# A TWIST OVER A MINIMAL ÉTALE GROUPOID THAT IS TOPOLOGICALLY NONTRIVIAL OVER THE INTERIOR OF THE ISOTROPY 

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#### Abstract

We present an example of a twist over a minimal Hausdorff étale groupoid such that the restriction of the twist to the interior of the isotropy is not topologically trivial; that is, the restricted twist is not induced by a continuous 2-cocycle.


## 1. Introduction

The theory of Cartan subalgebras in operator algebras deals with the question of recognising when an operator algebra together with a distinguished abelian subalgebra arises as the closure of the convolution algebra of compactly supported functions on a groupoid, possibly twisted by cohomological data. The study of this question began with Feldman and Moore [12, 13, 14] in the context of von Neumann algebras. Their theorem says that every Cartan pair of von Neumann algebras arises from a measured equivalence relation $R$ and a measurable circle-valued 2-cocycle on $R$. The corresponding question for $\mathrm{C}^{*}$-algebras took longer to answer. Renault [20, Theorem II.4.15] obtained an exact analogue of Feldman and Moore's results when the Cartan subalgebra is densely spanned by projections; but more general commutative subalgebras required Kumjian's notion of a twist (essentially a principal $\mathbb{T}$-bundle that is also a topological groupoid) over an étale groupoid [15]; and Renault [22] subsequently generalised Kumjian's work, which applied only to second-countable principal groupoids, to the more general situation of second-countable topologically principal groupoids. Raad [19] later generalised Renault's work to the most general possible situation (see [2, Theorem 3.1]) of twists over effective groupoids.

The key difference between the measurable and topological settings, as recognised by Kumjian, is the existence of circle bundles that are locally trivial but not globally trivial. In all of the constructions described above, there is a natural way to construct from a pair $(A, B)$ a twist: a principal circle bundle $\mathcal{E}$ (measurable in the setting of [12] or topological in the setting of $[15,22]$ ) over the groupoid $\mathcal{G}$ of germs for the action of the normaliser of $B$ in $A$ on the spectrum of $B$ in such a way that the reduced $\mathrm{C}^{*}$-completion of the convolution algebra of sections of the bundle coincides with $A$. For measurable bundles as in [12] one can always choose a measurable section of the bundle, and then the measurable 2-cocycle appearing in Feldman and Moore's theorem is the obstruction to this section being a homomorphism. Likewise when the spectrum of $A$ is totally disconnected, so is the groupoid $\mathcal{G}$, and so the bundle $\mathcal{E}$ admits a continuous global section, once again yielding the continuous 2-cocycle of Renault's result in [20]. Kumjian's innovation was to describe a $\mathrm{C}^{*}$-algebra constructed directly from the bundle $\mathcal{E}$ without the need to convert to a 2-cocycle.

[^0]Since then, the theory of twisted groupoid $\mathrm{C}^{*}$-algebras has largely concerned itself with twists and their $\mathrm{C}^{*}$-algebras rather than with continuous 2-cocycles. In particular, the first author proved an automatic-injectivity theorem [1, Theorem 6.3] for homomorphisms $\pi$ of $\mathrm{C}^{*}$-algebras of twists $(\mathcal{E}, \mathcal{G})$ over Hausdorff étale groupoids, whose key hypothesis is that $\pi$ should be injective on the subalgebra corresponding to the interior of the isotropy in $\mathcal{G}$. This raises a natural question, and one that the authors of this paper have been asked a number of times over the last few years: "I'd rather just work with cocycles; do I really need to deal with twists?" More precisely, is there an explicit example of a twist that satisfies the hypotheses of the injectivity theorem of [1] but can't be handled by results like those of [3] for continuous 2-cocycles?

The simplest example of a twist that does not arise from a 2-cocycle is due to Kumjiancommunicated to the second-named author in private correspondence, and described and generalised in [15, Section 4]. We give the details in Section 2, but the basic idea is as follows. Fix a nontrivial principal $\mathbb{T}$-bundle $p: S \rightarrow X$ over a locally compact Hausdorff space $X$. Let $\mathcal{G}$ be the transformation groupoid for $\mathbb{Z}_{2}$ interchanging two copies of $X$. The twist $\mathcal{E}$ comprises a copy of $(X \sqcup X) \times \mathbb{T}$ over the unit space of $\mathcal{G}$, and copies of $S$ and its conjugate bundle $\bar{S}$ over the nontrivial arrows. Multiplication is defined using pointwise multiplication in $\mathbb{T}$, the natural actions of $\mathbb{T}$ on $S$ and $\bar{S}$, and the standard isomorphisms $\left(S *_{X} \bar{S}\right) / \mathbb{T} \cong X \times \mathbb{T}$ and $\left(\bar{S} *_{X} S\right) / \mathbb{T} \cong X \times \mathbb{T}$. This twist cannot arise from a 2-cocycle because it contains a copy of the topologically nontrivial bundle $S$, whereas twists arising from cocycles are topologically trivial.

However, this twist is not minimal (each orbit has just two points), and its reduction to the interior of the isotropy of $\mathcal{G}$ is the trivial twist

$$
(X \sqcup X) \times \mathbb{T} \rightarrow(X \sqcup X) \times \mathbb{T} \rightarrow X \sqcup X
$$

so arises from a (trivial) continuous 2-cocycle. In particular, this example does not answer the question above. Since twisted groupoid C*-algebras have attracted significant recent interest $([1,2,3,4,5,8,9,10,11,16,17,23,24])$, we provide here an example that does answer the question (see Section 3), establishing the following result.

Theorem A. There exists a twist $\mathcal{E}$ over a minimal Hausdorff étale groupoid $\mathcal{G}$ such that the induced twist $\mathcal{I}^{\mathcal{E}}$ over the interior $\mathcal{I}^{\mathcal{G}}$ of the isotropy of $\mathcal{G}$ does not come from a 2 -cocycle.

## 2. Kumjian's example

In this section, we present Kumjian's example of a twist that does not arise from a 2-cocycle (see [15, Section 4]). While, as discussed in the introduction, Kumjian's construction does not provide an example of a situation in which the uniqueness theorem of [1] is applicable but a similar theorem for effective groupoids (such as [25, Theorem 10.2.7] or [21, Corollary 4.9]) would not suffice, it conveys the central idea of our construction later.

Kumjian's example is as follows. Let $p: S \rightarrow X$ be a nontrivial principal $\mathbb{T}$-bundle over a locally compact Hausdorff space (for example, the Hopf fibration). Let $\bar{p}: \bar{S} \rightarrow X$ be the conjugate bundle; so as a set, we have $\bar{S}:=\{\bar{t}: t \in S\}$, and the map $t \mapsto \bar{t}$ is a homeomorphism, but the $\mathbb{T}$-action on $\bar{S}$ is the conjugate action $z \cdot \bar{t}=\overline{\bar{z} \cdot t}$. Define

$$
S *_{X} \bar{S}:=\{(s, \bar{t}) \in S \times \bar{S}: p(s)=\bar{p}(\bar{t})\}
$$

and let $\left(S *_{X} \bar{S}\right) / \mathbb{T}$ be the quotient of $S *_{X} \bar{S}$ by the equivalence relation given by $(z \cdot s, \bar{t}) \sim$ $(s, z \cdot \bar{t})$ for all $s \in S, \bar{t} \in \bar{S}$, and $z \in \mathbb{T}$. Define $\left(\bar{S} *_{X} S\right) / \mathbb{T}$ analogously.

Let $R_{2}$ be the full equivalence relation on the two-point set $\{0,1\}$, regarded as a discrete groupoid with two units; we identify the unit space of $R_{2}$ with $\{0,1\}$. So $R_{2}$ has two elements that are not units: the element $(0,1)$ with range 0 and source 1 , and its inverse $(1,0)$ with range 1 and source 0 . Regarding $X$ as a topological groupoid consisting entirely of units,
let $G:=X \times R_{2}$, the product groupoid. So $G$ is a locally compact Hausdorff étale groupoid with unit space $G^{(0)}=X \times R_{2}^{(0)}=X \times\{0,1\}$, and remaining elements $X \times\{(0,1),(1,0)\}$, with range and source maps given by $r(x, a)=(x, r(a))$ and $s(x, a)=(x, s(a))$, and with multiplication given by $(x, a)(x, b)=(x, a b)$ whenever $(a, b) \in R_{2}^{(2)}$. Now consider the set

$$
E:=(X \times \mathbb{T} \times\{0\}) \sqcup(S \times\{(0,1)\}) \sqcup(\bar{S} \times\{(1,0)\}) \sqcup(X \times \mathbb{T} \times\{1\})
$$

Let $E^{(0)}=X \times\{0,1\}$. Define the range map $r: E \rightarrow E^{(0)}$ by

$$
\begin{aligned}
r(x, z, i) & =(x, i), & & x \in X, z \in \mathbb{T}, i \in\{0,1\}, \\
r(t,(0,1)) & =(p(t), 0), & & t \in S, \\
r(\bar{t},(1,0)) & =(\bar{p}(\bar{t}), 1)=(p(t), 1), & & \bar{t} \in \bar{S} .
\end{aligned}
$$

Define inversion by

$$
\begin{aligned}
(x, z, i)^{-1} & =(x, \bar{z}, i), & & x \in X, z \in \mathbb{T}, i \in\{0,1\}, \\
(t,(0,1))^{-1} & =(\bar{t},(1,0)), & & t \in S, \\
(\bar{t},(1,0))^{-1} & =(t,(0,1)), & & \bar{t} \in \bar{S},
\end{aligned}
$$

and the source map by $s(\eta)=r\left(\eta^{-1}\right)$. Finally, let $\theta_{l}:\left(S *_{X} \bar{S}\right) / \mathbb{T} \rightarrow \mathbb{T}$ and $\theta_{r}:\left(\bar{S} *_{X} S\right) / \mathbb{T} \rightarrow \mathbb{T}$ be the maps $\theta_{l}([z \cdot t, \bar{t}]):=z$ and $\theta_{r}([\bar{t}, z \cdot t]):=z$, and define multiplication on $E$ by

$$
\begin{aligned}
(x, w, i)(x, z, i) & =(x, w z, i), & & x \in X, w, z \in \mathbb{T}, i \in\{0,1\}, \\
(p(t), w, 0)(t,(0,1)) & =(w \cdot t,(0,1))=(t,(0,1))(p(t), w, 1), & & t \in S, w \in \mathbb{T}, \\
(\bar{p}(\bar{t}), w, 1)(\bar{t},(1,0)) & =(w \cdot \bar{t},(1,0))=(\bar{t},(1,0))(\bar{p}(\bar{t}), w, 0), & & \bar{t} \in \bar{S}, w \in \mathbb{T}, \\
(t,(0,1))(\bar{s},(1,0)) & =\left(p(t), \theta_{l}([t, \bar{s}]), 0\right), & \text { and } & \\
(\bar{s},(1,0))(t,(0,1)) & =\left(p(t), \theta_{r}([\bar{s}, t]), 1\right), & & t \in S, \bar{s} \in \bar{S}, p(t)=\bar{p}(\bar{s}) .
\end{aligned}
$$

