étale groupoids*, Proc. Amer. Math. Soc. **139** (2011), 897–907.

[ER07] R. Exel and J.N. Renault, *Semigroups of local homeomorphisms and interaction groups*, Ergodic Theory Dynam. Systems **27** (2007), 1737–1771.

[Fol99] G.B. Folland, *Real analysis* Modern techniques and their applications. Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1999. xvi+386 pp. ISBN: 0–471–31716–0.

[Fra] M.D. Francis, *Question #2375956*, Math StackExchange (2017) (https://math.stackexchange.com/questions/2375956/lifting-a-convergent-net-through-a-quotient-map; accessed January 21, 2023).

[GN43] I. Gelfand and M. Neumark, *On the imbedding of normed rings into the ring of operators in Hilbert space. (Russian summary)*, Rec. Math. [Mat. Sbornik] N.S. **12** (1943), 197–213.

[Gre80] P. Green, *The structure of imprimitivity algebras*, J. Funct. Anal. **36** (1980), 88–104.

[GMR18] R. Grigorchuk, M. Musat, and M. Rørdam, *Just-infinite C^*-algebras*, Comment. Math. Helv. **93** (2018), no. 1, pp. 157–201.

[HRSW13] R. Hazlewood, I. Raeburn, A. Sims and S.B.G. Webster, *On some fundamental results about higher-rank graphs and their C^*-algebras*, Proc. Edinburgh Math. Soc. **56** (2013), 575–597.

[HS04] J.H. Hong and W. Szymański, *The primitive ideal space of the C^*-algebras of infinite graphs*, J. Math. Soc. Japan **56** (2004), 45–64.

[aHNS21] A. an Huef, A.C.S. Ng, and A. Sims, *Stably finite extensions of rank-two graph C^*-algebras*, J. Operator Th., to appear (arXiv:2111.15165 [math.OA]).

[aHR97] A. an Huef and I. Raeburn, *The ideal structure of Cuntz–Krieger algebras*, Ergodic Theory Dynam. Systems **17** (1997), 611–624.

[Kat04] T. Katsura, *On C^*-algebras associated with C^*-correspondences*, J. Funct. Anal., **217** (2004), 366–401.

[Kat21] T. Katsura *Ideal structure of C^*-algebras of singly generated dynamical systems*, preprint 2021 (arXiv:2107.01389 [math.OA]).

[Kir95] E. Kirchberg, *Exact C^*-algebras, tensor products, and the classification of purely infinite algebras*, Proc. ICM (1995), pp. 943–954.

[Kot12] M. Koto, *Answer #193984*, Math StackExchange (2012) (https://math.stackexchange.com/a/193984; accessed May 12, 2022).

[KP00] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.

[KPR98] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.

[KPS16] A. Kumjian, D. Pask and A. Sims, *Simplicity of twisted C^*-algebras of higher-rank graphs and crossed products by quasifree actions*, J. Noncommut. Geom. **10** (2016), 515–549.

[Mac51] G.W. Mackey, *On induced representations of groups*, Amer. J. Math. **73** (1951), 576–592.

[MRW87] P.S. Muhly, J.N. Renault and D.P. Williams, *Equivalence and Isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987), 3–22.

[Phi00] N.C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114.

[Rae05] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics, 103. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005. vi+113 pp. ISBN: 0–8218–3660–9.

[RSY03] I. Raeburn, A. Sims and T. Yeend, *Higher-rank graphs and their C^*-algebras*, Proc. Edinburgh Math. Soc. **46** (2003), 99–115.

[RW98] I. Raeburn and D.P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, 60. American Mathematical Society, Providence, RI, 1998. xiv+327 pp. ISBN: 0–8218–0860–5.

[Ren80] J.N. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics **793**, Springer, Berlin (1980).

[Ren91] J.N. Renault, *The ideal structure of groupoid crossed product C^*-algebras*, J. Operator Th. **25** (1991), 3–36.

[Rie74] M.A. Rieffel, *Induced representations of C^*-algebras*, Adv. Math. **13** (1974), 176–257.

[Sim20] A. Sims, *Hausdorff étale groupoids and their C^*-algebras*, in Operator algebras and dynamics: groupoids, crossed products and Rokhlin Dimension (F. Perera, Ed.) in *Advanced Coursed in Mathematics. CRM Barcelona*, Birkhäuser, 2020.

[SW16] A. Sims and D.P. Williams, *The primitive ideals of some étale groupoid C^*-algebras*, Algebras and Representation Theory **19** (2016), 255–276.

[Siw71] F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, General Topology and Appl. **1** (1971), 143–154.

[TWW17] A. Tikuisis, S. White and W. Winter, *Quasidiagonality of nuclear C^*-algbebras*, Ann. Math. **185** (2017), 229–284.

[Wi07] D.P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, 134. American Mathematical Society, Providence, RI, 2007. xvi+528 pp. ISBN: 978–0–8218–4242–3; 0–8218–4242–0.

[WvW21] D.P. Williams and D.W. van Wyk, *The primitive ideal space of groupoid C^*-algebras for groupoids with abelian isotropy*, Indiana Univ. Math. J. **71** (2022), no. 1, 359–390.

[Win17] , W. Winter, *Structure of nuclear C^*-algebras: From quasidiagonality to classification, and back again*, Proc. ICM (2017), pp. 1797–1820.

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