These operations make $E$ into a groupoid, and we obtain a twist

$$
G^{(0)} \times \mathbb{T} \simeq X \times \mathbb{T} \times\{0,1\} \xrightarrow{\imath} E \xrightarrow{\pi} G
$$

where $\imath$ is the inclusion map, and $\pi$ is given by

$$
\begin{aligned}
\pi(x, z, i) & =(x, i), & & x \in X, z \in \mathbb{T}, i \in\{0,1\}, \\
\pi(t,(0,1)) & =(p(t),(0,1)), & & t \in S, \\
\pi(\bar{t},(1,0)) & =(\bar{p}(\bar{t}),(1,0)), & & \bar{t} \in \bar{S} .
\end{aligned}
$$

We claim that this twist does not arise from a continuous $\mathbb{T}$-valued 2-cocycle on $G$. To see why, suppose otherwise. Then there is a continuous global section $\Sigma: G \rightarrow E$. Restriction of this section to the subspace $X \times\{(0,1)\}$ gives a section

$$
\left.\Sigma\right|_{X \times\{(0,1)\}}: X \times\{(0,1)\} \rightarrow S \times\{(0,1)\} .
$$

Let $\pi_{1}: S \times\{(1,0)\} \rightarrow S$ be the projection map and define $i_{1}: X \rightarrow X \times\{(0,1)\}$ by $i_{1}(x)=(x,(0,1))$. Then $\pi_{1} \circ \Sigma \circ i_{1}$ is a global section of the bundle $p: S \rightarrow X$. This contradicts the nontriviality of $p: S \rightarrow X$. So $E$ is not topologically trivial. However,

$$
\operatorname{Iso}(G)=\{\gamma \in G: r(\gamma)=s(\gamma)\}=X \times\{0,1\}=G^{(0)}
$$

so the restriction of the twist to the interior of the isotropy is trivial.

## 3. The proof of Theorem A

3.1. Defining the twisted groupoid as a topological space. Let $p: B \rightarrow X$ be any nontrivial principal $\mathbb{T}$-bundle over a locally compact Hausdorff space $X$ such that $p$ is locally trivial and there exists a minimal homeomorphism $\sigma: X \rightarrow X$. For example, take $\sigma$ to be an irrational rotation map on $X=\mathbb{T}^{2}$ : by [18, Theorem 6.22], since $\mathbb{T}^{2}$ has second integral cohomology group $\mathbb{Z}$, it admits a nontrivial principal $\mathbb{T}$-bundle $p: B \rightarrow \mathbb{T}^{2}$. For each $x \in X$, let $B_{x}$ be the fibre $p^{-1}(x)$, and note that for each fixed $b \in B_{x}$, the map $\mathbb{T} \ni z \mapsto z \cdot b \in B_{x}$ is a homeomorphism.

Let $\mathbb{F}_{2}=\left\langle a, a^{-1}, b, b^{-1}\right\rangle$ be the free group with two generators $a$ and $b$. For each $w \in \mathbb{F}_{2}$, we will define an associated principal $\mathbb{T}$-bundle $p^{(w)}: B^{(w)} \rightarrow X$, and we will then define a groupoid structure on the disjoint union $\bigsqcup_{w \in \mathbb{F}_{2}} B^{(w)}$.

Let $\alpha$ be the group action of $\mathbb{F}_{2}$ on $X$ that is defined on generators by

$$
\alpha_{a}=\sigma \quad \text { and } \quad \alpha_{b}=\operatorname{id}_{X} .
$$

Let $\varepsilon$ be the identity in $\mathbb{F}_{2}$ - the empty word. Define $p^{(\varepsilon)}: C^{(\varepsilon)} \rightarrow X$ and $p^{(a)}: C^{(a)} \rightarrow X$ each to be the trivial bundle $X \times \mathbb{T} \rightarrow X$ with $\mathbb{T}$-action $z \cdot(x, w):=(x, z w)$. Let $p^{\left(a^{-1}\right)}: C^{\left(a^{-1}\right)} \rightarrow X$ be the trivial bundle $X \times \mathbb{T} \rightarrow X$ with the conjugate $\mathbb{T}$-action $z \cdot(x, w)=(x, \bar{z} w)$. Let $p^{(b)}: C^{(b)} \rightarrow X$ be the nontrivial bundle $p: B \rightarrow X$. Let $\bar{B}$ be a copy of $B$ and let $b \mapsto \bar{b}$ be the map that takes an element of $B$ to its copy in $\bar{B}$. Let $\bar{p}: \bar{B} \rightarrow X$ be given by $\bar{p}(\bar{b}):=p(b)$, and define a $\mathbb{T}$-action on $\bar{B}$ by $z \cdot \bar{b}:=\overline{\bar{z} \cdot b}$. Then $\bar{p}: \bar{B} \rightarrow X$ with this action is the conjugate principal $\mathbb{T}$-bundle of $p: B \rightarrow X$. Let $p^{\left(b^{-1}\right)}: C^{\left(b^{-1}\right)} \rightarrow X$ be this conjugate bundle.
Notation 3.1. For $d \in\left\{b, b^{-1}\right\}$, if $c \in C^{(d)}$, then $\bar{c}$ denotes the copy of $c$ in $C^{\left(d^{-1}\right)}$. And for $d \in\left\{\varepsilon, a, a^{-1}\right\}$, if $(x, z) \in C^{(d)}$, then we define $\overline{(x, z)}:=\left(\alpha_{d}(x), \bar{z}\right) \in C^{\left(d^{-1}\right)}$. So for all $d \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$ and $c \in C^{(d)}$, we have $\overline{\bar{c}}=c$, and

$$
\begin{equation*}
p^{\left(d^{-1}\right)}(\bar{c})=\alpha_{d}\left(p^{(d)}(c)\right) . \tag{3.1}
\end{equation*}
$$

Moreover, the map $C^{(d)} \ni c \mapsto \bar{c} \in C^{\left(d^{-1}\right)}$ is a homeomorphism.
For $w \in \mathbb{F}_{2} \backslash\{\varepsilon\}$, let $|w|$ denote the word length of $w$, and write $w=w_{1} w_{2} \cdots w_{|w|}$; we define $|\varepsilon|=0$. For $|w| \leq 1$, we define $B^{(w)}:=C^{(w)}$. For $|w| \geq 2$, let $B^{(w)}$ be the quotient of

$$
\begin{aligned}
C^{(w)}:=\left\{\left(c_{1}, \ldots, c_{|w|}\right):\right. & c_{i} \in C^{\left(w_{i}\right)} \text { for each } i \in\{1, \ldots,|w|\}, \text { and } \\
& \left.p^{\left(w_{i}\right)}\left(c_{i}\right)=\alpha_{w_{i+1}}\left(p^{\left(w_{i+1}\right)}\left(c_{i+1}\right)\right) \text { for each } i \in\{1, \ldots,|w|-1\}\right\}
\end{aligned}
$$

by the equivalence relation

$$
\left(c_{1}, \ldots, z \cdot c_{i}, \ldots, c_{|w|}\right) \sim\left(c_{1}, \ldots, z \cdot c_{j}, \ldots, c_{|w|}\right), \quad \text { for all } z \in \mathbb{T}, i, j \in\{1, \ldots,|w|\}
$$

We denote the equivalence class of $c=\left(c_{1}, \ldots, c_{|w|}\right) \in C^{(w)}$ by $[c]=\left[c_{1}, \ldots, c_{|w|}\right] \in B^{(w)}$.
We give $C^{(w)} \subseteq \prod_{i=1}^{|w|} C^{\left(w_{i}\right)}$ the subspace topology, and $B^{(w)}$ the quotient topology.
Lemma 3.2. For each $w \in \mathbb{F}_{2}, C^{(w)}$ is a locally compact Hausdorff space.
Proof. When $w=\varepsilon$, we have $C^{(w)}=X \times \mathbb{T}$, which is a locally compact Hausdorff space. If $w \neq \varepsilon$, and $\left(c_{1}^{i}, \ldots, c_{|w|}^{i}\right)_{i \in I}$ is a net in $C^{(w)}$ that converges to $\left(c_{1}, \ldots, c_{|w|}\right) \in \prod_{j=1}^{|w|} C^{\left(w_{j}\right)}$, then continuity of the $p^{\left(w_{j}\right)}$ and $\sigma$ ensure that $\left(c_{1}, \ldots, c_{|w|}\right) \in C^{(w)}$. So $C^{(w)}$ is a closed subspace of $\prod_{j=1}^{|w|} C^{\left(w_{j}\right)}$ and hence is locally compact and Hausdorff.

Lemma 3.3. For each $w \in \mathbb{F}_{2}, B^{(w)}$ is a locally compact Hausdorff space.
Proof. For $w=\varepsilon$ the result is trivial, so suppose $w \neq \varepsilon$. For each $n \in \mathbb{N} \backslash\{0\}$, define $K_{n}:=\left\{\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{T}^{n}: z_{1} \cdots z_{n}=1\right\}=\left\{\left(z_{1}, \ldots, z_{n-1}, \overline{z_{1} \cdots z_{n_{1}}}\right): z_{1}, \ldots, z_{n-1} \in \mathbb{T}\right\}$.

Then $K_{n} \cong \mathbb{T}^{n-1}$ is compact for $n \in \mathbb{N} \backslash\{0\}$. Also, $K_{|w|}$ is a group under coordinatewise multiplication, and acts coordinatewise on $C^{(w)}$. For $c=\left(c_{1}, \ldots, c_{|w|}\right) \in C^{(w)}$,

$$
B^{(w)} \ni[c]=\left\{\left(z_{1} \cdot c_{1}, \ldots, z_{|w|} \cdot c_{|w|}\right):\left(z_{1}, \ldots, z_{|w|}\right) \in K_{|w|}\right\}=K_{|w|} \cdot c .
$$

Thus $B^{(w)}$ is the quotient of $C^{(w)}$ by the action of $K_{|w|}$. Since $C^{(w)}$ is locally compact Hausdorff by Lemma 3.2 and $K_{|w|}$ is compact, [7, Proposition 2 and Corollary 1 of Section III.4.1 and Proposition 3 of Section III.4.2] imply that $B^{(w)}$ is locally compact Hausdorff.

Lemma 3.4. For each $w \in \mathbb{F}_{2} \backslash\{\varepsilon\}$, the map $p^{(w)}: B^{(w)} \rightarrow X$ given by

$$
p^{(w)}\left(\left[c_{1}, \ldots, c_{|w|}\right]\right)=p^{\left(w_{|w|}\right)}\left(c_{|w|}\right)
$$

defines a locally trivial principal $\mathbb{T}$-bundle over $X$ with $\mathbb{T}$-action given by

$$
z \cdot\left[c_{1}, \ldots, c_{|w|}\right]:=\left[c_{1}, \ldots, z \cdot c_{|w|}\right] \text { for } z \in \mathbb{T} .
$$

Moreover, for each $\left[c_{1}, \ldots, c_{|w|}\right] \in B^{(w)}$ and $i \in\{1, \ldots,|w|-1\}$, we have

$$
\begin{equation*}
p^{\left(w_{i}\right)}\left(c_{i}\right)=\alpha_{w_{i+1} \cdots w_{|w|}}\left(p^{(w)}\left(\left[c_{1}, \ldots, c_{|w|}\right]\right)\right) . \tag{3.2}
\end{equation*}
$$

Proof. Fix $w \in \mathbb{F}_{2} \backslash\{\varepsilon\}$. Since $p^{\left(w_{|w|}\right)}: C^{\left(w_{|w|}\right)} \rightarrow X$ is a continuous surjection, the map $C^{(w)} \ni\left(c_{1}, \ldots, c_{|w|}\right) \mapsto p^{\left(w_{|w|}\right)}\left(c_{|w|}\right) \in X$ is a continuous surjection, and it follows that it descends to a continuous surjection on the quotient $B^{(w)}$ of $C^{(w)}$.

Similarly, since the action of $\mathbb{T}$ on $B^{(d)}$ is continuous and fibre-preserving for each $d \in$ $\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$, the formula $z \cdot\left[c_{1}, \ldots, c_{|w|}\right]:=\left[c_{1}, \ldots, z \cdot c_{|w|}\right]$ defines a continuous fibrepreserving action of $\mathbb{T}$ on $C^{(w)}$.

For each $\left[c_{1}, \ldots, c_{|w|}\right] \in B^{(w)}$ and $i \in\{1, \ldots,|w|-1\}$, we have

$$
p^{\left(w_{i}\right)}\left(c_{i}\right)=\alpha_{w_{i+1}}\left(p^{\left(w_{i+1}\right)}\left(c_{i+1}\right)\right)
$$

by the definition of $B^{(w)}$, and Equation (3.2) follows by the definition of $p^{(w)}$.
We show that the $\mathbb{T}$-bundle $p^{(w)}: B^{(w)} \rightarrow X$ is principal. Fix $x \in X$, and fix $[c]=$ $\left[c_{1}, \ldots, c_{|w|}\right] \in B^{(w)}$ such that $p^{(w)}([c])=x$. We must show that $z \mapsto z \cdot[c]$ defines a homeomorphism $\varphi_{[c]}: \mathbb{T} \rightarrow\left(B^{(w)}\right)_{x}=\left(p^{(w)}\right)^{-1}(x)$. Since $\mathbb{T}$ is compact and $\left(B^{(w)}\right)_{x}$ is Hausdorff by Lemma 3.3, it suffices to show that $\varphi_{[c]}$ is a continuous bijection. Continuity of $\varphi_{[c]}$ follows from continuity of the $\mathbb{T}$-action. For injectivity, suppose that $z \cdot[c]=z^{\prime} \cdot[c]$ for some $z, z^{\prime} \in \mathbb{T}$. Then $[c]=\left(\bar{z} z^{\prime}\right) \cdot[c]=\left[c_{1}, \ldots,\left(\bar{z} z^{\prime}\right) \cdot c_{|w|}\right]$, so by definition of the equivalence relation used to define $B^{(w)}$, we have $\bar{z} z^{\prime}=1$. Thus $z=z^{\prime}$, and $\varphi_{[c]}$ is injective. For surjectivity, fix $\left[c^{\prime}\right]=\left[c_{1}^{\prime}, \ldots, c_{|w|}^{\prime}\right] \in\left(B^{(w)}\right)_{x}$. Then Equation (3.2) implies that for each $i \in\{1, \ldots,|w|\}$,

$$
\begin{equation*}
p^{\left(w_{i}\right)}\left(c_{i}\right)=\alpha_{w_{i+1} \cdots w_{|w|}}\left(p^{(w)}([c])\right)=\alpha_{w_{i+1} \cdots w_{|w|}}(x)=\alpha_{w_{i+1} \cdots w_{|w|}}\left(p^{(w)}\left(\left[c^{\prime}\right]\right)\right)=p^{\left(w_{i}\right)}\left(c_{i}^{\prime}\right) . \tag{3.3}
\end{equation*}
$$

(Note that for $i=|w|$, we have $w_{i+1} \cdots w_{|w|}=\varepsilon$, so the above equation says that $p^{\left(w_{|w|}\right)}\left(c_{i}\right)=$ $\alpha_{\varepsilon}(x)=x=p^{\left(w_{|w|}\right)}\left(c_{i}^{\prime}\right)$ in this case.) By Equation (3.3) and by principality of the $\mathbb{T}$-bundle $p^{\left(w_{i}\right)}: C^{\left(w_{i}\right)} \rightarrow X$, for each $i \in\{1, \ldots,|w|\}$, there exists a unique element $z_{i} \in \mathbb{T}$ such that $c_{i}^{\prime}=z_{i} \cdot c_{i}$. Therefore,

$$
\left[c^{\prime}\right]=\left[c_{1}^{\prime}, \ldots, c_{|w|}^{\prime}\right]=\left[z_{1} \cdot c_{1}, \ldots, z_{|w|} \cdot c_{|w|}\right]=\left(z_{1} \cdots z_{|w|}\right) \cdot[c]=\varphi_{[c]}\left(z_{1} \cdots z_{|w|}\right),
$$

so $\varphi_{[c]}$ is surjective. Hence $p^{(w)}: B^{(w)} \rightarrow X$ is a principal $\mathbb{T}$-bundle.
To complete the proof, we must show that $p^{(w)}: B^{(w)} \rightarrow X$ is locally trivial. Since it is principal, it suffices to show that it admits continuous local sections: these induce local trivialisations because the $\mathbb{T}$-action is free and transitive on each fibre of $B^{(w)}$. Fix $x \in X$. For each $i \in\{1, \ldots,|w|\}$, the principal $\mathbb{T}$-bundle $p^{\left(w_{i}\right)}: C^{\left(w_{i}\right)} \rightarrow X$ is locally trivial, so there exists an open neighbourhood $U_{x}^{(i)} \subseteq X$ of $\alpha_{w_{i+1} \cdots w_{|w|}}(x)$ on which the bundle $p^{\left(w_{i}\right)}: C^{\left(w_{i}\right)} \rightarrow X$
admits a continuous local section $S_{x}^{\left(w_{i}\right)}: U_{x}^{(i)} \rightarrow C^{\left(w_{i}\right)}$. Then $p^{\left(w_{i}\right)} \circ S_{x}^{\left(w_{i}\right)}=\mathrm{id}_{U_{x}^{(i)}}$ for each $i \in\{1, \ldots,|w|\}$. Define

$$
U_{x}:=\bigcap_{i=1}^{|w|} \alpha_{w_{i+1} \cdots w_{|w|}}^{-1}\left(U_{x}^{(i)}\right) .
$$

Then $U_{x} \subseteq X$ is an open neighbourhood of $x$ since $\alpha_{w^{\prime}}$ is a homeomorphism for each $w^{\prime} \in \mathbb{F}_{2}$. For each $u \in U_{x}$ and each $i \in\{1, \ldots,|w|\}$, define

$$
c_{i}^{u}:=S_{w_{i}}^{(x)}\left(\alpha_{w_{i+1} \cdots w_{|w|}}(u)\right),
$$

so that

$$
\begin{equation*}
p^{\left(w_{i}\right)}\left(c_{i}^{u}\right)=\alpha_{w_{i+1} \cdots w_{|w|}}(u)=\alpha_{w_{i+1}}\left(p^{\left(w_{i+1}\right)}\left(c_{i+1}^{u}\right)\right) . \tag{3.4}
\end{equation*}
$$

Then $\left[c_{1}^{u}, \ldots, c_{|w|}^{u}\right] \in B^{(w)}$, and we define $S_{w}^{(x)}: U_{x} \rightarrow B^{(w)}$ by $S_{w}^{(x)}(u):=\left[c_{1}^{u}, \ldots, c_{|w|}^{u}\right]$. All the maps involved in the definition of $S_{w}^{(x)}$ are continuous on $U_{x}$, so $S_{w}^{(x)}$ is continuous. Moreover, by letting $i=|w|-1$ in Equation (3.4) for the final equality, we see that

$$
p^{(w)}\left(S_{w}^{(x)}(u)\right)=p^{(w)}\left(\left[c_{1}^{u}, \ldots, c_{|w|}^{u}\right]\right)=p^{\left(w_{|w|}\right)}\left(c_{|w|}^{u}\right)=u,
$$

which proves that $S_{w}^{(x)}: U_{x} \rightarrow B^{(w)}$ is a local section of $p^{(w)}: B^{(w)} \rightarrow X$. Therefore, $p^{(w)}: B^{(w)} \rightarrow X$ is a locally trivial principal $\mathbb{T}$-bundle.

Let

$$
\begin{equation*}
\mathcal{E}:=\bigsqcup_{w \in \mathbb{F}_{2}} B^{(w)}=\left\{(w, c): w \in \mathbb{F}_{2}, c \in B^{(w)}\right\} \tag{3.5}
\end{equation*}
$$

under the disjoint union topology. For each $w \in \mathbb{F}_{2}$, define $\mathcal{E}_{w}:=\{w\} \times B^{(w)}$. We will give $\mathcal{E}$ the structure of a topological groupoid.
3.2. Defining multiplication on the twisted groupoid. In this section we define a continuous partially defined multiplication on the space $\mathcal{E}$ defined in Equation (3.5), that is $\mathbb{F}_{2}$-graded in the sense that it carries $\mathcal{E}_{w} s^{*}{ }_{r} \mathcal{E}_{w^{\prime}}$ to $\mathcal{E}_{w w^{\prime}}$. The basic idea is that if there is cancellation in the product of $w$ and $w^{\prime}$ in $\mathbb{F}_{2}$, say $w=u v$ and $w^{\prime}=v^{-1} u^{\prime}$, then we can eliminate the corresponding entries in $B^{(w)}$ and $B^{\left(w^{\prime}\right)}$ of composable pairs in $\mathcal{E}_{w} \times \mathcal{E}_{w^{\prime}}$.

For $d, e \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$ we describe a Baer-sum-like product $B^{(d, e)}$ of $B^{(d)}$ and $B^{(e)}$, called their balanced fibred product. Specifically, we define $B^{(d, e)}$ as the quotient of

$$
C^{(d, e)}:=B^{(d)}{ }_{p^{(d)}} *_{\alpha_{e} \circ p^{(e)}} B^{(e)}=\left\{\left(c_{1}, c_{2}\right) \in B^{(d)} \times B^{(e)}: p^{(d)}\left(c_{1}\right)=\alpha_{e}\left(p^{(e)}\right)\left(c_{2}\right)\right\}
$$

by the equivalence relation

$$
\begin{equation*}
\left(c_{1}, z \cdot c_{2}\right) \sim\left(z \cdot c_{1}, c_{2}\right), \quad \text { for } z \in \mathbb{T} . \tag{3.6}
\end{equation*}
$$

We endow $C^{(d, e)}$ with the subspace topology inherited from $C^{(d)} \times C^{(e)}$, and $B^{(d, e)}$ with the quotient topology. Note that if $d, e \neq \varepsilon$ and $d \neq e^{-1}$, then $C^{(d, e)}=C^{(d e)}$ and $B^{(d, e)}=B^{(d e)}$. In Lemma 3.6 we show that for all $d, e \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}, B^{(d, e)}$ is homeomorphic to $B^{(d e)}$.

We denote the equivalence class of $\left(c_{1}, c_{2}\right) \in C^{(d, e)}$ by $\left[c_{1}, c_{2}\right] \in B^{(d, e)}$. There is a continuous action of $\mathbb{T}$ on $B^{(d, e)}$ given by

$$
z \cdot\left[c_{1}, c_{2}\right]:=\left[z \cdot c_{1}, c_{2}\right]=\left[c_{1}, z \cdot c_{2}\right], \quad \text { for } z \in \mathbb{T} \text { and }\left[c_{1}, c_{2}\right] \in B^{(d, e)} .
$$

Lemma 3.5. Fix $d \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$. For $c \in B^{(d)}$, let $\bar{c}$ be as in Notation 3.1. Then

$$
\begin{equation*}
B^{\left(d, d^{-1}\right)}=\left\{z \cdot[c, \bar{c}]: z \in \mathbb{T}, c \in C^{(d)}\right\} . \tag{3.7}
\end{equation*}
$$

Proof. For $c \in C^{(d)}$ and $z \in \mathbb{T}$, we have $p^{\left(d^{-1}\right)}(\bar{c})=\alpha_{d}\left(p^{(d)}(c)\right)$ by Equation (3.1). So $p^{(d)}(z \cdot c)=p^{(d)}(c)=\alpha_{d^{-1}}\left(p^{\left(d^{-1}\right)}(\bar{c})\right)$. Hence $z \cdot[c, \bar{c}]=[z \cdot c, \bar{c}] \in B^{\left(d, d^{-1}\right)}$. For the reverse containment, fix $\left[c_{1}, c_{2}\right] \in B^{\left(d, d^{-1}\right)}$. Then $p^{(d)}\left(c_{1}\right)=\alpha_{d^{-1}}\left(p^{\left(d^{-1}\right)}\left(c_{2}\right)\right)$, so $p^{\left(d^{-1}\right)}\left(c_{2}\right)=$ $\alpha_{d}\left(p^{(d)}\left(c_{1}\right)\right)=p^{\left(d^{-1}\right)}\left(\overline{c_{1}}\right)$. Since $p^{\left(d^{-1}\right)}: C^{\left(d^{-1}\right)} \rightarrow X$ is a principal $\mathbb{T}$-bundle there is therefore a unique $z \in \mathbb{T}$ such that $c_{2}=z \cdot \overline{c_{1}}$. Thus $\left[c_{1}, c_{2}\right]=\left[c_{1}, z \cdot \overline{c_{1}}\right]=z \cdot\left[c_{1}, \overline{c_{1}}\right]$, as required.

We now show that $B^{(\varepsilon)}$ acts as an identity under the balanced fibred product, in the sense that for each $d \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$, we have $B^{(\varepsilon, d)} \simeq C^{(d)} \simeq B^{(d, \varepsilon)}$ and $B^{\left(d, d^{-1}\right)} \simeq C^{(\varepsilon)}$.

Lemma 3.6. (a) For each $d \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$, the maps

$$
\begin{aligned}
\psi_{\varepsilon, d}: B^{(\varepsilon, d)} \ni\left[\left(\alpha_{d}\left(p^{(d)}(c)\right), z\right), c\right] \mapsto z \cdot c \in C^{(d)} & =B^{(d)} \quad \text { and } \\
\psi_{d, \varepsilon}: B^{(d, \varepsilon)} \ni\left[c,\left(p^{(d)}(c), z\right)\right] \mapsto z \cdot c \in C^{(d)} & =B^{(d)}
\end{aligned}
$$

are $\mathbb{T}$-equivariant homeomorphisms.
(b) For each $d \in\left\{a, a^{-1}, b, b^{-1}\right\}$, the map

$$
\psi_{d, d^{-1}}: B^{\left(d, d^{-1}\right)} \ni z \cdot[c, \bar{c}] \mapsto z \cdot\left(\alpha_{d}\left(p^{(d)}(c)\right), 1\right)=z \cdot\left(p^{\left(d^{-1}\right)}(\bar{c}), 1\right)
$$

is a $\mathbb{T}$-equivariant homeomorphism.
To prove Lemma 3.6, we need some notation and preliminary results.
Notation 3.7. Given two elements $s_{1}$ and $s_{2}$ in the same fibre of a principal $\mathbb{T}$-bundle, we write $\left\langle s_{1}, s_{2}\right\rangle$ for the unique element of $\mathbb{T}$ such that $\left\langle s_{1}, s_{2}\right\rangle \cdot s_{2}=s_{1}$.

Remark 3.8. Fix $z \in \mathbb{T}$, and let $s_{1}$ and $s_{2}$ be in the same fibre of a principal $\mathbb{T}$-bundle. Then $\left\langle z \cdot s_{1}, s_{2}\right\rangle \cdot s_{2}=z \cdot s_{1}=z\left\langle s_{1}, s_{2}\right\rangle \cdot s_{2}$, and so $\left\langle z \cdot s_{1}, s_{2}\right\rangle=z\left\langle s_{1}, s_{2}\right\rangle$. Similarly, $\left\langle s_{1}, z \cdot s_{2}\right\rangle=$ $\bar{z}\left\langle s_{1}, s_{2}\right\rangle$, and $\left\langle z \cdot s_{1}, s_{1}\right\rangle=z=\left\langle s_{1}, \bar{z} \cdot s_{1}\right\rangle$. Furthermore, $\overline{\left\langle s_{1}, s_{2}\right\rangle}=\left\langle s_{2}, s_{1}\right\rangle=\left\langle\overline{s_{1}}, \overline{s_{2}}\right\rangle$.

Lemma 3.9. Let $q: T \rightarrow Y$ be a principal $\mathbb{T}$-bundle. For each $y \in Y$, the map $T_{y} \times T_{y} \ni$ $\left(s_{1}, s_{2}\right) \mapsto\left\langle s_{1}, s_{2}\right\rangle \in \mathbb{T}$ is continuous and surjective.

Proof. Fix $y \in Y$ and $t \in T_{y}$. Since $q: T \rightarrow Y$ is principal, the map $\varphi_{t}: \mathbb{T} \ni z \mapsto z \cdot t \in T_{y}$ is a homeomorphism. Fix $s_{1}, s_{2} \in T_{y}$. Then $s_{1}=\varphi_{t}^{-1}\left(s_{1}\right) \cdot t$ and $s_{2}=\varphi_{t}^{-1}\left(s_{2}\right) \cdot t$, and so

$$
\left(\varphi_{t}^{-1}\left(s_{1}\right) \overline{\varphi_{t}^{-1}\left(s_{2}\right)}\right) \cdot s_{2}=s_{1} .
$$

Thus $\left\langle s_{1}, s_{2}\right\rangle=\varphi_{t}^{-1}\left(s_{1}\right) \overline{\varphi_{t}^{-1}\left(s_{2}\right)}$, showing continuity. For surjectivity, fix $z \in \mathbb{T}$. For any $s \in T_{y}$, we have $z \cdot s \in T_{y}$, and $\langle z \cdot s, s\rangle=z$ by Remark 3.8.
Proof of Lemma 3.6. For (a), we prove the statement about $\psi_{\varepsilon, d}$; the proof of the statement about $\psi_{d, \varepsilon}$ is similar. Fix $d \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$. For $((x, z), c) \in C^{(\varepsilon, d)}$, we have $x=$ $p^{(\varepsilon)}(x, z)=\alpha_{d}\left(p^{(d)}(c)\right)$, so

$$
C^{(\varepsilon, d)}=\left\{((x, z), c): z \in \mathbb{T}, c \in C^{(d)}, x=\alpha_{d}\left(p^{(d)}(c)\right)\right\} .
$$

We claim that the map

$$
\Psi_{\varepsilon, d}: C^{(\varepsilon, d)} \ni\left(\left(\alpha_{d}\left(p^{(d)}(c)\right), z\right), c\right) \mapsto z \cdot c \in C^{(d)}
$$

respects the equivalence relation (3.6). Indeed, for $u \in \mathbb{T}$,

$$
\Psi_{\varepsilon, d}\left(\left(\alpha_{d}\left(p^{(d)}(c)\right), z\right), u \cdot c\right)=z \cdot(u \cdot c)=u z \cdot c=\Psi_{\varepsilon, d}\left(u \cdot\left(\alpha_{d}\left(p^{(d)}(c)\right), z\right), c\right) .
$$

Thus $\Psi_{\varepsilon, d}$ descends to a map

$$
\psi_{\varepsilon, d}: B^{(\varepsilon, d)} \ni\left[\left(\alpha_{d}\left(p^{(d)}(c)\right), z\right), c\right] \mapsto z \cdot c \in C^{(d)} .
$$

For all $u \in \mathbb{T}$ and $[(x, z), c] \in B^{(\varepsilon, d)}$, we have

$$
\psi_{\varepsilon, d}(u \cdot[(x, z), c])=\psi_{\varepsilon, d}([(x, u z), c])=u z \cdot c=u \cdot(z \cdot c)=u \cdot \psi_{\varepsilon, d}([(x, z), c]),
$$

so $\psi_{\varepsilon, d}$ is $\mathbb{T}$-equivariant. To see that $\psi_{\varepsilon, d}$ is a continuous surjection, note that $\Psi_{\varepsilon, d}: C^{(d, \varepsilon)} \rightarrow$ $C^{(d)}$ is continuous because $\mathbb{T}$ acts continuously on $C^{(d)}$. Also, $\Psi_{\varepsilon, d}$ is surjective because $\psi_{\varepsilon, d}\left(\left(\alpha_{d}\left(p^{(d)}(c)\right), 1\right), c\right)=c$ for all $c \in C^{(d)}$. Write $q_{\varepsilon, d}: B^{(\varepsilon, d)} \rightarrow C^{(\varepsilon, d)}$ for the quotient map. Since $\psi_{\varepsilon, d} \circ q_{\varepsilon, d}=\Psi_{\varepsilon, d}$, the map $\psi_{\varepsilon, d}$ is a continuous surjection. For injectivity, suppose that

$$
\psi_{\varepsilon, d}([(x, u), c])=\psi_{\varepsilon, d}([(y, z), e])
$$

for some $[(x, u), c],[(y, z), e] \in B^{(\varepsilon, d)}$. Then

$$
x=\alpha_{d}\left(p^{(d)}(c)\right)=\alpha_{d}\left(p^{(d)}(e)\right)=y \quad \text { and } \quad u \cdot c=z \cdot e,
$$

so

$$
[(x, u), c]=u \cdot[(x, 1), c]=[(x, 1), u \cdot c]=[(y, 1), z \cdot e]=z \cdot[(y, 1), e]=[(y, z), e],
$$

which proves that $\psi_{\varepsilon, d}$ is injective. Thus $\psi_{\varepsilon, d}$ is bijective with inverse given by

$$
\psi_{\varepsilon, d}^{-1}(c)=\left[\left(\alpha_{d}\left(p^{(d)}(c)\right), 1\right), c\right] .
$$

Since $q_{\varepsilon, d}, \alpha_{d}$, and $p^{(d)}$ are continuous, so is $\psi_{\varepsilon, d}^{-1}$, so $\psi_{\varepsilon, d}$ is a $\mathbb{T}$-equivariant homeomorphism.
For part (b), fix $d \in\left\{a, a^{-1}, b, b^{-1}\right\}$, and consider the map

$$
\Psi_{d, d^{-1}}: C^{\left(d, d^{-1}\right)} \ni\left(c_{1}, \overline{c_{2}}\right) \mapsto\left(p^{\left(d^{-1}\right)}\left(\overline{c_{2}}\right),\left\langle c_{1}, c_{2}\right\rangle\right) \in C^{(\varepsilon)} .
$$

For all $z \in \mathbb{T}$ and $\left(c_{1}, \overline{c_{2}}\right) \in C^{\left(d, d^{-1}\right)}$, we have $\left(z \cdot c_{1}, \overline{c_{2}}\right) \sim\left(c_{1}, z \cdot \overline{c_{2}}\right)=\left(c_{1}, \overline{\bar{z} \cdot c_{2}}\right)$, and by Remark 3.8, $\left\langle z \cdot c_{1}, c_{2}\right\rangle=z\left\langle c_{1}, c_{2}\right\rangle=\left\langle c_{1}, \bar{z} \cdot c_{2}\right\rangle$. Hence

$$
\Psi_{d, d^{-1}}\left(z \cdot c_{1}, \overline{c_{2}}\right)=\left(p^{\left(d^{-1}\right)}\left(\overline{c_{2}}\right),\left\langle z \cdot c_{1}, c_{2}\right\rangle\right)=\left(p^{\left(d^{-1}\right)}\left(\overline{\bar{z} \cdot c_{2}}\right),\left\langle c_{1}, \bar{z} \cdot c_{2}\right\rangle\right)=\Psi_{d, d^{-1}}\left(c_{1}, \overline{\bar{z} \cdot c_{2}}\right),
$$

and so $\Psi_{d, d^{-1}}$ is constant on equivalence classes. Thus $\Psi_{d, d^{-1}}$ descends to a map

$$
\psi_{d, d^{-1}}: B^{\left(d, d^{-1}\right)} \ni\left[c_{1}, \overline{c_{2}}\right] \mapsto\left(p^{\left(d^{-1}\right)}\left(\overline{c_{2}}\right),\left\langle c_{1}, c_{2}\right\rangle\right) \in C^{(\varepsilon)} .
$$

Recall from Lemma 3.5 that $B^{\left(d, d^{-1}\right)}=\left\{z \cdot[c, \bar{c}]: z \in \mathbb{T}, c \in C^{(d)}\right\}$, and recall from Equation (3.1) that $\alpha_{d}\left(p^{(d)}(c)\right)=p^{\left(d^{-1}\right)}(\bar{c})$ for all $c \in C^{(d)}$. For all $z \in \mathbb{T}$ and $c \in C^{(d)}$, we have $\langle z \cdot c, c\rangle=z$ by Remark 3.8, and it follows that
$\psi_{d, d^{-1}}(z \cdot[c, \bar{c}])=\psi_{d, d^{-1}}([z \cdot c, \bar{c}])=\left(p^{\left(d^{-1}\right)}(\bar{c}),\langle z \cdot c, c\rangle\right)=z \cdot\left(p^{\left(d^{-1}\right)}(\bar{c}), 1\right)=z \cdot \psi_{d, d^{-1}}([c, \bar{c}])$, so $\psi_{d, d^{-1}}$ is $\mathbb{T}$-equivariant. To see that $\psi_{d, d^{-1}}$ is a continuous surjection, first note that $p^{\left(d^{-1}\right)}: C^{\left(d^{-1}\right)} \rightarrow X$ is a continuous surjection by definition. Hence Lemma 3.9 implies that $\Psi_{d, d^{-1}}: C^{\left(d, d^{-1}\right)} \rightarrow C^{(\varepsilon)}$ is a continuous surjection. Writing $q_{d, d^{-1}}: C^{\left(d, d^{-1}\right)} \rightarrow B^{\left(d, d^{-1}\right)}$ for the quotient map and using that $\psi_{d, d^{-1}} \circ q_{d, d^{-1}}=\Psi_{d, d^{-1}}$, it follows that $\psi_{d, d^{-1}}$ is a continuous surjection. To see that $\psi_{d, d^{-1}}$ is injective, suppose that

$$
\psi_{d, d^{-1}}(u \cdot[c, \bar{c}])=\psi_{d, d^{-1}}(z \cdot[e, \bar{e}]),
$$

for some $u, z \in \mathbb{T}$ and $c, e \in C^{(d)}$. Then $\left(p^{\left(d^{-1}\right)}(\bar{c}), u\right)=\left(p^{\left(d^{-1}\right)}(\bar{e}), z\right)$, so $u=z$, and since $p^{\left(d^{-1}\right)}: C^{\left(d^{-1}\right)} \rightarrow X$ is a principal $\mathbb{T}$-bundle, there is a unique element $v \in \mathbb{T}$ such that $\bar{c}=v \cdot \bar{e}$. Hence $c=\overline{v \cdot \bar{e}}=\bar{v} \cdot e$, and so

$$
u \cdot[c, \bar{c}]=z \cdot[\bar{v} \cdot e, v \cdot \bar{e}]=z \bar{v} v \cdot[e, \bar{e}]=z \cdot[e, \bar{e}],
$$

which proves that $\psi_{d, d^{-1}}$ is injective. Thus $\psi_{d, d^{-1}}$ is bijective with inverse given by

$$
\psi_{d, d^{-1}}^{-1}\left(p^{\left(d^{-1}\right)}(\bar{c}), z\right)=z \cdot[c, \bar{c}] .
$$

For any $(x, z) \in C^{(\varepsilon)}$, since $p^{\left(d^{-1}\right)}$ is locally trivial it has a continuous section on a neighbourhood $x$. So $\psi_{d, d^{-1}}^{-1}$ is continuous at $(x, z)$. Thus $\psi_{d, d^{-1}}$ is a $\mathbb{T}$-equivariant homeomorphism.

Notation 3.10. For $d, e \in\left\{a, a^{-1}, b, b^{-1}\right\}$ with $d \neq e^{-1}$, we define $\psi_{d, e}:=\operatorname{id}_{B^{(d, e)}}$.
We use the next lemma to see that the maps $\psi_{d, d^{\prime}}$ determine an associative operation.
Lemma 3.11. Fix $d, d^{\prime} \in\left\{\varepsilon, a, a^{-1}, b, b^{-1}\right\}$. For all $e, e^{\prime}, e^{\dagger} \in C^{(\varepsilon)}, c, c^{\prime \prime} \in C^{(d)}, c^{\prime} \in C^{\left(d^{\prime}\right)}$, and $c^{\dagger} \in C^{\left(d^{-1}\right)}$ such that the following operations are defined, we have
(a) $\psi_{d, \varepsilon}\left(\left[\psi_{\varepsilon, d}([e, c]), e^{\prime}\right]\right)=\psi_{\varepsilon, d}\left(\left[e, \psi_{d, \varepsilon}\left(\left[c, e^{\prime}\right]\right)\right]\right)$;
(b) $\psi_{d, d^{\prime}}\left(\left[\psi_{d, \varepsilon}\left(\left[c, e^{\prime}\right]\right), c^{\prime}\right]\right)=\psi_{d, d^{\prime}}\left(\left[c, \psi_{\varepsilon, d^{\prime}}\left(\left[e^{\prime}, c^{\prime}\right]\right)\right]\right)$;
(c) $\psi_{d, d^{-1}}\left(\left[\psi_{\varepsilon, d}([e, c]), c^{\dagger}\right]\right)=\psi_{\varepsilon, \varepsilon}\left(\left[e, \psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right)\right]\right)$;
(d) $\psi_{\varepsilon, \varepsilon}\left(\left[\psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right), e^{\dagger}\right]\right)=\psi_{d, d^{-1}}\left(\left[c, \psi_{d^{-1}, \varepsilon}\left(\left[c^{\dagger}, e^{\dagger}\right]\right)\right]\right)$; and
(e) $\psi_{\varepsilon, d}\left(\left[\psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right), c^{\prime \prime}\right]\right)=\psi_{d, \varepsilon}\left(\left[c, \psi_{d^{-1}, d}\left(\left[c^{\dagger}, c^{\prime \prime}\right]\right)\right]\right)$.

Proof. Fix $e, e^{\prime}, e^{\dagger} \in C^{(\varepsilon)}, c, c^{\prime \prime} \in C^{(d)}, c^{\prime} \in C^{\left(d^{\prime}\right)}$, and $c^{\dagger} \in C^{\left(d^{-1}\right)}$ such that $(e, c) \in C^{(\varepsilon, d)}$, $\left(c, e^{\prime}\right) \in C^{(d, \varepsilon)},\left(e^{\prime}, c^{\prime}\right) \in C^{\left(\varepsilon, d^{\prime}\right)},\left(c, c^{\dagger}\right) \in C^{\left(d, d^{-1}\right)},\left(c^{\dagger}, e^{\dagger}\right) \in C^{\left(d^{-1}, \varepsilon\right)}$, and $\left(c^{\dagger}, c^{\prime \prime}\right) \in C^{\left(d^{-1}, d\right)}$. Then there exist $x, x^{\prime}, x^{\dagger} \in X$ and $z, z^{\prime}, z^{\dagger} \in \mathbb{T}$ such that $e=(x, z), e^{\prime}=\left(x^{\prime}, z^{\prime}\right)$, and $e^{\dagger}=\left(x^{\dagger}, z^{\dagger}\right)$. It then follows from Lemma 3.6(a) that
$\psi_{\varepsilon, d}([e, c])=z \cdot c, \quad \psi_{d, \varepsilon}\left(\left[c, e^{\prime}\right]\right)=z^{\prime} \cdot c, \quad \psi_{\varepsilon, d^{\prime}}\left(\left[e^{\prime}, c^{\prime}\right]\right)=z^{\prime} \cdot c^{\prime}, \quad$ and $\quad \psi_{d^{-1}, \varepsilon}\left(\left[c^{\dagger}, e^{\dagger}\right]\right)=z^{\dagger} \cdot c^{\dagger}$.
For part (a), we use Lemma 3.6(a) to see that

$$
\psi_{d, \varepsilon}\left(\left[\psi_{\varepsilon, d}([e, c]), e^{\prime}\right]\right)=\psi_{d, \varepsilon}\left(\left[z \cdot c, e^{\prime}\right]\right)=z z^{\prime} \cdot c=\psi_{\varepsilon, d}\left(\left[e, z^{\prime} \cdot c\right]\right)=\psi_{\varepsilon, d}\left(\left[e, \psi_{d, \varepsilon}\left(\left[c, e^{\prime}\right]\right)\right]\right)
$$

For part (b), we have

$$
\psi_{d, d^{\prime}}\left(\left[\psi_{d, \varepsilon}\left(\left[c, e^{\prime}\right]\right), c^{\prime}\right]\right)=\psi_{d, d^{\prime}}\left(\left[z^{\prime} \cdot c, c^{\prime}\right]\right)=\psi_{d, d^{\prime}}\left(\left[c, z^{\prime} \cdot c^{\prime}\right]\right)=\psi_{d, d^{\prime}}\left(\left[c, \psi_{\varepsilon, d^{\prime}}\left(\left[e^{\prime}, c^{\prime}\right]\right)\right]\right)
$$

For part (c), we use Lemma 3.6(a) to see that

$$
\psi_{d, d^{-1}}\left(\left[\psi_{\varepsilon, d}([e, c]), c^{\dagger}\right]\right)=\psi_{d, d^{-1}}\left(\left[z \cdot c, c^{\dagger}\right]\right)=z \cdot \psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right)=\psi_{\varepsilon, \varepsilon}\left(\left[e, \psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right)\right]\right)
$$

For part (d), we have
$\psi_{\varepsilon, \varepsilon}\left(\left[\psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right), e^{\dagger}\right]\right)=z^{\dagger} \cdot \psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right)=\psi_{d, d^{-1}}\left(\left[c, z^{\dagger} \cdot c^{\dagger}\right]\right)=\psi_{d, d^{-1}}\left(\left[c, \psi_{d^{-1}, \varepsilon}\left(\left[c^{\dagger}, e^{\dagger}\right]\right)\right]\right)$.
For part (e), note that since $\left(c, e^{\prime}\right) \in C^{(d, \varepsilon)}$, we have $p^{(d)}(c)=p^{(\varepsilon)}\left(e^{\prime}\right)=x^{\prime}$, and since $(e, c) \in C^{(\varepsilon, d)},\left(c, c^{\dagger}\right) \in C^{\left(d, d^{-1}\right)}$, and $\left(c^{\dagger}, c^{\prime \prime}\right) \in C^{\left(d^{-1}, d\right)}$, we have

$$
x=p^{(\varepsilon)}(e)=\alpha_{d}\left(p^{(d)}(c)\right)=\alpha_{d}\left(\alpha_{d^{-1}}\left(p^{\left(d^{-1}\right)}\left(c^{\dagger}\right)\right)\right)=p^{\left(d^{-1}\right)}\left(c^{\dagger}\right)=\alpha_{d}\left(p^{(d)}\left(c^{\prime \prime}\right)\right)
$$

So $p^{(d)}\left(c^{\prime \prime}\right)=p^{(d)}(c)=x^{\prime}$, and by Equation (3.1), $p^{\left(d^{-1}\right)}(\bar{c})=\alpha_{d}\left(p^{(d)}(c)\right)=p^{\left(d^{-1}\right)}\left(c^{\dagger}\right)=x$. Since $p^{(d)}: C^{(d)} \rightarrow X$ and $p^{\left(d^{-1}\right)}: C^{\left(d^{-1}\right)} \rightarrow X$ are principal $\mathbb{T}$-bundles, there are unique elements $u, u^{\dagger} \in \mathbb{T}$ such that $c^{\prime \prime}=u \cdot c$ and $c^{\dagger}=u^{\dagger} \cdot \bar{c}$. So Lemma 3.6(b) yields

$$
\psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right)=\left(x, u^{\dagger}\right) \quad \text { and } \quad \psi_{d^{-1}, d}\left(\left[c^{\dagger}, c^{\prime \prime}\right]\right)=\left(x^{\prime}, u^{\dagger} u\right)
$$

It then follows by Lemma 3.6(a) that

$$
\begin{aligned}
\psi_{\varepsilon, d}\left(\left[\psi_{d, d^{-1}}\left(\left[c, c^{\dagger}\right]\right), c^{\prime \prime}\right]\right) & =\psi_{\varepsilon, d}\left(\left[\left(x, u^{\dagger}\right), u \cdot c\right]\right)=u^{\dagger} u \cdot c \\
& =\psi_{d, \varepsilon}\left(\left[c,\left(x^{\prime}, u^{\dagger} u\right)\right]\right)=\psi_{d, \varepsilon}\left(\left[c, \psi_{d^{-1}, d}\left(\left[c^{\dagger}, c^{\prime \prime}\right]\right)\right]\right)
\end{aligned}
$$

We now extend the definition of the balanced fibred product to obtain an operation

$$
\left(B^{(w)}, B^{\left(w^{\prime}\right)}\right) \mapsto B^{\left(w, w^{\prime}\right)}
$$

for $w, w^{\prime} \in \mathbb{F}_{2}$. For each $w, w^{\prime} \in \mathbb{F}_{2}$, let $B^{\left(w, w^{\prime}\right)}$ be the quotient of the set

$$
C^{\left(w, w^{\prime}\right)}:=B_{p^{\left(w^{\prime}\right)}}^{(w)}{\alpha_{w^{\prime}} \circ p^{\left(w^{\prime}\right)}} B^{\left(w^{\prime}\right)}=\left\{\left(c_{1}, c_{2}\right) \in B^{(w)} \times B^{\left(w^{\prime}\right)}: p^{(w)}\left(c_{1}\right)=\alpha_{w^{\prime}}\left(p^{\left(w^{\prime}\right)}\right)\left(c_{2}\right)\right\}
$$

by the equivalence relation

$$
\left(c_{1}, z \cdot c_{2}\right) \sim\left(z \cdot c_{1}, c_{2}\right), \quad \text { for } z \in \mathbb{T}
$$

We endow $C^{\left(w, w^{\prime}\right)}$ with the subspace topology inherited from $B^{(w)} \times B^{\left(w^{\prime}\right)}$, and $B^{\left(w, w^{\prime}\right)}$ with the quotient topology. We denote the equivalence class of $\left(c_{1}, c_{2}\right) \in C^{\left(w, w^{\prime}\right)}$ by $\left[c_{1}, c_{2}\right] \in B^{\left(w, w^{\prime}\right)}$. There is a continuous action of $\mathbb{T}$ on $B^{\left(w, w^{\prime}\right)}$ given by

$$
z \cdot\left[c_{1}, c_{2}\right]:=\left[z \cdot c_{1}, c_{2}\right]=\left[c_{1}, z \cdot c_{2}\right], \quad \text { for } z \in \mathbb{T} \text { and }\left[c_{1}, c_{2}\right] \in B^{\left(w, w^{\prime}\right)}
$$

It follows from Lemmas 3.6 and 3.11 (by an induction argument on $|w|+\left|w^{\prime}\right|$ ) that for all $w, w^{\prime} \in \mathbb{F}_{2}$, there is a $\mathbb{T}$-equivariant homeomorphism

$$
\psi_{w, w^{\prime}}: B^{\left(w, w^{\prime}\right)} \rightarrow B^{\left(w w^{\prime}\right)}
$$

that coincides with the $\mathbb{T}$-equivariant homeomorphisms defined in Lemma 3.6 and Notation 3.10 when $|w|,\left|w^{\prime}\right| \leq 1$, and satisfies the following two properties for all $w, w^{\prime}, w^{\prime \prime} \in \mathbb{F}_{2}$ :
(i) for all $\left[c, c^{\prime}\right] \in B^{\left(w, w^{\prime}\right)}$,

$$
\begin{equation*}
p^{\left(w w^{\prime}\right)}\left(\psi_{w, w^{\prime}}\left(\left[c, c^{\prime}\right]\right)\right)=p^{\left(w^{\prime}\right)}\left(c^{\prime}\right) ; \quad \text { and } \tag{3.8}
\end{equation*}
$$

(ii) for all $c \in B^{(w)}, c^{\prime} \in B^{\left(w^{\prime}\right)}$, and $c^{\prime \prime} \in B^{\left(w^{\prime \prime}\right)}$ with $\left[c, c^{\prime}\right] \in B^{\left(w, w^{\prime}\right)}$ and $\left[c^{\prime}, c^{\prime \prime}\right] \in B^{\left(w^{\prime}, w^{\prime \prime}\right)}$,

$$
\begin{equation*}
\psi_{w w^{\prime}, w^{\prime \prime}}\left(\left[\psi_{w, w^{\prime}}\left(\left[c, c^{\prime}\right]\right), c^{\prime \prime}\right]\right)=\psi_{w, w^{\prime} w^{\prime \prime}}\left(\left[c, \psi_{w^{\prime}, w^{\prime \prime}}\left(\left[c^{\prime}, c^{\prime \prime}\right]\right)\right]\right) \tag{3.9}
\end{equation*}
$$

With $\mathcal{E}$ as in Equation (3.5), we define the set of composable pairs to be the space

$$
\mathcal{E}^{(2)}:=\left\{\left((w, c),\left(w^{\prime}, c^{\prime}\right)\right) \in \mathcal{E} \times \mathcal{E}:\left(c, c^{\prime}\right) \in C^{\left(w, w^{\prime}\right)}\right\} \subseteq \mathcal{E} \times \mathcal{E},
$$

under the subspace topology. We define multiplication on $\mathcal{E}$ by

$$
\mathcal{E}^{(2)} \ni\left((w, c),\left(w^{\prime}, c^{\prime}\right)\right) \mapsto\left(w w^{\prime}, \psi_{w, w^{\prime}}\left(\left[c, c^{\prime}\right]\right)\right) \in \mathcal{E}
$$

Equation (3.9) implies that this multiplication is associative. It is continuous because each $\psi_{w, w^{\prime}}$ is a homeomorphism onto the clopen set $B^{\left(w w^{\prime}\right)}$.
3.3. Defining inversion on the twisted groupoid. In this section we define a continuous inversion map on $\mathcal{E}$. Recall from Notation 3.1 the definition of the involutive homeomorphism $B^{(w)} \ni c \mapsto \bar{c} \in B^{\left(w^{-1}\right)}$ for $w \in \mathbb{F}_{2}$ with $|w|=1$. We first extend this map so that it is defined for all $w \in \mathbb{F}_{2}$ and $c \in B^{(w)}$.

Fix $w=w_{1} \cdots w_{|w|} \in \mathbb{F}_{2}$, and fix $c=\left[c_{1}, \ldots, c_{|w|}\right] \in B^{(w)}$ such that $c_{i} \in B^{\left(w_{i}\right)}$ for each $i \in\{1, \ldots,|w|\}$. By the definition of $B^{(w)}$ and by Equation (3.1), we have

$$
p^{\left(w_{i+1}^{-1}\right)}\left(\overline{c_{i+1}}\right)=\alpha_{w_{i+1}}\left(p^{\left(w_{i+1}\right)}\left(c_{i+1}\right)\right)=p^{\left(w_{i}\right)}\left(c_{i}\right)=\alpha_{w_{i}^{-1}}\left(p^{\left(w_{i}^{-1}\right)}\left(\overline{c_{i}}\right)\right),
$$

and so $\left(\overline{c_{i+1}}, \overline{c_{i}}\right) \in B^{\left(w_{i+1}^{-1}, w_{i}^{-1}\right)}$, for each $i \in\{1, \ldots,|w|-1\}$. Thus

$$
\bar{c}:=\left[\overline{c_{|w|}}, \ldots, \overline{c_{1}}\right] \in B^{\left(w^{-1}\right)}
$$

For all $z \in \mathbb{T}$,

$$
\overline{z \cdot c}=\left[\overline{z \cdot c_{|w|}}, \ldots, \overline{c_{1}}\right]=\left[\bar{z} \cdot \overline{c_{|w|}}, \ldots, \overline{c_{1}}\right]=\bar{z} \cdot \bar{c} .
$$

Moreover, $\overline{\bar{c}}=c$, so $B^{(w)} \ni c \mapsto \bar{c} \in B^{\left(w^{-1}\right)}$ is a $\mathbb{T}$-contravariant involutive homeomorphism.
Lemma 3.12. For all $w \in \mathbb{F}_{2}$ and $c \in B^{(w)}$, we have $(c, \bar{c}) \in C^{\left(w, w^{-1}\right)}$ and $(\bar{c}, c) \in C^{\left(w^{-1}, w\right)}$, and $\psi_{w, w^{-1}}([c, \bar{c}])=\left(p^{\left(w^{-1}\right)}(\bar{c}), 1\right)$ and $\psi_{w^{-1}, w}([\bar{c}, c])=\left(p^{(w)}(c), 1\right)$.
Proof. Fix $w=w_{1} \cdots w_{|w|} \in \mathbb{F}_{2}$, and fix $c=\left[c_{1}, \ldots, c_{|w|}\right] \in B^{(w)}$ such that $c_{i} \in B^{\left(w_{i}\right)}$ for each $i \in\{1, \ldots,|w|\}$. By definition of the principal $\mathbb{T}$-bundle $p^{\left(w^{-1}\right)}: C^{\left(w^{-1}\right)} \rightarrow X$ of Lemma 3.4, we have $p^{\left(w^{-1}\right)}(\bar{c})=p^{\left(w_{1}^{-1}\right)}\left(\overline{c_{1}}\right)$, because $\bar{c}=\left[\overline{c_{|w|}}, \ldots, \overline{c_{1}}\right]$ and $\overline{c_{1}} \in C^{\left(w_{1}^{-1}\right)}$. Using Equation (3.2) for the second equality and Equation (3.1) for the third equality, we see that

$$
\begin{equation*}
\alpha_{w}\left(p^{(w)}(c)\right)=\alpha_{w_{1}}\left(\alpha_{w_{2} \cdots w_{|w|}}\left(p^{(w)}(c)\right)\right)=\alpha_{w_{1}}\left(p^{\left(w_{1}\right)}\left(c_{1}\right)\right)=p^{\left(w_{1}^{-1}\right)}\left(\overline{c_{1}}\right)=p^{\left(w^{-1}\right)}(\bar{c}) \tag{3.10}
\end{equation*}
$$

and so $p^{(w)}(c)=\alpha_{w^{-1}}\left(p^{\left(w^{-1}\right)}(\bar{c})\right)$. Thus $(c, \bar{c}) \in C^{\left(w, w^{-1}\right)}$ and $(\bar{c}, c) \in C^{\left(w^{-1}, w\right)}$.
We show that $\psi_{w^{-1}, w}([\bar{c}, c])=\left(p^{(w)}(c), 1\right)$ by induction on $|w|$. If $|w| \leq 1$, then the claim holds by Lemma 3.6. Suppose that the claim holds for $w \in \mathbb{F}_{2}$ with $|w| \leq n$. Fix $w \in \mathbb{F}_{2}$ with $|w|=n+1$ and $c \in B^{(w)}$. Write $w^{\prime}=w_{1} \cdots w_{n}$. Since $\psi_{w^{\prime}, w_{n+1}}: B^{\left(w^{\prime}, w_{n+1}\right)} \rightarrow$ $B^{(w)}$ is a bijection, there exist $c^{\prime}=\left[c_{1}, \ldots, c_{n}\right] \in B^{\left(w^{\prime}\right)}$ and $c_{n+1} \in B^{\left(w_{n+1}\right)}$ such that $c=$ $\psi_{w^{\prime}, w_{n+1}}\left(\left[c^{\prime}, c_{n+1}\right]\right)$. By the inductive hypothesis, since $\left(c_{n}, c_{n+1}\right) \in C^{\left(w_{n}, w_{n+1}\right)}$,

$$
\psi_{\left(w^{\prime}\right)^{-1}, w^{\prime}}\left(\left[\overline{c^{\prime}}, c^{\prime}\right]\right)=\left(p^{\left(w^{\prime}\right)}\left(c^{\prime}\right), 1\right)=\left(p^{\left(w_{n}\right)}\left(c_{n}\right), 1\right)=\left(\alpha_{w_{n+1}}\left(p^{\left(w_{n+1}\right)}\left(c_{n+1}\right)\right), 1\right) \in B^{(\varepsilon)} .
$$

Thus, Lemma 3.6(a) and Equation (3.9) yield

$$
\psi_{\left(w^{\prime}\right)^{-1}, w}\left(\left[\overline{c^{\prime}}, c\right]\right)=\psi_{\left(w^{\prime}\right)^{-1}, w^{\prime} w_{n+1}}\left(\left[\overline{c^{\prime}}, c\right]\right)=\psi_{\left(w^{\prime}\right)^{-1}, w^{\prime} w_{n+1}}\left(\left[\overline{c^{\prime}}, \psi_{w^{\prime}, w_{n+1}}\left(\left[c^{\prime}, c_{n+1}\right]\right)\right]\right)
$$

$$
\begin{aligned}
& =\psi_{\left(w^{\prime}\right)^{-1} w^{\prime}, w_{n+1}}\left(\left[\psi_{\left(w^{\prime}\right)^{-1}, w^{\prime}}\left(\left[\overline{c^{\prime}}, c^{\prime}\right]\right), c_{n+1}\right]\right) \\
& =\psi_{\varepsilon, w_{n+1}}\left(\left[\left(\alpha_{w_{n+1}}\left(p^{\left(w_{n+1}\right)}\left(c_{n+1}\right)\right), 1\right), c_{n+1}\right]\right)=c_{n+1} \in B^{\left(w_{n+1}\right)},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\psi_{w^{-1}, w}([\bar{c}, c]) & =\psi_{w_{n+1}^{-1},\left(w^{\prime}\right)^{-1} w}\left(\left[\overline{c_{n+1}}, \psi_{\left.\left(w^{\prime}\right)\right)^{-1}, w}\left(\left[\overline{c^{\prime}}, c\right]\right)\right]\right) \\
& =\psi_{w_{n+1}^{-1}, w_{n+1}}\left(\left[\overline{c_{n+1}}, c_{n+1}\right]\right)=\left(p^{\left(w_{n+1}\right)}\left(c_{n+1}\right), 1\right)=\left(p^{(w)}(c), 1\right)
\end{aligned}
$$

completing the induction. An analogous argument gives $\psi_{w, w^{-1}}([c, \bar{c}])=\left(p^{\left(w^{-1}\right)}(\bar{c}), 1\right)$.
For each $(w, c) \in \mathcal{E}=\bigsqcup_{w \in \mathbb{F}_{2}} B^{(w)}$, we define

$$
(w, c)^{-1}:=\left(w^{-1}, \bar{c}\right) .
$$

For each fixed $w \in \mathbb{F}_{2}$, the map $\{w\} \times B^{(w)} \ni(w, c) \mapsto(w, c)^{-1} \in\left\{w^{-1}\right\} \times B^{\left(w^{-1}\right)}$ is a homeomorphism, so inversion is continuous on $\mathcal{E}=\bigsqcup_{w \in \mathbb{F}_{2}}\{w\} \times B^{(w)}$.

Fix $(w, c) \in \mathcal{E}$. Then $\left((w, c)^{-1}\right)^{-1}=(w, c)$. Moreover, by Lemma 3.12, we have

$$
\left((w, c),\left(w^{-1}, \bar{c}\right)\right),\left(\left(w^{-1}, \bar{c}\right),(w, c)\right) \in \mathcal{E}^{(2)} .
$$

It follows from Lemma 3.12 that

$$
\begin{aligned}
r(w, c):=(w, c)(w, c)^{-1} & =(w, c)\left(w^{-1}, \bar{c}\right)=\left(w w^{-1}, \psi_{w, w^{-1}}([c, \bar{c}])\right) \\
& =\left(\varepsilon,\left(p^{\left(w^{-1}\right)}(\bar{c}), 1\right)\right)=\left(\varepsilon,\left(\alpha_{w}\left(p^{(w)}(c)\right), 1\right)\right)
\end{aligned}
$$

and

$$
s(w, c):=(w, c)^{-1}(w, c)=\left(w^{-1}, \bar{c}\right)(w, c)=\left(w^{-1} w, \psi_{w^{-1}, w}([\bar{c}, c])\right)=\left(\varepsilon,\left(p^{(w)}(c), 1\right)\right) .
$$

Since $p^{(w)}: B^{(w)} \rightarrow X$ is surjective, it follows that

$$
\mathcal{E}^{(0)}:=r(\mathcal{E})=s(\mathcal{E})=\{\varepsilon\} \times(X \times\{1\})
$$

Identify $\mathcal{E}^{(0)}$ with $X$. To see that $\mathcal{E}^{(0)}$ consists of multiplicative units, fix $x, y \in X$ such that

$$
r(w, c)=(\varepsilon,(x, 1)) \quad \text { and } \quad s(w, c)=(\varepsilon,(y, 1)) .
$$

Then by Lemma 3.6(a), we have

$$
r(w, c)(w, c)=(\varepsilon,(x, 1))(w, c)=\left(\varepsilon w, \psi_{\varepsilon, w}([(x, 1), c])\right)=(w, c),
$$

and

$$
(w, c) s(w, c)=(w, c)(\varepsilon,(y, 1))=\left(w \varepsilon, \psi_{w, \varepsilon}([c,(y, 1)])\right)=(w, c)
$$

as required. Hence $\mathcal{E}$ is a topological groupoid under the given operations.
3.4. Defining the quotient groupoid $\mathcal{G}$ and proving that $\mathcal{E} \rightarrow \mathcal{G}$ is a twist. Recall from Lemma 3.3 that for each $w \in \mathbb{F}_{2}, B^{(w)}$ is a locally compact Hausdorff space. Since $\mathcal{E}$ has the disjoint union topology, it follows that $\mathcal{E}$ is a locally compact Hausdorff groupoid.

We now define the quotient groupoid. Let

$$
\mathcal{G}:=X \rtimes_{\alpha} \mathbb{F}_{2}=\left\{\left(\alpha_{w}(x), w, x\right): x \in X, w \in \mathbb{F}_{2}\right\} \subseteq X \times \mathbb{F}_{2} \times X
$$

Then $\mathcal{G}$ is a groupoid under the multiplication and inversion operations

$$
(x, w, u)\left(u, w^{\prime}, y\right):=\left(x, w w^{\prime}, y\right) \quad \text { and } \quad(x, w, y)^{-1}:=\left(y, w^{-1}, x\right) .
$$

The collection

$$
\left\{\left\{\left(\alpha_{w}(x), w, x\right): x \in U\right\}: w \in \mathbb{F}_{2}, U \text { is an open subset of } X\right\}
$$

is a basis for a locally compact Hausdorff étale groupoid topology on $\mathcal{G}$. The unit space of $\mathcal{G}$ is $\mathcal{G}^{(0)}=\{(x, \varepsilon, x): x \in X\}$, which we identify with $X$. The range and source maps are given by $r(x, w, y)=x$ and $s(x, w, y)=y$. Since $\alpha_{a}=\sigma$ is minimal, $\mathcal{G}$ is minimal. The

1-cocycle $c_{\mathcal{G}}: \mathcal{G} \rightarrow \mathbb{F}_{2}$ defined by $c_{\mathcal{G}}\left(\alpha_{w}(x), w, x\right):=w$ is continuous and $\mathbb{F}_{2}$ is discrete, so $\mathcal{G}_{w}:=c_{\mathcal{G}}^{-1}(w)$ is clopen in $\mathcal{G}$ for each $w \in \mathbb{F}_{2}$. We now show that $\mathcal{E}$ is a twist over $\mathcal{G}$.

Proposition 3.13. Define $\imath: X \times \mathbb{T} \rightarrow \mathcal{E}$ by $\imath(x, z):=(\varepsilon,(x, z))$, and define $\pi: \mathcal{E} \rightarrow \mathcal{G}$ by $\pi(w, c):=\left(\alpha_{w}\left(p^{(w)}(c)\right), w, p^{(w)}(c)\right)$. Then $X \times \mathbb{T} \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \mathcal{G}$ is a twist over $\mathcal{G}$.
Proof. We use the definition of a twist given in [6, Definition 3.1] (see also [1, Remark 2.6]). We have $\mathcal{E}^{(0)}=\imath(X \times\{1\})$, and $\imath$ is a continuous groupoid homeomorphism onto the open set $\imath(X \times \mathbb{T})=\mathcal{E}_{\varepsilon}=\pi^{-1}\left(\mathcal{G}_{\varepsilon}\right)$ that restricts to a homeomorphism of unit spaces. To see that $\pi$ is a homomorphism, fix $\left((w, c),\left(w^{\prime}, c^{\prime}\right)\right) \in \mathcal{E}^{(2)}$. Then $\left(c, c^{\prime}\right) \in C^{\left(w, w^{\prime}\right)}$, so $p^{(w)}(c)=\alpha_{w^{\prime}}\left(p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)\right)$. Write $c c^{\prime}:=\psi_{w, w^{\prime}}\left(\left[c, c^{\prime}\right]\right)$. By Equation (3.8), $p^{\left(w w^{\prime}\right)}\left(c c^{\prime}\right)=p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)$, and so

$$
\begin{aligned}
\pi(w, c) \pi\left(w^{\prime}, c^{\prime}\right) & =\left(\alpha_{w}\left(p^{(w)}(c)\right), w, p^{(w)}(c)\right)\left(\alpha_{w^{\prime}}\left(p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)\right), w^{\prime}, p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)\right) \\
& =\left(\alpha_{w}\left(p^{(w)}(c)\right), w w^{\prime}, p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)\right)=\left(\alpha_{w w^{\prime}}\left(p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)\right), w w^{\prime}, p^{\left(w^{\prime}\right)}\left(c^{\prime}\right)\right) \\
& =\left(\alpha_{w w^{\prime}}\left(p^{\left(w w^{\prime}\right)}\left(c c^{\prime}\right)\right), w w^{\prime}, p^{\left(w w^{\prime}\right)}\left(c c^{\prime}\right)\right)=\pi\left(w w^{\prime}, c c^{\prime}\right)=\pi\left((w, c)\left(w^{\prime}, c^{\prime}\right)\right)
\end{aligned}
$$

and hence $\pi$ is a groupoid homomorphism. Since $p^{(w)}$ is a continuous surjection, $\pi$ is a continuous surjection. For all $x \in X, \pi(\varepsilon,(x, 1))=(x, \varepsilon, x)$, so $\pi$ restricts to a homeomorphism of unit spaces. To see that $\pi$ is an open map, first note that for each $w \in \mathbb{F}_{2}$, it follows by local triviality of the principal $\mathbb{T}$-bundle $p^{(w)}: B^{(w)} \rightarrow X$ that $p^{(w)}$ is an open map (using an argument similar to the proof of [1, Lemma 2.7(a)]). So for each fixed $w \in \mathbb{F}_{2}$ and any open set $V \subseteq B^{(w)}$, we have $\pi(\{w\} \times V)=\left\{\left(\alpha_{w}(x), w, x\right): x \in p^{(w)}(V)\right\}$, which is open in $\mathcal{G}$ because $p^{(w)}(V)$ is open in $X$. Since $\mathcal{E}$ has the disjoint union topology, it follows that $\pi$ is an open map. To see that the extension is central, fix $x, y \in X, z \in \mathbb{T}$, and $(w, c) \in \mathcal{E}$ such that $(\imath(x, z),(w, c)),((w, c), \imath(y, z)) \in \mathcal{E}^{(2)}$. Then $x=\alpha_{w}\left(p^{(w)}(c)\right)=\alpha_{w}(y)$, and by Lemma 3.6(a), $\psi_{\varepsilon, w}([(x, z), c])=z \cdot c=\psi_{w, \varepsilon}([c,(y, z)])$. Thus,

$$
\imath(x, z)(w, c)=(\varepsilon,(x, z))(w, c)=(w, z \cdot c)=(w, c)(\varepsilon,(y, z))=(w, c) \imath(y, z)
$$

so $\imath(X \times \mathbb{T})$ is central in $\mathcal{E}$.
Remark 3.14. For $(x, w, y) \in \mathcal{G}$, we have $\alpha_{w}(x)=y$. Hence $(x, w, y) \mapsto(w, y)$ is a homeomorphism $\mathcal{G} \rightarrow \mathbb{F}_{2} \times X$ that carries each $\mathcal{G}_{w}$ onto $\{w\} \times X$. Thus $s_{b}$ : $\mathcal{G}_{b} \ni(x, b, x) \mapsto x \in X$ and $j_{b}: B^{(b)} \ni c \mapsto(b, c) \in \mathcal{E}_{b}$ are homeomorphisms that satisfy $\left.s_{b} \circ \pi\right|_{\mathcal{E}_{b}} \circ j_{b}=p^{(b)}$, where $p^{(b)}: B^{(b)} \rightarrow X$ is our original nontrivial principal $\mathbb{T}$-bundle $p: B \rightarrow X$ from Section 3.1.
3.5. Studying the isotropy and proving the desired properties. In this section we prove that the twist of Proposition 3.13 is not induced by a 2 -cocycle, even when restricted to the interior of the isotropy. For a groupoid $G$, let $\mathcal{I}^{G}$ denote the topological interior of its isotropy $\operatorname{Iso}(G)=\{\gamma \in \mathcal{G}: r(\gamma)=s(\gamma)\}$. Define the " $a$-counting map" $\ell_{a}: \mathbb{F}_{2} \rightarrow \mathbb{Z}$ to be the homomorphism defined on generators $a, b \in \mathbb{F}_{2}$ by

$$
\ell_{a}(a)=1 \quad \text { and } \quad \ell_{a}(b)=0 .
$$

Lemma 3.15. For the groupoid $\mathcal{G}=X \rtimes_{\alpha} \mathbb{F}_{2}$ defined in Section 3.4, we have

$$
\mathcal{I}^{\mathcal{G}}=\operatorname{Iso}(\mathcal{G})=\bigsqcup_{w \in \operatorname{ker}\left(\ell_{a}\right)} \mathcal{G}_{w} \simeq X \times \operatorname{ker}\left(\ell_{a}\right) \quad \text { and } \quad \mathcal{I}^{\mathcal{E}}=\pi^{-1}\left(\mathcal{I}^{\mathcal{G}}\right)=\operatorname{Iso}(\mathcal{E})=\bigsqcup_{w \in \operatorname{ker}\left(\ell_{a}\right)} \mathcal{E}_{w}
$$

Proof. By definition,

$$
\operatorname{Iso}(\mathcal{G})=\left\{\left(\alpha_{w}(x), w, x\right): x \in X, w \in \mathbb{F}_{2}, \alpha_{w}(x)=x\right\} .
$$

Since the $\mathbb{T}$-bundle $p^{(b)}: B^{(b)} \rightarrow X$ is nontrivial, $X$ is not discrete, and in particular, every dense subset of $X$ is infinite. Thus, if $\sigma^{\ell(w)}=\alpha_{w}(x)=x$, then minimality of $\sigma$ implies that $\ell_{a}(w)=0$ (for if $\ell_{a}(w)>0$ then the orbit of $x$ under $\sigma$ is finite). Thus

$$
\operatorname{Iso}(\mathcal{G})=\left\{(x, w, x) \in \mathcal{G}: x \in X, w \in \mathbb{F}_{2}, \ell_{a}(w)=0\right\}=\bigsqcup_{w \in \operatorname{ker}\left(\ell_{a}\right)} \mathcal{G}_{w}
$$

is homeomorphic to $X \times \operatorname{ker}\left(\ell_{a}\right)$. Since $\mathcal{G}_{w}=c_{\mathcal{G}}^{-1}(w)$ is open in $\mathcal{G}$ for each $w \in \mathbb{F}_{2}, \operatorname{Iso}(\mathcal{G})$ is a union of open sets and is therefore open, and so $\mathcal{I}^{\mathcal{G}}=\operatorname{Iso}(\mathcal{G})$.

By the proof of [1, Corollary 2.11(b)], we have $\mathcal{I}^{\mathcal{E}}=\pi^{-1}\left(\mathcal{I}^{\mathcal{G}}\right)=\pi^{-1}(\operatorname{Iso}(\mathcal{G}))=\operatorname{Iso}(\mathcal{E})$. Since $\pi^{-1}\left(\mathcal{G}_{w}\right)=\mathcal{E}_{w}$ for each $w \in \mathbb{F}_{2}$, it follows that

$$
\mathcal{I}^{\mathcal{E}}=\pi^{-1}\left(\mathcal{I}^{\mathcal{G}}\right)=\bigsqcup_{w \in \operatorname{ker}\left(\ell_{a}\right)} \pi^{-1}\left(\mathcal{G}_{w}\right)=\bigsqcup_{w \in \operatorname{ker}\left(\ell_{a}\right)} \mathcal{E}_{w} .
$$

Lemma 3.16. The restricted twist $\left.\pi\right|_{\mathcal{I}^{\varepsilon}}: \mathcal{I}^{\mathcal{E}} \rightarrow \mathcal{I}^{\mathcal{G}}$ is not induced by a 2-cocycle.
Proof. Suppose for contradiction that $\left.\pi\right|_{\mathcal{I}^{\mathcal{E}}}: \mathcal{I}^{\mathcal{E}} \rightarrow \mathcal{I}^{\mathcal{G}}$ comes from a 2-cocycle. Then $\left.\pi\right|_{\mathcal{I}^{\mathcal{E}}}$ admits a continuous global section $\Sigma: \mathcal{I}^{\mathcal{G}} \rightarrow \mathcal{I}^{\mathcal{E}}$. By Lemma 3.15, $\mathcal{G}_{b} \subseteq \mathcal{I}^{\mathcal{G}}$ and $\mathcal{E}_{b}=$ $\pi^{-1}\left(\mathcal{G}_{b}\right) \subseteq \mathcal{I}^{\mathcal{E}}$. So $\left.\Sigma\right|_{\mathcal{G}_{b}}$ is a continuous section for $\left.\pi\right|_{\mathcal{E}_{b}}: \mathcal{E}_{b} \rightarrow \mathcal{G}_{b}$. By Remark 3.14, this gives a continuous section for $p: B \rightarrow X$ contradicting that $p: B \rightarrow X$ is nontrivial.

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